Short Communication

Steady flow through a porous region contained between two cylinders

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Abstract

Aim of this paper is to study the flow of a viscous liquid through a porous annular region between two slowly rotating cylinders, employing a generalized Darcy's law proposed by Brinkman. Expressions for the velocity, moment and drag acting on the cylinders are given.

As special cases (i) Poiseulle flow, (ii) Couette flow and (iii) Flow between two parallel plates are derived.

Key words : Steady flow, Porous region, Cylinders.

1. Introduction

Flows through porous media have important applications in hydrology, petroleum industry and in agricultural engineering, etc.

Flows of different fluids through various types of porous media are studied employing the classical Darcy's law, which states that the seepage velocity of the fluid is proportional to the pressure gradient. This law fails to explain the phenomena occurring in highly porous media such as fibreglass. The viscous stress at the surface is able to penetrate into the medium and produces flow near the surface even in the absence of the pressure gradient. Brinkman[1] generalized the Darcy law taking into account the effect of viscous stress.

Brinkman's law gave good results in the case of highly porous media. Tam[2] derived the law analytically to study flow at low Reynold's number past a swarm of particles Yamamoto[3, 4] investigated flow past porous bodies using the generalized law.

In this paper, the flow of a viscous liquid through a porous region in between two rotating cylinders is examined. Also some special cases (i) Poiseulle flow, (ii) Couette flow, and (iii) Flow between parallel plates are derived.

Momentum and drag acting on each cylinder are obtained.

2. Formulation and solution of the problem

The cylindrical co-ordinate system (r, θ, z) is chosen such that z-axis lies along the length of common axis of the cylinders, r in the radial direction. All the physical quantities are independent of θ due to the axial symmetry.

Let the two cylinders of radii a and b rotate slowly with angular velocities Ω_a and Ω_b respectively. A porous region is contained between the cylinders and we take $a \leqslant r \leqslant b$.

The equation of motion of a viscous liquid through a porous medium as proposed by Brinkman is

$$O = -\nabla p - \frac{\mu}{k}\vec{V} + \nabla^2 \vec{V} \tag{2.1}$$

together with the equation of continuity

$$\text{div}\ \vec{V} = 0 \tag{2.2}$$

Here p, $\vec{V}$ stand for pressure and velocity fields and μ, k stand for coefficient of viscosity of the fluid and the permeability constant of the medium respectively.

The choice of the velocity $\vec{V}\ \{o,\ v(r),\ w(r)\}$ satisfies the equation of continuity and the pressure

$$p = c - Gz + \int_0^r r^{-1}\,[v(r)]^2\ dr \tag{2.3}$$

is taken to balance the centrifugal force generated by the velocity component $v(r)$ where

$$-\frac{\partial p}{\partial z} = G,$$

a constant and c is constant of integration.

Equation (2.1) gives

$$\frac{d^2 v}{dr^2} + r^{-1}\frac{dv}{dr} - (r^2 + k^{-1})\,v = 0 \tag{2.4}$$

$$\frac{d^2 w}{dr^2} + r^{-1}\frac{dw}{dr} - k^{-1}\,w = -\,G/\mu \tag{2.5}$$

These equations are to be solved using the boundary conditions:

$$\left.\begin{aligned} w(r) = o,\ v(r) = a\,\Omega_a \text{ at } r = a \\ w(r) = o,\ v(r) = b\,\Omega_b \text{ at } r = b \end{aligned}\right\} \tag{2.6}$$

Equations (2.4) to (2.6) give

$$v(r) = [a\,\Omega_a\, T_1(r, b) - b\,\Omega_b\, T_1(r, a)] \times [T_1(a, b)]^{-1} \qquad (2.7)$$

$$w(r) = ka^* [T_0(r, b) - T_0(r, a) - T_0(a, b)] \times [T_0(a, b)]^{-1}. \qquad (2.8)$$

with

$$T_i(x, y) = I_i(x/k^{1/2})\, K_i(y/k^{1/2}) \quad - I_i(y/k^{1/2})\, K_i(x/k^{1/2}) \quad i = 0,\ 1 \qquad (2.9)$$
$$a^* = -\, G/\mu$$

and I_i, K_i are modified Bessel functions.

A. *When the permeability of the medium is very small*

i.e., when k is very small, $1/k$ is very large.

Therefore

$$T_i(x, y) = \tfrac{1}{2}(k/xy)^{1/2}\ [\exp\{(x-y)/k^{1/2}\} - \exp\{-(x-y)/k^{1/2}\}] \qquad (2.10)$$

and we get

$$v(r) = r^{-1/2}\,\langle a^{3/2}\,\Omega_a \exp\{-(r-a)/k^{1/2}\} + b^{3/2}\,\Omega_b \exp\{-(b-r)/k^{1/2}\}\rangle \qquad (2.11)$$

$$w(r) = ka^*\,\langle r^{-1/2}\ [a^{1/2} \exp\{-(r-a)/k^{1/2}\} - b^{1/2} \exp\{-(b-r)/k^{1/2}\}] - 1\rangle \qquad (2.12)$$

We observe that

$\delta_1 = r - a$, $\delta_2 = b - r$ becomes very large

$$v(r) = 0 \qquad (2.13)$$

$$w(r) = -\, ka^* = a \text{ constant} \qquad (2.14)$$

Equations (2.13) and (2.14) show that as the distance from the axis to the wall increases, the velocity w $(r) \to$a constant. The classical Darcy effect is felt in a core very near to the axis of the cylinders. $v(r) = 0$ shows that there exists a thin layer between the cylinders far away from the boundaries of the cylinders, where the velocity $v(r)$ is zero.

Moment acting on the cylinder $r = a$

$$M_a = -\,2\pi a^2\,\langle\Omega_a\,[3/2 + a^2 k^{-1/2}] - (b/a)^{1/2} \exp\{(a-b)/k^{1/2}\}\ [k^{-1/2} - 3b/a]\rangle \qquad (2.15)$$

Moment acting on the cylinder $r = b$

$$M_b = -\,2\pi b^2\,\langle (a/b)^{1/2}\ \Omega_a \exp\{-(b-a)/k^{1/2}\}\,[(3/2)\ (a/b) + k^{-1/2}] - \Omega_b\,[b^2 k^{-1/2} - 3/2]\rangle \qquad (2.16)$$

Drag on the cylinder $r = a$

$$D_a = 2\pi\mu ka^* (ak^{-1/2} + \tfrac{1}{2}) [(b/a)^{1/2} \exp\{-(b-a)/k^{1/2}\} - 1] \quad (2.17)$$

Drag on the cylinder $r = b$

$$D_b = 2\pi\mu ka^* (bk^{-1/2} + \tfrac{1}{2}) [1 - (a/b)^{1/2} \exp\{-(b-a)/k^{1/2}\}] \quad (2.18)$$

B. *When the permeability* $k \to \infty, \frac{1}{k} \to 0.$

Then

$$v(r) = \frac{(b^2\Omega_b - a^2\Omega_a)}{b^2 - a^2} r + \frac{(\Omega_a - \Omega_b)}{b^2 - a^2} a^2b^2 r^{-1}$$

$$w(r) = \frac{G}{4\mu} [a^2 - r^2 + (b^2 - a^2) \log(r/a)/\log(b/a)] \quad (2.19)$$

are the same as those obtained in the case of flow through two rotating cylinders without porous region.

3. Special cases

(i) *Poiseulle flow* when $\Omega_a = \Omega_b = 0$, we get from equations (2.11) and (2.12)

$$v(r) = 0 \quad (3.1)$$

$$w(r) = ka^* (r^{-1/2} [a^{1/2} \exp\{-(r-a)/k^{1/2}\} - b^{1/2} \exp\{-(b-r)/k^{1/2}\}] - 1) \quad (3.2)$$

(ii) *Couette flow*

We take the pressure gradient to be absent, then $G=0$ from (2.11) and (2.12)

$$w(r) = 0$$

$$v(r) = r^{-1/2} [a^{3/2}\Omega_a \exp\{-(r-a)/k^{1/2}\} + b^{3/2}\Omega_b \exp\{-(b-r)/k^{1/2}\}] \quad (3.3)$$

(*a*) *When the cylinders rotate with same angular velocities*

i.e., $\Omega_a = \Omega_b = \Omega$

$$v(r) = r^{-1/2} \Omega [a^{3/2} \exp\{-(r-a)/k^{1/2}\} + b^{3/2} \exp\{-(b-r)/k^{1/2}\}] \quad (3.4)$$

(*b*) *Flow in a rotating cylinder*

i.e., $a \to 0, \Omega_a \to 0$

$$v(r) = r^{1/2} \Omega_b b^{3/2} \exp\{-(b-r)/k^{1/2}\} \quad (3.5)$$

Equations (3.4) and (3.5) show that unlike in the case of the usual flow of a Newtonian fluid in a rotating cylinder, the motion is not rigid one.

(iii) *Flow between parallel plates*

Flow through the porous region bounded by two infinite parallel plates $y = \pm h$ can be derived by taking $a \sim b = 2h$, as the distance between the two plates with respect to the cartesian co-ordinate system (x, y, z). The velocity components are given by

$$v(r) = 0 \tag{3.6}$$

$$w(r) = ka^* \, [T_0(r, b) - T_0(r, a) - T_0(a, b)] \, [T_0(a, b)]^{-1} \tag{3.7}$$

$$= ka^* \left[-1 + \frac{b^{1/2} \sinh (r - a) + a^{1/2} \sinh (r - b)}{r^{1/2} \sinh (a - b)} \right] \tag{3.8}$$

Let

$$r = b + h + y,$$

we get

$$w(y) = ka^* \left[-1 + \frac{\cosh (y) \, k^{1/2}}{\cosh \, (h/k^{1/2})} \right] \tag{3.9}$$

(*a*) *When the permeability constant*, k, *is very small*

then $1/k$ is very large and we have

$$w(y) = ka^* \, [-1 + \exp \{-(h - y)/k^{1/2}\}] \tag{3.10}$$

(*b*) *When the permeability constant*, k, *is very large*

then $1/k$ is very small and we get

$$w(y) = \frac{a^* h^2}{2 + \frac{h^2}{k}} \left[\frac{y^2}{h^2} - 1 \right] \tag{3.11}$$

The effect of the porous region is thus to reduce the velocity by $(2 + h^2/k)$
If

$$k \to \infty, \; 1/k \to o$$

then

$$w(y) = \frac{a^* \, h^2}{2} \left[\frac{y^2}{h^2} - 1 \right] \tag{3.12}$$

which is same as in the case of flow between the infinite parallel plates without porous region.

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NOMENCLATURE

r, θ, z	...	Cylindrical co-ordinate system
Ω_a, Ω_b	...	Angular velocities of cylinders of radii a, b
p	...	Pressure field
$\vec{V}$	...	Velocity of field
$\{o, v(r), w(r)\}$	...	Velocity components in r, θ, z directions
μ	...	Coefficient of viscosity of the fluid
k	...	Permeability constant of the medium
∇	...	grad
∇^2	...	$\frac{\partial^2}{\partial r^2} + \frac{1\partial}{r\partial r} + \frac{\partial^2}{\partial z^2} + \frac{1\partial^2}{r^2\partial\theta^2}$
G, c	...	Constants
I_i, k_i	...	Modified Bessel functions
$2h$	...	Distance between two plates.

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