Inhomogeneous thermopiezoelectric problem

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Abstract

Seth's theory of finite deformation has been suitably applied here to work out the deformation, stress components and the electric field within an inhomogeneous piezoelectric body subjected to a temperature distribution. The governing equations of elasticity, Maxwell's electromagnetic equations and the equation of heat-flow along with the constitutive equations of the material are solved to achieve the solutions for the structure in the form of a uniform annular plate or a cylinder.

Key words : Anisotropy, Annular plate, Crystal, Cylinder, Electric field, Heterogeneity, Multilayer capacitor, Plane-strain, Thermal gradient, Thermopiezoelectricity.

Introduction

Natural crystals like quartz, Rochelle salt, tourmaline, etc., are widely used in the fabrication of electroacoustical devices, electronic devices and in the field of ultrasonic technology as the basic material because of their piezoelectric properties. Generally these crystal-controlled devices operate satisfactorily when the ambient temperature is low. But at the elevated temperature the performance and operational characteristics of the devices change to a great extent due to the changes in the physical properties of the matter.

Since the piezoelectric devices constructed from ordinary crystals like quartz, etc. fail to operate at high temperature, ceramics of barium titanate are used as a common piezoelectric material at high temperature region *vide* Pask and Copley[1], Quarrie and Buessem[2]. Haskins and Walsh[3] in a statical problem derived the constitutive equations of piezoelectric substance in the absence of thermal field, while Mindlin[4] extended the analysis by considering the problem under thermal influence. But they fail to consider in their problems that the piezoelectric body, in general, becomes inhomogeneous when electric field is applied upon it [Landau and Lifshitz[5]]. Appropriate electric field in such a body under elevated temperature influences its physical properties which are dependent upon the gradient of thermodynamic quantities which vary through the body [Landau and Lifshitz[5]]. So, in order to have a rigorous understanding of the problem

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one must take into consideration that the elastic and piezoelectric parameters as well as the permittivity of the material are, of course, some functions of the spatial co-ordinates while the temperature of the body is so.

In this paper, the displacement, the stress components and the electric field in an inhomogeneous piezoelectric body have been investigated in an annular plate and a cylinder under the simultaneous action of mechanical, electrical and thermal fields. Seth's[6] theory of finite deformations is found useful to tackle the problems. In the first part of the problem a uniform annular plate of piezoelectric material has been considered, while the latter part deals with the case of a uniform annular cylinder. This analysis may be of much interest to the designers handling with the aggregate of barium titanate cement mixture [Orchard[7]] or with the stratified media of piezoelectric bodies.

2. First problem

2.1. *Fundamental equations*

The annular plate under consideration is of uniform thickness and its inner and outer radii are r_1 and r_2 respectively. The annular plate becomes polarised radially when a static voltage is impressed between its inner and outer boundaries which are kept at temperatures T_1 and T_2 respectively.

Since the annular plate does not contain any volume distribution of charges, current and magnetic field, the Maxwell's equations become

$$\text{Curl}\ \vec{E} = 0 \tag{1}$$

$$\text{Div}\ \vec{D} = 0. \tag{2}$$

where $\vec{E}$ and $\vec{D}$ are the electric field intensity and the electric induction respectively.

Polar co-ordinates (r, θ) are used as the co-ordinate of reference with the centre of the annular plate as origin. σ_r and σ_θ denote the radial and tangential components of stress at (r, θ) the stress equation of equilibrium in elasticity is given by

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0. \tag{3}$$

The Gaussian divergence equation (2) in polar co-ordinates stands as

$$\frac{\partial D_r}{\partial r} + \frac{1}{r}\frac{\partial D_\theta}{\partial \theta} = 0. \tag{4}$$

The temperature at any point must satisfy the Laplace equation

$$\nabla^2 T = 0 \tag{5}$$

where ∇^2 is the Laplacian operator in two dimensions.

The above set of equations constitutes the fundamental equations for the problem. It is required to solve them keeping in view the following thermopiezoelectric constitutive relations relevant to the present problem.

$$S_r = s_{33}\sigma_r + s_{13}\sigma_\theta + d_{33}E_r + \mu_1^E T \quad \text{(i)}$$

$$S_\theta = s_{13}\sigma_r + s_{11}\sigma_\theta + d_{31}E_r + \mu_2^E T \quad \text{(ii)}$$

$$S_{r\theta} = s_{44}\sigma_{r\theta} + d_{15}E_\theta \quad \text{(iii)}$$

$$D_r = d_{33}\sigma_r + d_{31}\sigma_\theta + \varepsilon_{33}E_r + p_1^S T \quad \text{(iv)}$$

$$D_\theta = d_{15}\sigma_{r\theta} + \varepsilon_{11}E_\theta + p_2^S T \quad \text{(v)}$$

(6)

where S_r, S_θ and $S_{r\theta}$ are the strain components, s_{ij} are the elastic compliances, d_{ij} are the piezoelectric strain parameters, ε_{ij} are the dielectric permittivities, μ_1^E, μ_2^E are the thermopiezoelectric constants and p_1^S, p_2^S are the constant thermopiezoelectric permittivities.

In the presence of an applied electric field and an elevated temperature the body behaves inhomogeneously [Landau and Lifshitz[5]]. The inhomogeneity of such a body is characterized by the variations of s_{ij}, d_{ij} and ε_{ij} from point to point in a static problem [Bychawski and Piszczek[8]]. μ_1^E, μ_2^E, p_1^S and p_2^S are assumed to be constant in space because their changes with the co-ordinates of the point considered are negligible in relation to those of elastic compliances or piezoelectric strain parameters or dielectric permittivities. In particular, their variations, where radial symmetry is considered may be of the following form, [Greif and Chou[9]]

$$s_{ij} = c_{ij} f(r) \quad \text{(i)}$$

$$d_{ij} = b_{ij} f(r) \quad \text{(ii)}$$

$$\varepsilon_{ij} = v_{ij} f(r) \quad \text{(iii)}$$

$$i, j = 1, 2, 3, \quad (7)$$

where c_{ij}, b_{ij}, v_{ij} are the material constants in relation to its elastic piezoelectric and dielectric properties respectively.

Over and above, two types of boundary conditions are also to be taken into consideration. The mechanical boundary condition comes from the fact that the boundaries are free from stresses so that

$$\sigma_r = 0 \quad \text{at} \quad r = r_1 \quad \text{and} \quad r = r_2. \quad (8)$$

Reviewing the equation (1), one can write down the electrical boundary condition as

$$\int_{r_1}^{r_2} E_r \, dr = V \text{(constant)}. \quad (9)$$

2.2. *Method of solution*

A solution of the heat equation (5) consistent with the boundary conditions, $T = T_1$ on $r = r_1$ and $T = T_2$ on $r = r_2$ may be readily set as

$$T = T_1 - T_0 \ln r/r_1 \tag{10}$$

where

$$T_0 = \frac{T_1 - T_2}{\ln r_2 r_1}. \tag{10a}$$

For the radially symmetric case $\partial D_\theta / \partial \theta = 0$ and the divergence equation (4) yields

$$D_r = \text{Constant} = D_0 \text{ (say)}. \tag{11}$$

This relation and the equation 6 (iv) give

$$E_r = \frac{D_0 - p_1^S T - d_{33}\sigma_r - d_{31}\sigma_\theta}{\varepsilon_{33}}. \tag{12}$$

Following Seth[6] one can take the radial and tangential components of displacement as

$$u = r(1 - \phi), \quad v = 0 \tag{13}$$

where $\phi(r)$ is to be determined later on. The radial and tangential strain components may now be written as

$$S_r = \frac{du}{dr} = 1 - \phi - r\frac{d\phi}{dr} \tag{14}$$

and

$$S_\theta = \frac{u}{r} = 1 - \phi. \tag{15}$$

When these expressions of E_r, S_r, S_θ in equations (12), (14) and (15) along with those of s_{ij}, d_{ij} and ε_{ij} given by equations 7 (i, ii, iii) are incorporated in equations 6 (i) and 6 (ii) one may have

$$1 - \phi - r\frac{d\phi}{dr} - \frac{b_{33}}{v_{33}} D_0 - \left(\mu_1^E - \frac{b_{33}}{v_{33}} p_1^S\right) T = f(r)(\lambda_1 \sigma_r + \lambda_2 \sigma_\theta) \tag{16}$$

and

$$1 - \phi - \frac{b_{31}}{v_{33}} D_0 - \left(\mu_2^E - \frac{b_{31}}{v_{33}} p_1^S\right) T = f(r)(\lambda_2 \sigma_r + \lambda_3 \sigma_\theta) \tag{17}$$

where

$$\lambda_1 = c_{33} - \frac{b_{33}^2}{v_{33}}, \quad \lambda_2 = c_{13} - \frac{b_{31} b_{33}}{v_{33}}, \quad \lambda_3 = c_{11} - \frac{b_{31}^2}{v_{33}}. \tag{17a}$$

Solving (16) and (17) for σ_r and σ_θ one gets

$$\sigma_r = \left\{(1-\phi)(\lambda_3-\lambda_2) - r\lambda_3 \frac{d\phi}{dr} + C_1 D_0 + C_2 T\right\} \Big/ Mf(r) \tag{18}$$

and

$$\sigma_\theta = -\left\{(1-\phi)(\lambda_2-\lambda_1) - r\lambda_2 \frac{d\phi}{dr} + C_1' D_0 + C_2' T\right\} \Big/ Mf(r) \tag{19}$$

where

$$C_1 = \frac{b_{31}\lambda_2 - b_{33}\lambda_3}{v_{33}}, \quad C_2 = \mu_2^E \lambda_2 - \mu_1^E \lambda_3 + \frac{(b_{33}\lambda_3 - b_{31}\lambda_2)}{v_{33}} p_1^S$$

$$C_1' = \frac{b_{31}\lambda_1 - b_{33}\lambda_2}{v_{33}}, \quad C_2' = \mu_2^E \lambda_1 - \mu_1^E \lambda_2 + \frac{(b_{33}\lambda_2 - b_{31}\lambda_1)}{v_{33}} p_1^S$$

and

$$M = \lambda_1\lambda_3 - \lambda_2^2. \tag{19 a}$$

Substituting the expressions of σ_r and σ_θ as obtained in equations (18) and (19) respectively in the equation (3) one can have the following inhomogeneous differential equation in ϕ,

$$\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) + (1-2m)\, r \frac{d\phi}{dr} + n\phi = \Lambda_1 + \Lambda_2 \ln r, \tag{20}$$

for

$$f(r) = r^{2m}. \tag{20 a}$$

Here

$$n = \left(1 - \frac{\lambda_1}{\lambda_3}\right) - 2m\left(1 - \frac{\lambda_2}{\lambda_3}\right)$$

$$\lambda_3\Lambda_1 = \{(1-2m)\,C_1 + C_1'\}\, D_0 + \{(1-2m)\,C_2 + C_2'\}\,(T_1 + T_0 \ln r_1)$$
$$+ \lambda_3 - \lambda_1 + 2m\,(\lambda_2 - \lambda_3) - C_2 T_0 = \lambda_3\,(D_0\delta_1 + \delta_2) \text{ say,}$$

and

$$\lambda_3\Lambda_2 = -\{(1-2m)\,C_2 + C_2'\}\, T_0. \tag{20 b}$$

The form of ϕ suitable for the equation (20) may be set as

$$\phi(r) = Ar^{\alpha_1} + Br^{\alpha_2} + \frac{\Lambda_2}{n}\ln r + \frac{n\Lambda_1 + 2(m-1)\Lambda_2}{n^2} \tag{21}$$

where A, B are arbitrary constants and

$$\alpha_1, \alpha_2 = (m-1) \pm \sqrt{(m-1)^2 - n}. \tag{21 a}$$

The full form to the radial displacement u can now be written with the help of the equation (13),

$$u = r - Ar^{\alpha_1+1} - Br^{\alpha_2+1} - \frac{\Lambda_1}{n} r - \frac{\Lambda_2}{n} r \ln r - \frac{2(m-1)}{n^2} \Lambda_2 r. \tag{22}$$

On substituting the expression of $\phi(r)$; σ_r, σ_θ and E_r may be expressed as functions of the radial distance r.

$$\sigma_r = \Big[A(\lambda_2 - \lambda_3 - \lambda_3\alpha_1) r^{\alpha_1} + B(\lambda_2 - \lambda_3 - \lambda_3\alpha_2) r^{\alpha_2} + D_0 C_1$$
$$- (\lambda_2 - \lambda_3)\left\{1 - \frac{n\Lambda_1 + 2(m-1)\Lambda_2}{n^2} - \frac{\Lambda_2}{n}\ln r\right\} - \frac{\Lambda_2}{n}\lambda_3 + C_2 T\Big] \Big/ Mr^{\alpha} \tag{i}$$

$$\sigma_\theta = -\Big[A(\lambda_1 - \lambda_2 - \lambda_2\alpha_1) r^{\alpha_1} + B(\lambda_2 - \lambda_2 - \lambda_2\alpha_2) r^{\alpha_2} + D_0 C_1'$$
$$- (\lambda_1 - \lambda_2)\left\{1 - \frac{n\Lambda_1 + 2(m-1)\Lambda_2}{n^2} - \frac{\Lambda_2}{n}\ln r\right\} - \frac{\Lambda_2}{n}\lambda_2 + C_2' T\Big] \Big/ Mr^{\alpha} \tag{ii}$$

$$E_r = \Big[A\{b_{33}(\lambda_3 - \lambda_2) - b_{31}(\lambda_2 - \lambda_1) + \alpha_1(\lambda_3 b_{33} - \lambda_2 b_{31})\} r^{\alpha_1}$$
$$+ B\{b_{33}(\lambda_3 - \lambda_2) - b_{31}(\lambda_2 - \lambda_1) + \alpha_2(\lambda_3 b_{33} - \lambda_2 b_{31})\} r^{\alpha_2}$$
$$+ D_0(M - b_{33}C_1 + b_{31}C_1') - (b_{33}C_2 - b_{31}C_2' + Mp_1^S) T$$
$$+ \{b_{33}(\lambda_3 - \lambda_2) - b_{31}(\lambda_2 - \lambda_1)\}\left\{\frac{\Lambda_2}{n}\ln r + \frac{n\Lambda_1 + 2(m-1)\Lambda_2}{n^2} - 1\right\}$$
$$+ (b_{33}\lambda_3 - b_{31}\lambda_2)\frac{\Lambda_2}{n}\Big] \Big/ v_{33} M r^{2m}. \tag{iii (23)}$$

With the aid of the boundary conditions (8) and (9) the set of above equations 23 (i, ii, iii) gives the following three equations containing unknowns A, B and D_0.

$$l_{11}A + l_{12}B + l_{13}D_0 = K_1 \tag{i}$$
$$l_{21}A + l_{22}B + l_{23}D_0 = K_2 \tag{ii}$$
$$l_{31}A + l_{32}B + l_{33}D_0 = K_3 \tag{iii (24)}$$

where the values of the constants 'l_{ij}'s ($i, j = 1, 2, 3$) and 'K's are known and given in Appendix I.

From the equations 24 (i, ii, iii) it is found that

$$A = \frac{\Delta_1}{\Delta}, \quad B = \frac{\Delta_2}{\Delta} \quad \text{and} \quad D_0 = \frac{\Delta_3}{\Delta} \tag{25}$$

where Δ is the nonsingular value of the determinant

$$\begin{vmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{vmatrix} \tag{26}$$

and 'Δ_i's are obtained from Δ by replacing the ith column by $\begin{matrix} K_1 \\ K_2 \\ K_3 \end{matrix}$

The radial and tangential stresses and the electric field for the above mentioned boundary conditions (8) and (9) can now be fully expressed by the equations 23 (i, ii, iii) after inserting the known values of A, B and D_0 from the equations (25).

3. Second problem

3.1. *Fundamental equations*

In this section the analysis has been extended in the case of an annular cylinder having an internal and external radii of r_1 and r_2 respectively. The cylinder is under the state of plane-strain which is maintained by a uniform longitudinal extension α. Like the first problem, it is assumed that the boundary surfaces of the cylinder are free from the radial component of stress as well as the cylinder is polarised radially under the influence of a static voltage between the inner and outer boundaries which are kept at temperatures T_1 and T_2 respectively. The object of this problem is to find out a solution satisfying the equations (1), (2), (3), (5) keeping in view the following constitutive equations relevant to the present problem.

$$S_r = s_{33}\sigma_r + s_{13}\sigma_\theta + s_{13}\sigma_z + d_{33}E_r + \mu_1^E T \tag{i}$$

$$S_\theta = s_{13}\sigma_r + s_{11}\sigma_\theta + s_{12}\sigma_z + d_{31}E_r + \mu_2^E T \tag{ii}$$

$$S_z = s_{13}\sigma_r + s_{12}\sigma_\theta + s_{11}\sigma_z + d_{31}E_r + \mu_3^E T \tag{iii}$$

$$S_{rz} = s_{44}\sigma_{rz} + d_{15}E_z \tag{iv}$$

$$S_{r\theta} = s_{44}\sigma_{r\theta} + d_{15}E_\theta \tag{v}$$

$$S_{\theta z} = s_{66}\sigma_{\theta z} \tag{vi}$$

$$D_r = d_{33}\sigma_r + d_{31}\sigma_\theta + d_{31}\sigma_z + \varepsilon_{33}E_r + p_1^S T \tag{vii}$$

$$D_\theta = d_{15}\sigma_{r\theta} + \varepsilon_{11}E_\theta + p_2^S T \tag{viii}$$

$$D_z = d_{15}\sigma_{rz} + \varepsilon_{11}E_z + p_3^S T \tag{ix}$$

(27)

where the symbols carry their usual meaning.

3.2. *Method of solution*

Since the electric field is assumed radial $E_\theta = E_z = 0$. Because of the state of plane strain $S_{r\theta} = S_{rz} = 0$. Now from the equations 27 (iv) and 27 (v) it is evident that

$$\sigma_{r\theta} = \sigma_{\theta z} = 0.$$

Gaussian divergence equation in cylindrical co-ordinates can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{1}{r}\frac{\partial D_\theta}{\partial \theta} + \frac{\partial D_z}{\partial z} = 0.$$

Owing to the radial symmetry, the above equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r) = 0.$$

which indicates $D_r = D_1/r$, D_1 is any constant. The equation 27 (vii) now turns to

$$E_r = \left(\frac{D_1}{r} - d_{33}\sigma_r - d_{31}\sigma_\theta - d_{31}\sigma_z - p_1^S T\right)\Big/ \varepsilon_{33}. \tag{28}$$

In order to find out the deformations, Seth's[6] theory of deformation is again followed and the components of displacement are chosen as

$$u = r(1 - \psi), \qquad v = 0, \qquad w = \alpha z \tag{29}$$

where α is a constant.

The three strain components in 27 (i, ii, iii) now stand as

$$S_r = 1 - \psi - r\frac{d\psi}{dr} \tag{i}$$

$$S_\theta = 1 - \psi \tag{ii}$$

$$S_z = \alpha. \tag{iii}$$

(30)

On using the relations 7 (i, ii, iii), 27 (i, ii, iii), 28 and 30 (i, ii, iii), the following set of equations may be formed,

$$\lambda_1\sigma_r + \lambda_2\sigma_\theta + \lambda_2\sigma_z = A_1 \tag{i}$$

$$\lambda_2\sigma_r + \lambda_3\sigma_\theta + \lambda_4\sigma_z = A_2 \tag{ii}$$

$$\lambda_2\sigma_r + \lambda_4\sigma_\theta + \lambda_3\sigma_z = A_3 \tag{iii}$$

(31)

where

$$\lambda_1 = c_{33} - \frac{b_{33}^2}{\nu_{33}}, \quad \lambda_2 = c_{13} - \frac{b_{31}b_{33}}{\nu_{33}}$$

$$\lambda_3 = c_{11} - \frac{b_{31}^2}{v_{33}}, \quad \lambda_4 = c_{12} - \frac{b_{31}^2}{v_{33}} - \tag{31 a}$$

$$A_1 = \left\{1 - \psi - r\frac{d\psi}{dr} - \frac{b_{33}}{v_{33}}\frac{D_1}{r} - \left(\mu_1^E - \frac{b_{33}}{v_{33}}p_1^S\right)T\right\}\Big/ f(r)$$

$$A_2 = \left\{1 - \psi - \frac{b_{31}}{v_{33}}\frac{D_1}{r} - \left(\mu_2^E - \frac{b_{31}}{v_{33}}p_1^S\right)T\right\}\Big/ f(r)$$

$$A_3 = \left\{\alpha - \frac{b_{31}}{v_{33}}\frac{D_1}{r} - \left(\mu_3^E - \frac{b_{31}}{v_{33}}p_1^S\right)T\right\}\Big/ f(r). \tag{31 b}$$

From the equations 31 (i, ii, iii) one can have

$$\sigma_r = [A_1(\lambda_3 + \lambda_4) - A_2(\lambda_2 + \lambda_3)]/H \tag{i}$$

$$\sigma_\theta = [A_2(\lambda_1\lambda_3 - \lambda_2^2) - A_3(\lambda_1\lambda_4 - \lambda_2^2) - A_1\lambda_2(\lambda_3 - \lambda_4)]/H' \tag{ii}$$

$$\sigma_z = [A_3(\lambda_1\lambda_3 - \lambda_2^2) - A_2(\lambda_1\lambda_4 - \lambda_2^2) - A_1\lambda_2(\lambda_3 - \lambda_4)]/H' \tag{iii}$$

(32)

where

$$H = \lambda_1(\lambda_3 + \lambda_4) - 2\lambda_2^2 \quad \text{and} \quad H' = (\lambda_3 - \lambda_4)H. \tag{32 a}$$

By the use of equations 32 (i, ii, iii) and (20 a) the equation (28) may be put in the form

$$E_r = \left[\frac{D_1}{r^{2m+1}} - \frac{p_1^S T}{r^{2m}} - \frac{A_1\{b_{33}(\lambda_3 + \lambda_4) - 2b_{31}\lambda_2\} + A_2\{b_{31}\lambda_1 - b_{33}(\lambda_2 + \lambda_3)\} + A_3 b_{31}\lambda_1}{H}\right]\Big/ v_{33}. \tag{33}$$

The expressions of σ_r and σ_θ of equations 32 (i, ii) along with (31 b) and (20 a) are inserted in the equation (3), which is still valid in this problem, to obtain the following inhomogeneous differential equation

$$\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + (1 - 2m)r\frac{d\psi}{dr} + N\psi = a_0 + \frac{a_1}{r} + a_2 \ln r/r_1 \tag{34}$$

The constants N, a_0, a_1 and a_2 are

$$N = (1 - 2m) + \frac{2m\lambda_2}{\lambda_3 + \lambda_4} + \frac{(\lambda_1\lambda_3 - \lambda_2^2)}{(\lambda_4^2 - \lambda_3^2)}$$

$$a_0 = 2m\left\{\frac{\lambda_2(1+\alpha)}{\lambda_3+\lambda_4} - 1\right\} + \frac{(\lambda_1\lambda_3 - \lambda_2^2 - \lambda_3^2 + \lambda_4^2)}{(\lambda_4^2 - \lambda_3^2)} - \frac{\alpha(\lambda_1\lambda_4 - \lambda_2\lambda_3 + \lambda_2\lambda_4 - \lambda_2^2)}{(\lambda_4^2 - \lambda_3^2)}$$

$$+ \left[\left\{(2m-1) - \frac{\lambda_2}{\lambda_3+\lambda_4}\right\}\left(\mu_1^E - \frac{b_{33}}{v_{33}}p_1^S\right) - \frac{2m\lambda_2}{\lambda_3+\lambda_4}\left(\mu_2^E + \mu_3^E - 2\frac{b_{31}}{v_{33}}p_1^S\right)\right.$$

$$+\frac{(\lambda_1\lambda_3+\lambda_2\lambda_3-\lambda_2\lambda_4-\lambda_2^2)}{(\lambda_3^2-\lambda_4^2)}\left(\mu_2^E-\frac{b_{31}}{v_{33}}p_1^S\right)-\frac{(\lambda_1\lambda_4-\lambda_2\lambda_3+\lambda_2\lambda_4-\lambda_2^2)}{(\lambda_3^2-\lambda_4^2)}$$

$$\times\left(\mu_3^E-\frac{b_{31}}{v_{33}}p_1^S\right)\Big]T_0-\left\{\frac{\lambda_2}{\lambda_3+\lambda_4}\left(\mu_2^E+\mu_3^E-\frac{2b_{31}}{v_{33}}p_1^S\right)\right.$$

$$\left.+\left(\mu_1^E-\frac{b_{33}}{v_{33}}p_1^S\right)\right\}T_1$$

$$a_1=\left[b_{31}(4m\lambda_2-\lambda_1)+b_{33}\left(2m-\frac{\lambda_2}{\lambda_3+\lambda_4}\right)\right]\frac{D_1}{v_{33}}$$

$$a_2=\left[\frac{2m\lambda_2}{(\lambda_3+\lambda_4)}\left(\mu_2^E+\mu_3^E-\frac{2b_{31}}{v_{33}}p_1^S\right)-\left\{(2m-1)-\frac{\lambda_2}{\lambda_3+\lambda_4}\right\}\right.$$

$$\times\left(\mu_1^E-\frac{b_{33}}{v_{33}}p_1^S\right)+\frac{(\lambda_2\lambda_3+\lambda_1\lambda_3-\lambda_2\lambda_4-\lambda_2^2)}{(\lambda_4^2-\lambda_3^2)}\left(\mu_2^E-\frac{b_{31}}{v_{33}}p_1^S\right)$$

$$\left.-\frac{(\lambda_1\lambda_4-\lambda_2\lambda_3+\lambda_2\lambda_4-\lambda_2^2)}{(\lambda_4^2-\lambda_3^2)}\left(\mu_3^E-\frac{b_{31}}{v_{33}}p_1^S\right)\right]T_1. \tag{34a}$$

The general form of $\psi(r)$ satisfying the equation (34) may be found to be

$$\psi(r)=Cr^{\xi_1}+Dr^{\xi_2}+a_0'+\frac{a_1'}{r}+a_2'\ln r/r_1 \tag{35}$$

where ξ_1 and ξ_2 are the roots of the equation

$$\xi^2+2(1-m)\xi+N=0 \tag{35a}$$

C and D are to be determined along with the unknowns α and D_1 which are involved in a_0 and a_1 in (34 a). Also a_0', a_1' and a_2' are given by

$$a_0'=\left[a_0-\frac{2(1-m)a_2}{N}\right]\Big/N=\delta_3\alpha+\delta_4\text{ say,}$$

$$a_1'=\frac{a_1}{(2m+N-1)}=\delta_5 D_1\text{ say,} \tag{35b}$$

$$a_2'=\frac{a_2}{N}.$$

The radial displacement u can now be expressed as,

$$u=r-Cr^{\xi_1+1}-Dr^{\xi_2+1}-a_0'r-a_1'-a_2'r\ln r/r_1. \tag{36}$$

By the use of the equations (31 b) and (35) the stress-components in the equations 32 (i, ii, iii) and the equation for electric field (33) can be expressed in terms of the radial distance

$$\sigma_r = \Big[C(\Gamma_1 + \Gamma_2 \xi_1) r^{\xi_1 - 2m} + D(\Gamma_1 + \Gamma_2 \xi_2) r^{\xi_2 - 2m} + D_1 \Gamma_3 r^{-(2m+1)} + \alpha C_2 r^{-2m} - \Gamma_1 r^{-2m}\left(1 - a_0' - \frac{a_1'}{r} - d_2' \ln r/r_1\right) - \Gamma_2 r^{-2m}\left(\frac{a_1'}{r} - d_2'\right) - \Gamma_4 r^{-2m} T \Big] \Big/ (-H) \quad \text{(i)}$$

$$\sigma_\theta = \Big[C(\Gamma_5 + \Gamma_7 \xi_1) r^{\xi_1 - 2m} + D(\Gamma_5 + \Gamma_7 \xi_2) r^{\xi_2 - 2m} - D_1 \Gamma_6 r^{-(2m+1)} + \alpha \Gamma_8 r^{-2m} - \Gamma_5 r^{-2m}\left(1 - a_0' - \frac{a_1'}{r} - a_2' \ln r/r_1\right) - \Gamma_7 r^{-2m}\left(\frac{d_1'}{r} - d_2'\right) - \Gamma_9 r^{-2m} T \Big] \Big/ (-H') \quad \text{(ii)}$$

$$\sigma_z = \Big[C(\Gamma_{12} + \Gamma_7 \xi_1) r^{\xi_1 - 2m} + D(\Gamma_{12} + \Gamma_7 \xi_2) r^{\xi_2 - 2m} - D_1 \Gamma_6 r^{-(2m+1)} - \alpha \Gamma_{10} r^{-2m} - \Gamma_{12} r^{-2m}\left(1 - a_0' - \frac{a_1'}{r} - d_2' \ln r/r_1\right) - \Gamma_7 r^{-2m}\left(\frac{a_1'}{r} - a_2'\right) - \Gamma_{11} r^{-2m} T \Big] \Big/ (-H') \quad \text{(iii)} \tag{37}$$

and

$$E_r = C(f_3 + f_2 \xi_1) r^{\xi_1 - 2m} + D(f_3 + f_2 \xi_2) r^{\xi_2 - 2m} + D_1 f_0 r^{-(2m+1)} + \alpha f_4 r^{-2m} + \{f_2 a_2' + f_3(a_0' - 1)\} r^{-2m} + (f_3 - f_2) a_1' r^{-(2m+1)} - f_1 r^{-2m} T + f_3 a_2' r^{-2m} \ln r/r_1 \tag{38}$$

where the 'Γ's and 'f's are constants. Their values are written in Appendix II.

The constants C, D, α and D_1 can be evaluated from the above equations 37 (i, ii, iii) and (38) by using the two mechanical boundary conditions stated in (8) and the electrical boundary condition (9) along with the condition

$$\int_{r_1}^{r_2} \sigma_z \, r dr = 0. \tag{39}$$

Application of the conditions (8) in the equation 37 (i), (9) in the equation (38) and (39) in the equation 37 (iii) yields the following set of equations,

$$m_{11} C + m_{12} D + m_{13} D_1 + m_{14} \alpha = Q_1 \quad \text{(i)}$$

$$m_{21} C + m_{22} D + m_{23} D_1 + m_{24} \alpha = Q_2 \quad \text{(ii)}$$

$$m_{31}C + m_{32}D + m_{33}D_1 + m_{34}\alpha = Q_3 \quad \text{(iii)}$$

$$m_{41}C + m_{42}D + m_{43}D_1 + m_{44}\alpha = Q_4 \quad \text{(iv)}$$

(40)

where 'm_{ij}' and 'Q_i's are constants, values of which are given in Appendix II.

From the equations 40 (i, ii, iii, iv)

$$C = \frac{\square_1}{\square}, \quad D = \frac{\square_2}{\square}, \quad D_1 = \frac{\square_3}{\square} \text{ and } \alpha = \frac{\square_4}{\square} \quad (40\,a)$$

where

$$\square = \begin{vmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{vmatrix}$$

and $\square_s$ may be found from the determinant $\square$ by replacing its sth column by

$$\begin{matrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{matrix}$$

The stress components σ_r, σ_θ, σ_z and the electric field E_r of equations 37 (i, ii, iii) and (38) can now be expressed completely by substituting the values of C, D, D_1 and α from the equations (40 a). The stresses thus obtained tally well with those found by Roy[8] for purely elastic nonhomogeneous cylinder on making $b_{ij} = 0$, $v_{ij} = 0$, $\mu_i^E = 0$ and $p_i^S = 0$.

This analysis would be helpful to design multilayer capacitors using nonlinear dielectric aggregates [Tareev[11], and Orchard[7]].

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Appendix I

The following symbols are used in the first problem:

$$l_{11} = r_1^{\alpha_1}\{\lambda_2 - \lambda_3(1 + \alpha_1)\}, \quad l_{12} = r_1^{\alpha_2}\{\lambda_2 - \lambda_3(1 + \alpha_2)\},$$

$$l_{13} = C_1 + \frac{(\lambda_2 - \lambda_3)}{n}\delta_1$$

$$l_{21} = r_2^{\alpha_1}\{\lambda_2 - \lambda_3(1+\alpha_1)\}, \quad l_{22} = r_2^{\alpha_2}\{\lambda_2 - \lambda_3(1+\alpha_2)\}, \quad l_{23} = l_{13}$$

$$l_{31} = \{b_{33}(\lambda_3 - \lambda_2) - b_{31}(\lambda_2 - \lambda_1) + \alpha_1(\lambda_3 b_{33} - \lambda_2 b_{31})\} \times \frac{(r_2^{\alpha_1-2m+1} - r_1^{\alpha_1-2m+1})}{(\alpha_1 - 2m + 1)}$$

$$l_{32} = \{b_{33}(\lambda_3 - \lambda_2) - b_{31}(\lambda_2 - \lambda_1) + \alpha_2(\lambda_3 b_{33} - \lambda_2 b_{31})\} \times (r_2^{\alpha_2-2m+1} - r_1^{\alpha_2-2m+1})/(\alpha_2 - 2m + 1)$$

$$l_{33} = \left[M - b_{33}C_1 + b_{31}\left\{C_1' + \frac{(\lambda_3 - \lambda_1)}{n}\delta_1\right\}\right](r_2^{1-2m} - r_1^{1-2m})\Big/(1-2m)$$

$$K_1 = (\lambda_2 - \lambda_3)\left\{1 - \frac{\Lambda_2}{n}\ln r_1 - \frac{n\delta_2 + 2(m-1)\Lambda_2}{n^2}\right\} + \frac{\Lambda_2}{n}\lambda_3 - C_2 T_1$$

$$K_2 = (\lambda_2 - \lambda_3)\left\{1 - \frac{\Lambda_2}{n}\ln r_2 - \frac{n\delta_2 + 2(m-1)\Lambda_2}{n^2}\right\} + \frac{\Lambda_2}{n}\lambda_3 - C_2 T_2$$

$$\begin{aligned} K_3 = \Big[&(b_{33}C_2 - b_{31}C_2' + M p_1^S)(T_1 + T_0 \ln r_1) - \{b_{33}(\lambda_3 - \lambda_2) - b_{31}(\lambda_2 - \lambda_1)\} \\ &\times \left\{\frac{n\delta_2 + 2(m-1)\Lambda_2}{n^2} - 1\right\} + (\lambda_2 b_{31} - \lambda_3 b_{33})\frac{\Lambda_2}{n}\Big](r_2^{1-2m} - r_1^{1-2m})\Big/(1-2m) \\ &- \left[(b_{33}C_2 - b_{31}C_2' + M p_1^S)T_0 + \{b_{33}(\lambda_3 - \lambda_2) - b_{31}(\lambda_2 - \lambda_1)\}\frac{\Lambda_2}{n}\right] \\ &\times \{(r_2^{1-2m}\ln r_2 - r_1^{1-2m}\ln r_1)/(1-2m) - (r_2^{1-2m} - r_1^{1-2m})/(1-2m)^2\} \\ &+ MV\nu_{33} \end{aligned}$$

Appendix II

The following symbols are used in the second problem:

$$\Gamma_1 = (\lambda_3 + \lambda_4 - \lambda_2), \quad \Gamma_2 = (\lambda_3 + \lambda_4), \quad \Gamma_3 = \{(\lambda_3 + \lambda_4)b_{33} - 2\lambda_2 b_{31}\}/\nu_{33}$$

$$\Gamma_4 = \left\{\lambda_2\left(\mu_2^E + \mu_3^E - \frac{2b_{31}}{\nu_{33}}p_1^S\right) - (\lambda_3 + \lambda_4)\left(\mu_1^E - \frac{b_{33}}{\nu_{33}}p_1^S\right)\right\}$$

$$\Gamma_5 = (\lambda_1\lambda_3 - \lambda_2\lambda_3 + \lambda_2\lambda_4 - \lambda_2^2)$$

$$\Gamma_6 = \{(\lambda_4 - \lambda_3)(\lambda_1 b_{31} - \lambda_2 b_{33})\}/\nu_{33}$$

$$\Gamma_7 = \lambda_2(\lambda_4 - \lambda_3), \qquad \Gamma_8 = (\lambda_1\lambda_4 - \lambda_2^2)$$

$$\Gamma_9 = \lambda_2(\lambda_3 - \lambda_4)\mu_1^E - (\lambda_1\lambda_3 - \lambda_2^2)\mu_2^E + \{(\lambda_3 - \lambda_4)(\lambda_1 b_{31} - \lambda_2 b_{33})\}p_1^S/\nu_{33}$$

$$\Gamma_{10} = (\lambda_1\lambda_3 - \lambda_2^2)$$

$$\Gamma_{11} = \lambda_2(\lambda_3 - \lambda_4)\mu_1^E + (\lambda_1\lambda_4 - \lambda_2^2)\mu_2^E + \{(\lambda_3 - \lambda_4)(\lambda_1 b_{31} - \lambda_2 b_{33})\}p_1^S/\nu_{33}$$

$$\Gamma_{12} = (\lambda_2\lambda_4 - \lambda_2\lambda_3 - \lambda_1\lambda_4 + \lambda_2^2)$$

$$m_{11} = (\Gamma_1 + \Gamma_2\xi_1)\, r_1^{\xi_1}, \quad m_{12} = (\Gamma_1 + \Gamma_2\,\xi_2)\, r_1^{\xi_2},$$

$$m_{13} = \frac{\Gamma_3 + (\Gamma_2 - \Gamma_1)\,\delta_5}{r_1}, \quad m_{14} = C_2 - \Gamma_1\,\delta_3$$

$$m_{21} = (\Gamma_1 + \Gamma_2\xi_1)\, r_2^{\xi_1}, \quad m_{22} = (\Gamma_1 + \Gamma_2\,\xi_2)\, r_2^{\xi_2},$$

$$m_{23} = \frac{\Gamma_3 + (\Gamma_2 - \Gamma_1)\,\delta_5}{r_2}, \qquad m_{24} = m_{14}$$

$$m_{31} = (\xi_1 f_2 + f_3)\,(r_2^{\xi_1 - 2m + 1} - r_1^{\xi_1 - 2m + 1})/(\xi_1 - 2m + 1)$$

$$m_{32} = (\xi_2 f_2 + f_3)\,(r_2^{\xi_2 - 2m + 1} - r_1^{\xi_2 - 2m + 1})/(\xi_2 - 2m + 1)$$

$$m_{33} = \{f_0 + (f_3 - f_2)\,\delta_5\}\,(r_1^{-2m} - r_2^{-2m})/2m$$

$$m_{34} = (f_4 + f_3\,\delta_3)\,(r_2^{1-2m} - r_1^{1-2m})/(1 - 2m)$$

$$m_{41} = (\Gamma_{12} + \Gamma_7\xi_1)\,\{r_2^{\xi_1 + 2(1-m)} - r_1^{\xi_1 + 2(1-m)}\}/\xi_1 + 2\,(1 - m)$$

$$m_{42} = (\Gamma_{12} + \Gamma_7\xi_2)\,\{r_2^{\xi_2 + 2(1-m)} - r_1^{\xi_2 + 2(1-m)}\}/\xi_2 + 2\,(1 - m)$$

$$m_{43} = \{\Gamma_6 + (\Gamma_7 - \Gamma_{12})\,\delta_5\}\,(r_2^{1-2m} - r_1^{1-2m})/(2m - 1)$$

$$m_{44} = (\Gamma_{10} - \Gamma_{12}\,\delta_3)\,\{r_2^{2(1-m)} - r_1^{2(1-m)}\}/2\,(m - 1)$$

$$Q_1 = \Gamma_1\,(1 - \delta_4) - \Gamma_2 a_2' + \Gamma_4 T_1$$

$$Q_2 = \Gamma_1\,(1 - \delta_4 - a_2' \ln r_2/r_1) - \Gamma_2\, a_2' + \Gamma_4 T_2$$

$$Q_3 = v_{33}\, V - \{f_2\, a_2' + f_3\,(\delta_4 - 1)\}\,(r_2^{1-2m} - r_1^{1-2m})/(1 - 2m)$$
$$+ f_1\, T_0\,(r_2^{1-2m} - r_1^{1-2m})/(1 - 2m)$$
$$- \frac{(f_1\, T_1 + f_3\, a_2')}{2\,(1 - m)}\,[r_2^{2(1-m)} \ln r_2/r_1 - \{r_2^{2(1-m)} - r_1^{2(1-m)}\}]$$

$$Q_4 = \{\Gamma_{12}\,(1 - \delta_4) - \Gamma_7 a_2' + \Gamma_{11} T_0\}\,\{r_2^{2(1-m)} - r_1^{2(1-m)}\}/2\,(1 - m)$$
$$- \frac{(\Gamma_{11}\, T_1 + \Gamma_{12}\, a_2')}{2(1 - m)}\,[r_2^{2(1-m)} \ln r_2/r_1 - \{r_2^{2(1-m)} - r_1^{2(1-m)}\}/2\,(1 - m)]$$

$$f_0 = 1 + \frac{\Gamma_3}{H}\, b_{33} - \frac{2\Gamma_6}{H'}\, b_{31}$$

$$f_1 = p_1^S + \frac{\Gamma_4}{H}\, b_{33} + \frac{(\Gamma_9 + \Gamma_{11})}{H'}\, b_{31}$$

$$f_2 = \frac{\Gamma_2}{H} b_{33} + \frac{2\,\Gamma_7}{H'}\, b_{31}$$

$$f_3 = \frac{\Gamma_1}{H}\, b_{33} + \frac{(\Gamma_5 + \Gamma_{12})}{H'}\, b_{31}$$

$$f_4 = \frac{c_2}{H} b_{33} + \frac{(\Gamma_8 - \Gamma_{10})}{H'}\, b_{31}$$

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