A SPECTRAL THEOREM FOR A PAIR OF SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT

The present paper deals with the problem involving the nature of the spectrum of a second order matrix differential equation with a prescribed set of boundary conditions at an end point. The method used is Titchmarsh's complex variable method as initiated in his " Eigenfunction expansion associated with the second order differential equations ".

Key words: Spectrum, boundary-condition vector, Kronecker delta, L^2-solution, eigenvalue parameter, principle of reflection, entire functions.

1. INTRODUCTION

The object of the present paper is to study the nature of the spectrum associated with a second-order differential system

$$LU = \lambda U, \tag{1}$$

where

$$L = \begin{pmatrix} -\dfrac{d^2}{dx^2} + p(x) & r(x) \\ r(x) & -\dfrac{d^2}{dx^2} + q(x) \end{pmatrix}, \quad U = \{u, v\} = \begin{pmatrix} u \\ v \end{pmatrix}$$

and λ is the eigenvalue parameter; $p(x)$ and $q(x)$ are real valued functions each twice differentiable with respect to the variable x; $r(x)$ is real valued and continuous in $0 \leqslant x < \infty$; $p(x)$, $q(x)$ and or $r(x)$ tend to infinity while $p'(x)$, $P''(x)$, $q'(x)$, $q''(x)$ remain finite as x tends to infinity (the accent denotes differentiation with respect to x).

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The boundary conditions considered are

$$[U(x, \lambda), \phi_l](0) \equiv a_{j1} u(0) + a_{j2} u'(0) + a_{j3} v(0) + a_{j4} v'(0)$$
$$= 0 \; (j = 1, 2) \tag{2}$$

where

$$[\phi_1 \phi_2](0) = 0, \tag{3}$$

$U(x, \lambda)$ is a solution of (1.1) and $\phi_l \equiv \phi_l(0/x, \lambda)$ $(l = 1, 2)$ are the ' boundary-condition vectors ' at $x = 0$ (*See* Chakrabary [2]).

Put

$$u_j(0|0,/\lambda) = a_{j2}, \quad u'_j(0/0, \lambda) = -a_{j1}$$
$$(j = 1, 2) \tag{4}$$
$$v_j(0|0,/\lambda) = a_{j4}, \quad v'_j(0/0, \lambda) = -a_{j3}$$

where

(i) a_{jk} $(J = 1, 2;\ k = 1, 2, 3, 4)$ are real valued constants, independent of λ.

(ii) The set $\{a_{1k}\}$ is linearly independent of the set $\{a_{2k}\}$.

Let

$$\theta_k(x, \lambda) = \{x_k(0/x, \lambda),\ y_k(0/x, \lambda)\} \qquad (k = 1, 2)$$

which take real values independent of λ at $x = 0$ and satisfy (1) be determined by

$$[\theta_1 \theta_2] = 0, \qquad [\phi_j \theta_k] = \delta_{jk} \; (j, k = 1, 2), \tag{5}$$

δ_{jk} being the Kronecker delta.

Then θ_k are entire functions of λ and

$$x_k(0/0, \lambda) = (-1)^k a_{l4}, \qquad x_k'(0/0, \lambda) = (-1)^{k-1} a_{l3}$$
$$y_k(0/0, \lambda) = (-1)^k a_{l2}, \qquad y_k'(0/0, \lambda) = (-1)^{k-1} a_{l1}$$

(when $k = 1$, $l = 2$ and when $k = 2$, $l = 1$).

As and when necessary we shall use the results of Chakrabarty [2–3], Bhagat [1].

2. NOTATIONS AND ABBREVIATIONS

In what follows we write

$$(Y, Z) = y_1(t)z_1(t) + y_2(t)z_2(t)$$

for two vectors

$$Y = \{y_1(t), \quad y_2(t)\}, \qquad Z = \{(z_1(t), \quad z_2(t)\}$$

See Chakrabarty [2] and Naimark [5].

Further we represent the vector $\{\xi_{ij}, \eta_{ij}\}$ by $(\xi\eta)_{ij}$.

We make use of the following abbreviations in the present paper.

$$z(t) = \{\lambda - p(t)\}\{\lambda - q(t)\}$$

$$M(t) = -(1/4)z(t)^{-1/4}[p''(\lambda - p)^{-1} + q''(\lambda - q)^{-1} + $$
$$+ (5/4)p'^2(\lambda - p)^{-2} + (5/4)q'^2(\lambda - q)^{-2} + (1/2)p'q'z(t)^{-1}]$$

$$\xi_1(t) \equiv \xi_1(p, q, t) = z(t)^{1/2} - M(t)z(t)^{-1/4} + \{\lambda - p)/(t)\}^{1/2}/\{\lambda - q(t)\}^{1/2}$$

$$\eta_1(t) = -r(t)z(t)^{-1/2}$$

$$\zeta_1(t) = \xi_1(q, p, t)$$

$$\xi_{11}(t) = |\xi_1(t)/z(t)|$$

$$\xi_{12}(t) \equiv \xi_2(p, q, t) = [|z(t)|^{1/2} + |M(t)/z(t)^{1/4}|$$
$$+ |(\lambda - p(t)/\{\lambda - q(t)\}|^{1/2}]/|z(t)|$$

$$\xi_{13}(t) = |r(t)/z(t)^{-3/2}|$$

$$\eta_{11}(t) = |\eta_1(t)/z(t)|$$

$$\eta_{12}(t) = \xi_{13}(t)$$

$$\eta_{13}(t) = \xi_2(q, p, t)$$

$$N_1(t) = \begin{pmatrix} \xi_1(t) & \eta_1(t) \\ \eta_1(t) & \zeta_1(t) \end{pmatrix}$$

$$N_2(t) = \{z(t)^{-1}, \qquad z(t)^{-1}\}$$

$$\zeta(x) = \begin{pmatrix} \cos\xi(x) \\ \sin\xi(x) \end{pmatrix}$$

We write $|A|$ to represent the matrix whose elements are the moduli of the elements of the corresponding matrix A.

3. A Basic Transformation

The system (1) is equivalent to the system

$$\frac{d^2 u}{dx^2} + (\lambda - p(x))\, u = r(x)\, v$$

$$\frac{d^2 v}{dx^2} + \{\lambda - q(x)\}\, v = r(x)\, u \tag{6}$$

By means of the transformation

$$\xi(x) = i \int_0^x \big(\lambda - p(t)\big)^{1/2} \big(\lambda - q(t)\big)^{1/2}\, dt$$

$$\{\eta(x),\ \zeta(x)\} = i \big(\lambda - p(x)\big)^{1/4} \big(\lambda - q(x)\big)^{1/4} \{u(x),\ v(x)\} \tag{7}$$

the system (6) is transformed to

$$\frac{d^2 \eta}{d\xi^2} + \{K(x, \lambda) - 1/\big(\lambda - q(x)\big)\}\, \eta = r_1(x)\, \zeta \tag{8}$$

$$\frac{d^2 \zeta}{d\xi^2} + \{K(x, \lambda) - 1/\big(\lambda - p(x)\big)\}\, \zeta = r_1(x)\, \eta$$

i.e.,

$$(L_1 - K(x, \lambda))\, \Omega = 0, \tag{9}$$

where

$$\begin{aligned} K(x, \lambda) = -\, & [(1/4)\{p''(\lambda - p)^{-2}(\lambda - q)^{-1} + q''(\lambda - q)^{-2}(\lambda - p)^{-1}\} \\ & + (5/16)\{p'^2(\lambda - p)^{-3}(\lambda - q)^{-1} + q'^2(\lambda - q)^{-3}(\lambda - p)^{-1}\} \\ & + (1/8)\, p' q' z(x)^{-2}] \end{aligned}$$

$$r_1(x) = -\, r(x)\, z(x)^{-1},$$

and

$$L_1 \text{ is the operator } \begin{pmatrix} -\dfrac{d^2}{d\xi^2} + \dfrac{1}{\lambda - q(x)} & r_1(x) \\ r_1(x) & -\dfrac{d^2}{d\xi^2} + \dfrac{1}{\lambda - p(x)} \end{pmatrix},$$

$$\Omega = \Omega(x) = \begin{pmatrix} \eta(x) \\ \zeta(x) \end{pmatrix} = \begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \{\eta,\ \zeta\}.$$

The system (8) is of the same form as (6) where the co efficients of η and ζ tend to zero when x tends to infinity. This happens for example, for a given finite λ when

(i) $r(x)$ is bounded and negative for all x but $|p|$, $|q|$ tend to infinity as x tends to infinity.

(ii) $p(x)$, $q(x)$ satisfy the conditions stated in (i) but

$$r = 0(p) \qquad \text{or} \qquad r = 0(p).$$

If however λ tends to infinity, we can take $r = 0(pq)$, where p, q satisfy the conditions stated in (i).

Let

$$P(x) = z(x)^{1/4} H(x) \tag{10}$$

where

$$H(x) = \{H_1(x),\ H_2(x)\}$$

with

$$H_1(x) = \frac{d}{dx}\left[z(x)^{-1/2}\frac{d\eta}{dx}\right] - i\,z(x)^{-1/4}\frac{d^2 u}{dx^2}$$

$$H_2(x) = \frac{d}{dx}\left[z(x)^{-1/2}\frac{d\zeta}{dx}\right] - iz(x)^{-1/4}\frac{d^2 v}{dx^2}$$

Then from (10) and (6)

$$P(x) = M(x)\,\Omega(x). \tag{11}$$

Let $$I = \{I_1, I_2\} = \int_0^x z(t)^{-1/4} \sin\big(\xi(x) - \xi(t)\big)\, P(t)\, dt \tag{12}$$

so that

$$I_1 = \int_0^x \sin\big(\xi(x) - \xi(t)\big)\Big[\frac{d}{dt}\Big\{z(t)^{-1/2}\frac{d\eta}{dt}\Big\} +$$

$$+ \{i\big(\lambda - p(t)\big)\,u(t) - r(t)\,v(t)\}\,z(t)^{-1/4}\Big]\,dt$$

$$= \int_0^x \sin\big(\xi(x) - \xi(t)\big)\frac{d}{dt}\Big[z(t)^{-1/2}\frac{d\eta}{dt}\Big]\,dt +$$

$$+ \int_0^x \sin\big(\xi(x) - \xi(t)\big) \left[\frac{\{\lambda - p(t)\}^{1/2}}{\{\lambda - q(t)\}^{1/2}} \, \eta(t) \right.$$

$$\left. - \, r(t)\, \zeta(t)\, z(t)^{-1/2} \right] dt$$

$$= I_{11} + I_{12}, \text{ say.}$$

On integration by parts and subsequent simplification, we have

$$I_{11} = - \, z(0)^{-1/2} \eta'(0) \sin \xi(x) - i\eta(0) \cos \xi(x) + i\eta(x) +$$

$$+ \int_0^x \sin\big(\xi(x) - \xi(t)\big)\, z(t)^{1/2}\, \eta(t)\, dt.$$

Therefore,

$$I_1 = - \, z(0)^{-1/2} \sin \xi(x)\, \eta'(0) - i \cos \xi(x)\, \eta(0) + i\eta(x) +$$

$$+ \int_0^x \sin\big(\xi(x) - \xi(t)\big)\, z(t)^{1/2}\, dt +$$

$$+ \int_0^x \sin\big(\xi(x) - \xi(t)\big) \left[\frac{\{\lambda - p(t)\}^{1/2}}{\{\lambda - q(t)\}^{1/2}} \, \eta(t) \right.$$

$$\left. - \, r(t)\, \zeta(t)\, z(t)^{-1/2} \right] dt$$

with a similar result for I_2.

Hence if $l_1(t) \equiv \{\xi_1(t), \quad \eta_1(t)\}$, it follows that

$$\eta(x) = \eta(0) \cos \xi(x) - i\, \eta'(0)\, z(0)^{-1/2} \sin \xi(x) +$$

$$+ \, i \int_0^x \sin\big(\xi(x) - \xi(t)\big) \big(l_1(t), \quad \Omega(t)\big)\, dt. \tag{13}$$

Similarly,

$$\zeta(x) = \zeta(0) \cos \xi(x) - i\, \zeta'(0)\, z(0)^{-1/2} \sin \xi(x) +$$

$$+ \, i \int_0^x \sin\big(\xi(x) - \xi(t)\big) \big(l_2(t), \quad \Omega(t)\big)\, dt \tag{14}$$

where

$$l_2(t) \equiv \{\eta_1(t), \quad \zeta_1(t)\}.$$

Hence finally

$$\Omega(x) = N(0)\, S(x) + i \int_0^x \sin\big(\xi(x) - \xi(t)\big)\, N_1(t)\, \Omega(t)\, dt \qquad (1$$

where

$$N(0) = \begin{pmatrix} \eta(0) & -i\eta'(0) & z(0)^{-1/2} \\ \zeta(0) & -i\zeta'(0) & z(0)^{-1/2} \end{pmatrix}$$

4. Associated Lemmas

Lemma I

Let the coefficients $p(x)$, $q(x)$, $r(x)$ of (1) satisfy

(i) either $p(x),\ q(x) > Q(x),\ r(x) = 0(1)$

or,

$$p(x),\ q(x),\ r(x) > Q(x),\ r(x) = 0\big(p(x)\, q(x)\big)$$

where

$Q(x) \geqslant \delta > 0,\ x \geqslant 0$

(ii) $p'(x),\ q'(x) \geqslant 0$

(iii) $p'(x) = 0\,[p(x)]^c, \quad q'(x) = 0\,[q(x)]^c, \quad 0 < c < 5/2$

(iv) $p''(x)\, q''(x)$ are ultimately of one sign

(v) $1/Q(x) \in L\,[0, \infty)$.

Then

$$\int_0^\infty |N_1(t)|\,|N_2(t)|\, dt \text{ is uniformly}$$

convergent with respect to λ in any region for which

$$|\lambda - p(x)|,\ |\lambda - q(x)| \geqslant \delta_1 > 0 \qquad \text{for} \qquad 0 \leqslant x < \infty;$$

N_1, N_2 being defined as before.

Proof.—We have

$$\int_{x_1}^{x} [|\xi_1(t)| + |\eta_1(t)|]/|z(t)|\, dt$$

$$\leqslant \int_{X_0}^{X} |z(t)|^{-1/2}\,dt + \int_{X_0}^{X} |(\lambda - p(t))^{-1/2}(\lambda - q(t))^{-3/2}|\,dt$$

$$+ \int_{X_0}^{X} |r(t)\,z(t)^{-3/2}|\,dt + \int_{X_0}^{X} |M(t)\,z(t)^{-5/4}|\,dt \tag{16}$$

$$= J_{11} + J_{12} + J_{13} + J_{14}, \text{ say}$$

Now,

$$J_{11} = 0\,[\int_{X_0}^{X} 1/Q(t)\,dt] = 0\,(1), \text{ uniformly in } \lambda.$$

$$J_{12} = 0\,[\int_{X_0}^{X} 1/Q^2(t)\,dt] = 0\,(1), \text{ uniformly in } \lambda.$$

By condition (i) of the lemma it follows that

$$J_{13} = 0(1), \text{ uniformly in } \lambda.$$

$$J_{14} = 0\,[\int_{X_0}^{X} p''\,p^{-5/2}\,q^{-3/2}\,dt] + 0\,[\int_{X_0}^{X} q''\,p^{-3/2}\,q^{-5/2}\,dt] +$$

$$+ 0\,[\int_{X_0}^{X} p'q'\,p^{-5/2}\,q^{-5/2}\,dt] + 0\,[\int_{X_0}^{X} p'^2\,p^{-7/2}\,q^{-3/2}\,dt] +$$

$$+ 0\,[\int_{X_0}^{X} q'^2\,q^{-7/2}p^{-3/2}\,dt].$$

$$= J_1 + J_2 + J_3 + J_4 + J_5, \text{ say}$$

On integration by parts and using conditions (iii) of the problem, it follows that

$$J_1 = 0\,[\int_{X_0}^{X} p''\,p^{-5/2}\,dt]$$

$$= 0\,(1), \text{ uniformly in } \lambda,$$

where

$$0 < c < 5/2.$$

Similarly,

$$J_2 = 0\,(1), \quad \text{uniformly in } \lambda, \text{ where } 0 < c < 5/2.$$

By the Schwartz inequality,

$$J_3 = 0\,[\int_{X_0}^{X} p'^2 p^{-5}\,dt \int_{X_0}^{X} q'^2 q^{-5}\,dt]^{1/2}$$

$$= 0\,[\int_{X_0}^{X} p'\, p^{c-5}\,dt \int_{X_0}^{X} q'\, q^{c-5}\,dt]^{1/2}$$

$$= 0\,(1), \text{ uniformly in } \lambda, \quad \text{where} \quad 0 < c < 4.$$

Again,

$$J_4 = 0\,[\int_{X_0}^{X} p'^2 p^{-7/2}\,dt$$

$$= 0\,[\int_{X_0}^{X} p'\, p^{c-7/2}\,dt]$$

$$= 0\,(1), \text{ uniformly in } \lambda, \quad \text{where} \quad 0 < c < 5/2.$$

Similarly.

$$J_5 = 0\,(1), \text{ uniformly in } \lambda, \text{ where } 0 < c < 5/2.$$

Therefore,

$$\int_{X_0}^{X} M\,(t)\, z\,(t)^{-5/4}\,dt = 0\,(1), \text{ uniformly in } \lambda,$$

where $0 < c < 5/2$.

It follows therefore from (16) that $\int_0^\infty [|\,\xi_1\,| + |\,\eta_1\,|]/|\,z\,|\,dt$ converge uniformly with respect to λ (real or complex). Similar result holds for

$$\int_0^\infty [|\,\eta_1\,| + |\,\zeta_1\,|]/|\,z\,|\,dt.$$

The lemma therefore follows.

Lemma II.

If $Im\,\lambda > 0$, $0 < arg\,\lambda < \pi$, then

$$\exp\,[i\xi\,(x)] \to \infty \text{ as } x \to \infty, \ \xi\,(x)$$

being defined in (7).

Proof.—Since $p(t),\ q(t) > 0$,

$$(1/2)\ arg\ \lambda \leqslant arg\ [(\lambda - p(t))^{1/2}] \leqslant \pi/2$$

$$(1/2)\ arg\ \lambda \leqslant arg\ [(\lambda - q(t))^{1/2}] \leqslant \pi/2$$

so that

$$arg\ \lambda \leqslant arg\ [z(t)^{1/2}] \leqslant \pi$$

Again,

$$\pi/2 + arg\ \lambda \leqslant arg\ [iz(t)^{1/2}] \leqslant 3\pi/2.$$

It follows from the definition of $\xi(x)$ that

$$Im\ \xi(x) > 0.$$

Also, if $|\lambda|$ is bounded as x tends to infinity, then

$$i\xi\ (x) \sim i^2 \int_0^x i(p(t))^{1/2}\, i(q(t))^{1/2}\, dt$$

$$= \int_0^x (pq)^{1/2}\, dt.$$

Hence the lemma follows.

5. Some Order Relations

We assume that all the conditions of the lemma I are satisfied.

Let

$$\Omega_2(x) = \{\eta_2(x),\ \zeta_2(x)\} = z(x)\,\Omega(x) \exp[i\xi(x)].$$

Therefore from (15),

$$\Omega_2(x) = (\tfrac{1}{2})\, z(x)\, [1 + \exp(-2i\xi(x))]\, \Omega(0) -$$
$$- (\tfrac{1}{2})\, z(x)\, z(0)^{-1/2}\, [1 - \exp(-2i\,\xi(x))]\, \Omega'(0) +$$
$$+ (\tfrac{1}{2})\, z(x) \int_0^x [1 - \exp(2 - i(\xi(x) - \xi(t)))]\, z(t)^{-1} \times$$
$$\times N_1(t)\, \Omega_1(t)\, dt.$$

Then

$$|\eta_2(x)/z(x)| \leqslant |\eta(0)| + |z(0)^{-1/2}\eta'(0)| + \int_0^x [\xi_{11}(t)\, |\eta_2(t)|$$
$$+ \eta_{11}(t)\, |\zeta_2(t)|]\, dt$$

for large x. Hence,

$$| \eta_2 (x)/z (x) | \leqslant |\eta (0) | + |z (0)^{1/2} + \eta' (0) | + \langle (\xi \eta)_{12}, (\eta \zeta)_{22} \rangle 0, x \tag{17}$$

where

$$\langle Y, Z \rangle 0, x = \int_0^\infty (Y, Z)\, dt$$

for two vectors

$$Y = \{y_1, y_2\} \qquad Z = \{z_1, z_2\}$$

See Chakrabarty [3].

Similarly,

$$| \zeta_2 (x)/z (x) | \leqslant | \zeta (0) | + | z (0)^{-1/2} \zeta' (0) | + \langle (\xi \eta)_{12}, (\eta\zeta)_{22} \rangle 0, x. \tag{18}$$

Since the same arguments hold if p, q be interchanged in the differential system (1), we can, without loss of generality, assume that $p (x) > q (x)$ in the discussion which follows.

Then

$$\eta_{13} (t) < \xi_{12} (t).$$

Therefore,

$$| \eta_2 (x)/z (x) | \leqslant | \eta (0) | + | z (0)^{-1/2} \eta' (0) | + \langle (\xi\eta)_{12}, (\eta\zeta)_{22} \rangle 0, x$$

$$| \zeta_2 (x)/z (x) | \leqslant | \zeta (0) | + | z (0)^{-1/2} \zeta' (0) | + \langle (\eta\xi)_{12}, (\eta\zeta)_{22} \rangle 0, x$$

Hence,

$$| \eta_2 (x)/z (x) |, \ | \zeta_2 (x)/z (x) | \leqslant K_1 + \langle s_1, s_2 \rangle 0, x \tag{19}$$

where

$$K_1 = \max [| \eta (0) | + | z (0)^{-1/2} \eta' (0) |, \ | \zeta (0) | + | z (0)^{-1/2} \zeta' (0) |]$$

$$(s_1, s_2) = \max [\big((\xi\eta)_{12}, (\eta\zeta)_{22}\big), \big((\eta\xi)_{12}, (\eta\zeta)_{22}\big)]$$

with

$$s_1 = \{c_1, c_2\} \qquad s_2 = \{|\eta_2(t)|, \quad |\zeta_2(t)|\}, \text{ say.}$$

It follows that

$$c_1 + c_2 = \xi_{12} + \eta_{12}. \tag{20}$$

From (19),

$$|\eta_2(x)/z(x)|, \ |\zeta_2(x)/z(x)| \leqslant K_1 + \int_0^x [c_{11}|\eta_2(t)/z(t)| + c_{21}|\zeta_2(t)/z(t))|]\,dt \tag{21}$$

where

$$c_{11} = c_1 |z(t)|, \qquad c_{21} = c_2 |z(t)|.$$

Making use of the following well known result, *viz.*,

"If h_1, h_2, g_1, g_2 be non-negative functions of x over the interval $0 \leqslant x \leqslant X$ and if h_1, h_2 be continuous and g_1, g_2 be integrable over this interval then,

$$h_1(x), \ h_2(x) \leqslant B \exp\left[\int_0^x \{g_1(s) + g_2(s)\}\,ds\right], \ 0 \leqslant x \leqslant X,$$

where

$$h_1(x), h_2(x) \leqslant B + \int_0^x [h_1(t)\,g_1(t) + h_2(t)\,g_2(t)]\,dt,$$

B constant" Conte and Sangren [4],

it follows from (21) that

$$|\eta_2(x)/z(x)|, \ |\zeta_2(x)/z(x)| \leqslant K_1 \exp\left[\int_0^x (c_{11} + c_{21})\,dt\right]$$

$$= K_1 \exp\left[\int_0^x (c_1 + c_2)\,|z(t)|\,dt\right].$$

Thus

$$|\eta(x)|, \ |\zeta(x)| \leqslant K_1 \exp\left[\int_0^x (c_1 + c_2)\,|z(t)|\,dt\right] |\exp(i\xi(x))|$$

$$= K_1 \exp\left[|z(x)| \int_0^x (\xi_{12} + \eta_{12})\,|z(t)/z(x)|\,dt\right] |\exp(i\xi(x))|$$

so that

$$| \eta(x) |,\ | \zeta(x) | \leqslant K_1 \exp [| z(x) | \int_0^\infty (\xi_{12} + \eta_{12})\, dt] \\ | \exp (i\xi(x)) | \tag{22}$$

Since

$$\xi_{12} + \eta_{12} = 0\,(1/Q(t)),$$

$$\int_0^\infty (\xi_{12}(t) + \eta_{12}(t))\, dt = 0\,(1).$$

Therefore,

$$| \eta(x) |,\ | \zeta(x) | = 0\,[| \exp \big(K_2\, p(x)\, q(x)\big) || \exp (i\xi(x)) |]. \tag{23}$$

6. Asymptotic Relations

We have from (15),

$$\begin{aligned} \Omega(x) &= (\tfrac{1}{2}) [\exp \big(i\xi(x)\big) + \exp \big(-i\xi(x)\big)]\, \Omega(0) - \\ &\quad - (\tfrac{1}{2})\, z(0)^{-1/2} [\exp \big(i\xi(x)\big) - \exp \big(-i\xi(x)\big)]\, \Omega'(0) + \\ &\qquad + (\tfrac{1}{2}) \int_0^x [\exp (i\big(\xi(x) - \xi(t)\big) \\ &\quad - \exp (-i\big(\xi(x) - \xi(t)\big))]\, N_1(t)\, \Omega(t)\, dt \\ &= (\tfrac{1}{2}) \exp \big(i\xi(x)\big) \{[1 + \exp \big(-2i\xi(x)\big)]\, \Omega(0) - z(0)^{-1/2} \\ &\qquad [1 - \exp \big(-2i\xi(x)\big)]\, \Omega'(0) + z(x) \exp \big(K_2\, p(x)\, q(x)\big) \\ &\qquad \int_0^x z(x)^{-1} \exp [-i\xi(t) + K_2\, p(x)\, q(x)] \\ &\qquad \times [1 - \exp \big(-2i(\xi(x) - \xi(t))\big)]\, N_3(t)\, dt\} \end{aligned}$$

where

$$N_3(t) = \{(l_1, \Omega),\ (l_2, \Omega)\}.$$

Now, by (23)

$$| \exp \big(-i\xi(t)\big) [1 - \exp \big(-2i(\xi(x) - \xi(t))\big)]\, (l_1, \Omega) / \\ z(x) \exp \big(K_2 p(x)\, q(x)\big) |$$

$$\leqslant |\exp(-i\xi(t))||(l_1, \Omega)||z(t)|^{-1}|\exp(K_2 p(t) q(t))|$$
$$\leqslant K_3(|\xi_1(t)| + |\eta_1(t)|)/|z(t)|,$$

where K_3 is a constant.

Also, by (23)

$$|\exp(-i\xi(t))[1 - \exp(-2i(\xi(x) - \xi(t)))](l_2, \Omega)/$$
$$z(x)\exp(K_2 p(x) q(x))| \leqslant K_4(|\eta_1(t)| + |\zeta_1(t)|)/|z(t)|,$$

where K_4 is a constant.

But

$\int_0^\infty |N_1(t)||N_2(t)|\,\mathrm{dt}$ is convergent (uniformly with respect to λ) by lemma 1.

Hence,

$$\Omega(x) - (\tfrac{1}{2}) z(x) \exp[i\xi(x) + K_1 p(x) q(x)] R \tag{24}$$

where

$$R = \{R_1, R_2\} = \lim_{x\to\infty} \int_0^x \exp(-i\xi(t))[1 - 2i(\xi(x) - \xi(t))] z(x)^{-1}$$
$$\times \exp(-K_2 p(x) q(x)) N_2(t)\,dt \tag{25}$$

which is finite.

Let

$$\begin{pmatrix} X_k(x) = iz(x)^{1/4}\,\Omega_k(x, \lambda) \\ Y_k(x) = iz(x)^{1/4}\,\phi_k(x, \lambda) \end{pmatrix} (k = 1, 2) \tag{26}$$

where

$$X_k = \{X_{k1}, X_{k2}\}, \qquad Y_k = \{Y_{k1}, Y_{k2}\}, \text{ say.}$$

Then

$$X_k(0) = (-1)^k z(0)^{1/4}\{a_{l4}, a_{l2}\}$$
$$X'_k(0) = (-1)^{k-1} z(0)^{1/4}\{a_{l3}, a_l\} + (1|4)[p'(0)$$
$$(\lambda - p(0))^{-3/4}(\lambda - q(0))^{+1/4} + q'(0)\ (a - q(0)^{-3/4}$$
$$(\lambda - p(0))^{\mathrm{I}1/4}\ (-1)^k\{a_{l4}, a_{l2}\} \tag{27}$$

(when $k = 1$, $l = 2$ and when $k = 2$, $l = 1$), with a similar expressions for $Y_k(0)$, $Y_k'(0)$ $(k = 1, 2)$.

It follows from (26), (27), (22)

$$|x_k(x, \lambda)|, \; |y_k(x, \lambda)| = 0\left[\,|z(0)^{1/4}|\,|\exp\big(K_2 p(x)\, q(x)\big)| \times \right.$$
$$\left. \times \,|\exp\big(i\xi(x)\big)|\,|z(x)^{-1/4}|\,\right]$$

for all x and $Im\ \lambda > 0$.

There are similar expressions for $u_k(x, \lambda)$, $v_k(x, \lambda)$. Thus we have for a fixed λ, as x tends to infinity,

$$\begin{aligned} X_k(x) &\sim (\tfrac{1}{2})\, z(x) \exp[i\xi(x) + K_2 p(x)\, q(x)]\, T_k(x) \\ Y_k(x) &\sim (\tfrac{1}{2})\, z(x) \exp[i\xi(x) + K_2 p(x)\, q(x)]\, S_k(x) \end{aligned} \quad (k = 1, 2) \tag{28}$$

where

$$T_k = \{R_{1k}, R_{2k}), \qquad S_k = \{S_{1k}, S_{2k}\}$$

and R_{ik}, S_{ik} $(i, k = 1, 2)$ are independent of λ.

7. BASIC THEOREM

We now establish the following theorem.

THEOREM. If all the conditions of the lemma I are satisfied then the spectrum of the system (1) with boundary conditions (2)–(3) at the end point $x = 0$, is discrete over the whole range $(-\infty, \infty)$ except possibly at the point at infinity.

Proof.—Let

$$\psi_1 = \theta_1(x, \lambda) + \sum_{r=1}^{2} m_{1r}(\lambda)\, \phi_r(x, \lambda)$$
$$= \begin{pmatrix} x_1 + m_{11} u_1 + m_{12} u_2 \\ y_1 + m_{11} v_1 + m_{12} v_2 \end{pmatrix}$$
$$= -\,iz(x)^{-1/4} \begin{pmatrix} X_{11} + m_{11} Y_{11} + m_{12} Y_{21} \\ X_{12} + m_{11} Y_{12} + m_{12} Y_{22} \end{pmatrix}$$

so that

$$\psi_1 \sim (\tfrac{1}{2} i)\{\exp[i\xi(x) + K_2 p(x)\, q(x)]\, z(x)^{3/4}\}\, W(\lambda) \tag{29}$$

where

$$W(\lambda) = \{W_1(\pi), \; W_2(\lambda)\}$$

with

$$W_j(\lambda) = R_{j1} + m_{11} S_{j1} + m_{12} S_{j2} \; (j = 1, 2). \tag{30}$$

Now in the singular case $0 \leqslant x < \infty$, for values of λ other than real values, there exists at least two linearly independent solutions of (1), say $\chi_r(x, \lambda)$ $(r = 1, 2)$ such that

$$\chi_r(x, \lambda) \in L^2[0, \infty)$$

(compare Chakrabarty [3]).

In order that ψ_1 may be an L^2-solution of (1) we must have

$$W_1(\lambda) = 0, \quad W_2(\lambda) = 0,$$

since $(1/2i) \exp[i\xi(x) + K_2 p(x) q(x)] z(x)^{34}$ does not belong to $L^2[0, \infty)$.

Therefore,

$$\begin{aligned} m_{11}S_{11} + m_{12} S_{12} &= -R_{11} \\ m_{11}S_{21} + m_{12} S_{22} &= -R_{21}. \end{aligned} \tag{31}$$

Similarly for the solution

$$\psi_2 = \theta_2(x, \lambda) + \sum_{r-1}^{2} m_{2r}(\lambda) \phi_r(x, \lambda)$$

it follows that

$$\psi_2 \sim (1/2i) \exp[i\xi(x) + K_2 p(x) q(x)] z(x)^{/34} V(\lambda)$$

where

$$V(\lambda) = \{v_1(\lambda), v_2(\lambda)\}$$

with

$$V_j(\lambda) = R_{j2} + m_{21} S_{j1} + m_{22} S_{j2} \qquad (j = 1, 2).$$

By arguments similar to those given as before we obtain

$$\begin{aligned} m_{21} S_{11} + m_{22}S_{12} &= -R_{12} \\ m_{21}S_{21} + m_{22} S_{22} &= -R_{22} \end{aligned} \tag{32}$$

Solving (31) for m_{11}, m_{12} and (32) for m_{21}, m_{22} we have

$$m_{rs}(\lambda) = N_{rs}(\lambda)/D(\lambda) \quad (r, s = 12)$$

where

$$N_{rs}(\lambda) = R_{2r} S_{12} - R_{1r} S_{22} \text{ if } s = 1, \ r = 1, 2$$
$$= R_{1r} S_{21} - R_{2r} S_{11} \text{ if } s = 1, \ r = 12,$$

and

$$D(\lambda) = S_{11} S_{22} - S_{12} S_{21}.$$

From (23) we have

$$\eta(x), \zeta(x) = 0 \left[| \exp\left(i\xi(x)\right) | | \exp\left(K_2 p(x) q(x)\right) | \right]$$

uniformly with respect to λ as λ approaches any point in the interval of the negative real axis. Then ξ_1, η_1, ζ_1, R_{ik}, S_{ik} are all real there, $\xi(t)$ being purely imaginary. Finally, $N_{rs}(\lambda)$, $D(\lambda)$ are real and continuous. Therefore it follows that the numerator and denominator of each element of the matrix (m_{ij}) are real and continuous up to any point on the negative real axis. Similar arguments hold if λ approaches any point in an interval of the positive real axis. For, let β be the right hand end point of the interval under consideration and let λ tend to β. Then the cases $p(x)$, $q(x) < \beta$ or $> \beta$ lead to the same behaviour of the elements of (m_{ij}).

Since the numerator and the denominator of each element of the matrix (m_{ij}) are regular in the upper-half plane, it follows from the principle of reflection that $N_{rs}(\lambda)$, $D(\lambda)$ are entire functions of λ so that each element of (m_{ij}) is a meromorphic function of λ.

Therefore, the spectrum of the system (1), (2)–(3) is discrete over the whole range $(-\infty, \infty)$ if $p(x)$, $q(x) < \beta$ or $> \beta$.

If λ tends to α and $q(\alpha) < \beta < p(\alpha)$, α being a fixed real number, then the whole argument can be repeated by changing the interval $[0, \infty)$ by $[X, \infty)$ (so that $p(X)$, $q(X) > \beta$) and the limits 0, x in the expressions R_{ik}, S_{ik} $(i, k = 1, 2)$ by X, x.

The spectrum is then discrete over the whole range $(-\infty, \infty)$.

To examine the point at infinity on the real λ-axis we note that

$$R_{ik} = \lim_{x \to \infty} \int_0^x 0 \left[\left(\xi_1(t) + \eta_1(t) \right) / z(t) \right] dt$$

$$= \lim_{x \to \infty} \int^{x} 0 \ [z(t)^{-1/2}]\, dt$$

$$= 0 \ [\lim_{x \to \infty} \int_0^x (pq)^{-1/2}\, dt]$$

$$= 0\,(1).$$

Similarly,

$$S_{ik} = 0\,(1)$$

Again,

$$\xi_1(t) + \eta_1(t) \geqslant z(t)^{1/2} \ (\lambda \text{ large but fixed})$$

so that

$$\big(\xi_1(t) + \eta_1(t)\big)/z(t) \geqslant z(t)^{-1/2}.$$

Therefore,

$$\int_0^\infty \big(\xi_1(t) + \eta_1(t)\big)/z(t)\, dt \geqslant \int_0^\infty (p\,q)^{-1/2}\ dt = \text{a constant}$$

independent of λ. It follows that

$$R_{1k} \geqslant g_{1k},\ S_{1k} \geqslant h_{1k},$$

g_{1k} and h_{1k} being positive constants.

Also,

$$\eta_1(t) + \zeta_1(t) \geqslant z(t)^{1/2} \qquad (\lambda \text{ large but fixed}).$$

It follows similarly that

$$R_{2k} \geqslant G_{2k},\ S_{2k} \geqslant H_{2k},$$

G_{2k}, H_{2k} being positive constants.

Therefore,

$$1/D(\lambda) = 0\,(1).$$

Hence,

$$\lim_{\lambda \to \infty} m_{rs}(\lambda) = \lim_{\lambda \to \infty} N_{rs}(\lambda)/D(\lambda)$$
$$= 0(1)$$

Therefore the point at infinity is a regular point.

Thus the theorem is proved.

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