# Vibrations of irregular-shaped orthotropic plates resting on elastic foundation under inplane forces

PARITOSH BISWAS
Department of Mathematics, P. D. Women's College, Jalpaiguri, West Bengal, India

Received on October 27, 1977

**Abstract**

A unified method for determining the lowest natural frequency of linear vibrations of orthotropic plates of any shape under inplane forces and placed on elastic foundation is given. Conformal mapping technique is introduced and Galerkin's method is used to calculate approximate values of lowest natural frequency.

Key words: Orthotropic, elastic foundation, inplane forces, mapping function.

## Introduction

A change in the natural frequencies of a plate can be achieved by altering its stiffness, mass or by the influence of inplane forces in an elastic plate. The natural frequencies are thus lowered or remain unchanged depending on the nature of the inplane forces applied. Such type of problems are of great practical use to the engineers dealing with thermal stresses and with panels of rockets at take off conditions.

In this paper the author makes an attempt to present an analysis of calculating the natural frequencies of orthotropic plates of any shape under inplane forces and placed on elastic foundation of the Winkler type. The boundary of the plate is transformed conformally onto a unit circle and the solution is obtained with the help of error function and Galerkin procedure.

### *Equation of motion*

The Lagrangian differential equation for small amplitude vibrations of orthotropic plates in the presence of inplane forces and placed on elastic foundation, in their usual notations, reduces to

$$D_x \frac{\partial^4 \omega}{\partial x^4} + 2D_{xy} \frac{\partial^4 \omega}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 \omega}{\partial y^4} + k\omega = -\rho h \frac{\partial^2 \omega}{\partial t^2} + N\nabla^2 \omega \qquad (1)$$

where $\rho$ is the mass density per unit volume, $\omega(x, y; t)$ is the transverse deflection, $h$ the plate thickness, $k$ the foundation modulus, $N$ the uniform inplane tensile force, and $D_x = E_x h^3/12(1 - \nu_x \nu_y)$, $D_y = E_y h^3/12(1 - \nu_x \nu_y)$, $D_{xy} = \nu_y D_x + Gh^3/6$; $E_x$, $E_y$, $\nu_x$, $\nu_y$ and $G$ being material constants of an orthotropic material.

If

$$\omega(x, y; t) = W(x, y)\, e^{i\tilde{\omega} t} \tag{2}$$

where $\tilde{\omega}$ is the circular frequency, and if

$$z = x + iy, \qquad \bar{z} = x - iy.$$

so that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}; \qquad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

then equation (1) transforms into

$$(D_x + D_y - 2D_{xy})\left(\frac{\partial^4 w}{\partial z^4} + \frac{\partial^4 w}{\partial \bar{z}^4}\right) + (6D_x + 6D_y + 4D_{xy})\frac{\partial^4 w}{\partial z^2 \partial \bar{z}^2}$$
$$+ 4(D_x - D_y)\left(\frac{\partial^4 w}{\partial z^3 \partial \bar{z}} + \frac{\partial^4 w}{\partial z \partial \bar{z}^3}\right) + (k - \rho h\tilde{\omega}^2)\, W$$
$$- 4N\frac{\partial^2 W}{\partial z \partial \bar{z}} = 0. \tag{3}$$

Let $z = f(\xi)$ be the analytic function which maps the given shape in the $\xi$-plane onto a unit circle. Thus equation (3) transforms into complex coordinates as

$$(D_x - 2D_y + D_y)\left\{\frac{\partial^4 w}{\partial \xi^4}\left(\frac{dz}{d\xi}\right)^3\left(\frac{dz}{d\xi}\right)^7 - 6\frac{\partial^3 w}{\partial \xi^3}\frac{d^2 z}{d\xi^2}\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^7\right.$$
$$+ 15\frac{\partial^2 w}{d\xi^2}\left(\frac{d^2 z}{d\xi^2}\right)^2\frac{dz}{d\xi}\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^7 - 4\frac{\partial^2 w}{\partial \xi^2}\frac{d^3 z}{d\xi^3}\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^7$$
$$+ \frac{\partial w}{\partial \xi}\left[10\frac{d^3 z}{d\xi^3}\frac{d^2 z}{d\xi^2}\frac{dz}{d\xi}\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^7 - 15\left(\frac{d^2 z}{d\xi^2}\right)^3\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^7 + \frac{d^4 z}{d\xi^4}\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^7\right]$$
$$+ \frac{\partial^4 w}{\partial \xi^4}\left(\frac{dz}{d\xi}\right)^7\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3 - 6\frac{\partial^3 w}{\partial \bar{\xi}^3}\frac{d^2 \bar{z}}{d\bar{\xi}^2}\left(\frac{dz}{d\bar{\xi}}\right)^7\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2$$
$$+ 15\frac{\partial^2 w}{\partial \bar{\xi}^2}\left(\frac{d^2 z}{d\bar{\xi}^2}\right)^2\left(\frac{dz}{d\xi}\right)^7\left(\frac{d\bar{z}}{d\bar{\xi}}\right) - 4\frac{\partial^2 w}{\partial \bar{\xi}^2}\frac{d^3 z}{d\bar{\xi}^3}\left(\frac{dz}{d\xi}\right)^7\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2$$
$$+ \frac{\partial w}{\partial \bar{\xi}}\left[10\frac{d^3 z}{d\bar{\xi}^3}\frac{d^2 z}{d\bar{\xi}^2}\left(\frac{dz}{d\xi}\right)^7\frac{d\bar{z}}{d\bar{\xi}} - 15\left(\frac{d^2 \bar{z}}{d\bar{\xi}^2}\right)^2\left(\frac{dz}{d\xi}\right)^7\right.$$

$$+ \frac{d^4 \bar{z}}{d\bar{\xi}^4}\left(\frac{dz}{d\xi}\right)^7\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2\Big]\Big\} + (6D_x + 4D_{xy} + 6D_y)\Big\{\frac{\partial^4 w}{\partial \xi^2 \partial \bar{\xi}^2}\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2$$

$$- \frac{\partial^3 w}{\partial \xi^2 \partial \bar{\xi}}\frac{d^2 z}{d\xi^2}\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^7 + \frac{\partial^3 w}{\partial \xi \partial \bar{\xi}^2}\frac{d^2 z}{d\bar{\xi}^2}\left(\frac{dz}{d\xi}\right)^7\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2$$

$$+ \frac{\partial^2 w}{\partial \xi \partial \bar{\xi}}\frac{d^2 z}{d\xi^2}\frac{d^2 \bar{z}}{d\bar{\xi}^2}\left(\frac{dz}{d\xi}\right)^4\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^4$$

$$+ 4(D_x - D_y)\frac{\partial^4 w}{\partial \xi^3 \partial \bar{\xi}}\left(\frac{dz}{\partial \xi}\right)^4\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^6 - 3\frac{\partial^3 w}{\partial \xi^2 \partial \bar{\xi}}\frac{d^2 z}{d\xi^2}\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^6$$

$$+ \frac{\partial^2 w}{\partial \xi \partial \bar{\xi}}\left[3\left(\frac{d^2 z}{d\xi^2}\right)^2\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^6 - \frac{d^3 z}{d\xi^3}\left(\frac{dz}{d\xi}\right)^3\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^6\right]$$

$$+ \frac{\partial^4 w}{\partial \xi \partial \bar{\xi}^3}\left(\frac{dz}{d\xi}\right)^6\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^6 - 3\frac{\partial^3 w}{\partial \xi \partial \bar{\xi}^2}\frac{d^2 \bar{z}}{d\bar{\xi}^2}\left(\frac{dz}{d\xi}\right)^6\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3$$

$$+ \frac{\partial^2 w}{\partial \xi \partial \bar{\xi}}\left[3\left(\frac{d^2 \bar{z}}{d\bar{\xi}^2}\right)^4\left(\frac{dz}{d\xi}\right)^6\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2 - \frac{d^3 \bar{z}}{d\bar{\xi}^3}\left(\frac{dz}{d\xi}\right)^6\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3\right]\Big\}$$

$$+ (k - \rho h\tilde{\omega}^2)\, W\left(\frac{dz}{d\xi}\right)^7\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^7 - 4N\frac{\partial^2 w}{\partial \xi \partial \bar{\xi}}\left(\frac{dz}{d\xi}\right)^6\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^6 = 0. \tag{4}$$

***Method of solution***

Since an exact solution of equation (4) is, at best, very difficult, it is convenient to use an approximate method to solve it. We shall use Galerkin's method in this study.

The solution of equation (4) can be expressed in the forms.[3]

$$W \approx \sum_{n=1}^{N} l_n \{1 - (\xi\bar{\xi})^n\}^2 \tag{5}$$

or

$$W \approx \sum_{n=1}^{N} l_n (1 - \xi\bar{\xi})\left[\frac{4n + 1 + a_1}{1 + a_1} - (\xi\bar{\xi})^n\right] \tag{6}$$

where $a_1 = D_1/D_n$ (for isotropy, $a_1 = \nu$, the Poisson's ratio), and $\xi = re^{i\theta}$, $\xi\bar{\xi} = r^2$.

The form of $W$ in equation (5) satisfies—

$$W = 0 = \frac{dW}{dr} \quad \text{at} \quad r = 1$$

and can be taken as an admissible function for the clamped edge conditions. Also the form of $W$ in equation (6) satisfies $W = 0$ at $r = 1$, and moment $M_n = 0$ at $r = 1$ and can be taken as an admissible function for simply-supported edge conditions.[3]

Substituting equation (5) or (6) into equation (4) yields the error function $\varepsilon_N(\xi, \bar{\xi})$ which does not vanish, in general, since equation (5) or equation (6) is not an exact solution.

Galerkin's procedure requires that the error function $\varepsilon_N$ be orthogonal over the domain under consideration, *i.e*,

$$\int\int_S \varepsilon_N(\xi, \bar{\xi})\, W(\xi, \bar{\xi})\, ds = 0, \quad (n = 1, 2, \ldots, N). \tag{7}$$

From equation (7) a system of linear homogeneous equations is obtained. Such a system can have non-trivial solution only if the determinant of the coefficients of the unknowns vanishes identically. For fundamental frequency the lowest root of this is to be taken.

*Applications*

Let us apply the procedure explained above to the case of a clamped circular plate for which the mapping function is given by

$$z = a\xi. \tag{8}$$

Taking the first term approximation of $W$ given by (5) and substituting this along with (8) into equation (4) one gets the error function, $\varepsilon_1$, as:

$$\varepsilon_1 = \frac{b_1 a^{10}}{D_{xy}} \{4(6D_1 + 6D_2 + 4) + (\bar{K} - \Omega^2)(1 - r^2)^2 + 8\bar{N}(1 - 2r^2)\} \tag{9}$$

where

$$D_1 = D_x / D_{xy}, \qquad D_2 = D_y / D_{xy}, \qquad \Omega^2 = \frac{\rho h \tilde{\omega}^2 a^4}{D_{xy}},$$

$$\bar{N} = \frac{N a^2}{D_{xy}}, \qquad \bar{K} = \frac{a^4 k}{D_{xy}}, \qquad \xi\bar{\xi} = r^2.$$

Introducing $\varepsilon_1$ in equation (7) one gets

$$\int_0^1 \int_0^{2\pi} [4(6D_1 + 6D_2 + 4) + (\bar{K} - \Omega^2)(1 - r^2)^2 + 8\bar{N}(1 - 2r^2)]\,(1 - r^2)^2\, r\, d\theta\, dr = 0. \tag{10}$$

Equation (10) gives rise to the frequency equation as

$$3\Omega^2 = 20(6D_1 + 6D_2 + 4) + 3\bar{K} + 20\bar{N}. \tag{11}$$

Equation (11) shows that the frequency of the plate is greater when placed on elastic foundation than when not.

The values of natural frequencies obtained from (11) by attributing different values to $D_1$, $D_2$, $\bar{K}$ and $\bar{N}$ can be improved by considering first two terms of the series (5). Therefore

$$\begin{aligned}\varepsilon_2 \doteq \frac{a^{10}}{D_{xy}} [& 24 (D_1 + D_2 - 2) l_2 (\xi^4 + \bar{\xi}^4) + (6D_1 + 6D_2 + 4) \\ & \times (l_1 - 2l_2 + 36 \, l_2 \xi^2 \bar{\xi}^2) + 384 (D_1 - D_2) l_2 \xi\bar{\xi} (\xi^2 + \bar{\xi}^2) \\ & + (\bar{K} - \Omega^2) \{l_1 (1 - \xi\bar{\xi})^2 + l_2 (1 - \xi^2 \bar{\xi}^2)^2\} + 8\bar{N} \{l_1 (1 - 2\xi\bar{\xi}) \\ & + 4l_2 (\xi\bar{\xi} - 2 \xi^3 \bar{\xi}^3)\}]. \end{aligned} \tag{12}$$

Since from (7),

$$\int_0^1 \int_0^{2\pi} \varepsilon_2 \, Wr \, d\theta dr = 0 \tag{13}$$

one gets

$$\begin{aligned}\int_0^1 & [A \{4l_1 + 8l_2 (18 r^4 - 1)\} + (\bar{K} - \Omega^2) \{1 - 2r^2 + r^4) \\ & + l_2 (1 - 2r^4 + r^4)\} - \bar{N} \{8l_1 (2r^2 - 1) + 36 l_2 (2r^6 - r^2)\}] \\ & \times [ \sum_{n=1}^{N} l_n (r - 2r^{2n+1} + r^{4n+1})] \, dr = 0 \end{aligned} \tag{13 i}$$

where

$$A = 6D_1 + 6D_2 + 4.$$

Integrating 13 (i) and taking $n = 1$ and $n = 2$, one gets the following two simultaneous equations

$$\left(\frac{2}{3} A + \frac{\bar{K} - \Omega^2}{10} + \frac{2}{3} \bar{N} \right) l_1 + \left(\frac{16}{15} A + \frac{29 (\bar{K} - \Omega^2)}{210} + \frac{4}{5} \bar{N} \right) l_2 = 0 \tag{14}$$

$$\left(\frac{16}{15} A + \frac{29 (\bar{K} - \Omega^2)}{210} + \frac{4}{5} \bar{N}\right) l_1 + \left(\frac{352}{105} A + \frac{64 (\bar{K} - \Omega^2)}{315} + \frac{4}{3} \bar{N}\right) l_2 = 0. \tag{15}$$

For non-trivial solution one must have

$$\begin{vmatrix} \left(\frac{2}{3} A + \frac{\bar{K} - \Omega^2}{10} + \frac{2}{3} \bar{N}\right) & \left(\frac{16}{15} A + \frac{29 (\bar{K} - \Omega^2)}{210} + \frac{4}{5} \bar{N}\right) \\ \left(\frac{16}{15} A + \frac{29 (\bar{K} - \Omega^2)}{210} + \frac{4}{5} \bar{N}\right) & \left(\frac{352}{105} A + \frac{64 (\bar{K} - \Omega^2)}{315} + \frac{4}{3} \bar{N}\right) \end{vmatrix} = 0 \tag{16}$$

from which the lowest root gives the fundamental frequency.

The following table (Table I) is constructed showing the variation of fundamental frequency for different values of inplane forces and foundation modulus, considering the set of values for plywood material,

$$D_1 = 4\cdot341, \qquad D_2 = 1\cdot6136.$$

Table I

| | $\bar{K} = 0$ | $\bar{K} = 400$ |
|---|---|---|
| $\Omega\,(\bar{N} = 0)$ | 73·35 | 96·83 |
| $\Omega\,(\bar{N} = 25)$ | 85·41 | 87·72 |
| $\Omega\,(\bar{N} = -25)$ | 64·13 | 67·19 |

**Observation**

The frequency of vibration of a plate is greater when placed on elastic foundation than when not. Also, frequency is greater when the forces are tensile than when compressive.

**Discussion**

Finding the exact mapping function is usually out of the question. Several approximate techniques, however, are available for the determination of accurate mapping function (1). Also mapping functions of plates of different shape are given in (2). Thus after knowing the proper mapping functions the natural frequencies of any desired plate can be obtained under different conditions imposed on the plate following the above method.

**Acknowledgement**

The author acknowledges the receipt of financial assistance from the U.G.C., New Delhi, in support of this research work (Grant Code No. 7589).

**References**

1. Kantorovich L. V. and Krylov, V. I. *Approximate Methods of Higher Analysis*, 1958, Inter-science Pub., New York.
2. Laura, P A. and Shahady, P. Complex variable Theory and elastic stability problems, *Journal of Eng. Mech. Division*, 1969, **95**, EMI.
3. Biswas, P. Thermal Buckling of Orthotropic Plates, *Journal of Applied Mechanics, ASME*, 1976, **43**, No. 2.