Bivariate Laplace transforms for some *H*-functions

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Abstract

The object of the present paper is to have computation of certain bivariate Laplace transforms for the *H*-functions. A general theorem is established, which gives the correspondences, involving Fox's *H*-function, between the original and the image in two variables. A few particular cases of interest are also discussed.

Key words : Computation, Laplace transform, Fox's *H*-function, image and original functions, operational correspondences, integral equation.

1. Introduction

The Laplace Carson transform in two variables is defined and represented by the integral equation[3] (p. 39)

$$F(p,q) \doteqdot pq \int_0^\infty \int_0^\infty e^{-px-qy} f(x,y)\,dx\,dy\ ;\quad \mathrm{Re}\ (p,q)>0\ ; \tag{1}$$

where $F(p, q)$ and $f(x, y)$ are said to be operationally related to each other. $F(p, q)$ is called the image and $f(x, y)$ the original.

Symbolically we can write

$$F(p,q) \doteqdot f(x,y) \text{ or } \textit{vice versa}, \tag{2}$$

where the symbol $\doteqdot$ is termed as operational.

The *H*-function[5] is defined and represented in the notation of Braaksma[1] as follows :

$$H_{r,s}^{m,n}\left[x \,\middle|\, \begin{matrix}(a, A)\\(b, B)\end{matrix}\right] = H_{r,s}^{m,n}\left[x \,\middle|\, \begin{matrix}(a_1, A_1), \ldots, (a_r, A_r)\\(b_1, B_1), \ldots, (b_s, B_s)\end{matrix}\right]$$

$$= (2i\pi)^{-1} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - wB_j) \prod_{j=1}^{n} \Gamma(1 - a_j + wA_j)}{\prod_{j=m+1}^{s} \Gamma(1 - b_j + wB_j) \prod_{j=n+1}^{r} \Gamma(a_j - wA_j)} x^w\,dw, \tag{3}$$

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where x may be real or complex but is not equal to zero and an empty product is interpreted as unity, m, n, r, s are integers satisfying the inequalities $1 \leq m \leq s$, $0 \leq n \leq r$, $A_j (j = 1, \ldots, r)$, $B_j (j = 1, \ldots, s)$ are positive numbers and $a_j (j = 1, \ldots, r)$, $b_j (j = 1, \ldots, s)$ are complex numbers such that no pole of $\Gamma(b_h - wB_h) (h = 1, \ldots, m)$ coincide with any pole of $\Gamma(1 - a_j + wA_j) (j = 1, \ldots, n)$; *i.e.*,

$$A_j (b_h + \varepsilon) \neq B_h (a_j - \eta - 1)$$
$$(\varepsilon, \eta = 0, 1, 2, \ldots; \; h = 1, \ldots, m \; ; \; j = 1, \ldots, n). \tag{4}$$

The contour L runs from $\sigma - i\infty$ to $\sigma + i\infty$ (σ real) such that the poles of $\Gamma(b_j - wB_j)$ $(j = 1, \ldots, m)$ lie on the right-hand side of L and those of $\Gamma(1 - a_j + wA_j)$ $(j = 1, \ldots, n)$ lie on the left-hand side of L. Such a contour is possible on account of (4).

The integral in (3) converges for

$$|\arg x| < \tfrac{1}{2} \lambda \pi, \; \lambda > 0, \tag{5}$$

where

$$\lambda = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{r} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{s} B_j. \tag{6}$$

These conditions are assumed to hold good throughout this paper.

In this paper we shall obtain correspondences, involving Fox's H-function, between the original and the image in two variables.

In what follows we shall denote the original variables by x and y and the transformed variables by p and q. The notations employed are those of Ditkin and Prudnikov's Operational Calculus[3]. The results obtained here provide a generalization to the results given earlier by Dahiya[2].

2. Theorem

If (i) $0 \leq n \leq r, \; 1 \leq m \leq s, \; \delta > 0, \; \text{Re}(p) > 0,$

(ii) $|\arg u| < \tfrac{1}{2}\pi\lambda, \; \lambda > 0,$

(iii) $\text{Re}(v) > 0, \; \text{Re}[-\beta + \delta b_j/B_j] > -1 \; (j = 1, \ldots, m),$
$\text{Re}[\beta + v - \delta(a_j - 1)/A_j] < 3/4 \; (j = 1, \ldots, n),$

(iv) $A_j (b_h + \varepsilon) \neq B_h (a_j - \eta - 1)$
$(\varepsilon, \eta = 0, 1, 2, \ldots \; ; \; h = 1, \ldots, m \; ; \; j = 1, \ldots, n),$

$$p^{-\frac{1}{2}}(pq)^{(3/2)-\frac{1}{2}\beta-v}\, H^{m,\,n+1}_{r+1,\,s}\left[(u^2 pq)^{\frac{1}{2}\delta}\,\middle|\,\begin{matrix}(\beta,\delta),\ (a, A)\\ (b, B)\end{matrix}\right]$$

$$\doteqdot (\pi y)^{-\frac{1}{2}}(4xy)^{v+\frac{1}{2}\beta-1}\, H^{m,\,n+1}_{r+2,\,s}\left[(u^{-2}\,4xy)^{-\frac{1}{2}\delta}\,\middle|\,\begin{matrix}(\beta,\delta),\ (a, A),\ (\beta+2v-1,\delta)\\ (b, B)\end{matrix}\right]. \tag{7}$$

Proof: The Laplace transform of H-function is given by[6] [p. 140, Eqn. (2.4)]

$$\int_0^\infty e^{-pt}\, t^{-\beta}\, H^{m,\,n}_{r,\,s}\left[(ut)^{\delta}\,\middle|\,\begin{matrix}(a, A)\\ (b, B)\end{matrix}\right] dt$$

$$= p^{\beta-1}\, H^{m,\,n+1}_{r+1,\,s}\left[\left(\frac{u}{p}\right)^{\delta}\,\middle|\,\begin{matrix}(\beta,\delta),\ (a, A)\\ (b, B),\end{matrix}\right], \tag{8}$$

valid for $\operatorname{Re}(p)>0$, $\delta>0$, $\lambda>0$, $|\arg u|<\lambda\pi/2$, $\operatorname{Re}[-\beta+\delta(b_j/B_j)]>-1$ $(j=1,2,\ldots,m)$.

On writing $(pq)^{-\frac{1}{2}}$ for p, multiplying both sides of (8) by $(pq)^{1-v}p^{-\frac{1}{2}}$ and then interpreting it with the help of the known result[3] [p. 144 (3.26)], we get

$$(\pi y)^{-\frac{1}{2}}(4xy)^{(2v-1)/4}\int_0^\infty t^{(1-2\beta-2v)/2}\, J_{2v-1}\left[(64\, xyt^2)^{\frac{1}{4}}\right]$$

$$\times\, H^{m,\,n}_{r,\,s}\left[(ut)^{\delta}\,\middle|\,\begin{matrix}(a, A)\\ (b, B)\end{matrix}\right] dt$$

$$\doteqdot p^{-\frac{1}{2}}(pq)^{(3-\beta-2v)/2}\, H^{m,\,n+1}_{r+1,\,s}\left[(u^2 pq)^{\frac{1}{2}\delta}\,\middle|\,\begin{matrix}(\beta,\delta),\ (a, A)\\ (b, B)\end{matrix}\right], \tag{9}$$

provided $\operatorname{Re}(v)>0$.

Now on evaluating the left hand side integral with the help of a known formula[4] [p. 326 (2)], we obtain the desired result valid under the conditions (i)–(iv) stated with the theorem.

3. Particular cases

By taking proper choice of the parameters in (7) and on using the known results[7] (p. 54–68), we obtain the following two class of results :

(A) *The named image functions expressed in terms of the H-function*

For the sake of brevity we shall use the following abbreviations in this section :

$$X=(pq)^{\frac{1}{2}},\quad Y=(4xy)^{\frac{1}{2}}\quad \text{and}\quad \theta=(p\delta^2)^{-\frac{1}{2}}.$$

$\theta X^{6-4v} I_v(X) K_\mu(X)$

$$\doteqdot \frac{Y^{(4v-3)/2}}{4\pi y^{\frac{1}{2}}} H^{2,2}_{3,4}\left[Y^{-\delta} \middle| \begin{matrix} (\frac{1}{2}, \delta), (1, \delta), (2v - \frac{1}{2}, \delta) \\ \left(\frac{\mp\mu + v + 1}{2}, \delta\right), \left(\frac{1-\mu \mp v}{2}, \delta\right) \end{matrix} \right].$$

$\theta X^{5-4v} H^{(1)}_v(X) H^{(2)}_v(X)$

$$\doteqdot \frac{(\cos v\pi)}{2\pi^3 y^{\frac{1}{2}}} Y^{2v+1} H^{3,1}_{2,3}\left[Y^{-\delta} \middle| \begin{matrix} (1, \delta), (2v, \delta) \\ (\frac{1}{2} \pm v, \delta), (\frac{1}{2}, \delta) \end{matrix} \right].$$

$\theta X^{2k-4-4v} M_{k,m}(2X) W_{-k,m}(2X)$

$$\doteqdot \frac{\Gamma(1+2m)\, Y^{2v-k-\frac{1}{2}}}{\pi y^{\frac{1}{2}} \Gamma(m+k+\frac{1}{2})} H^{3,1}_{3,4}\left[Y^{-\delta} \middle| \begin{matrix} (\frac{3}{2} \pm k, \delta), (2v - k + \frac{1}{2}, \delta) \\ (1 \pm m, \delta), (\frac{3}{2}, \delta), (1, \delta) \end{matrix} \right].$$

$\theta X^{4-2k-4v} e^{\frac{1}{2}X^2} W_{k,m}(X^2)$

$$\doteqdot \frac{Y^{2v+k} (\pi y)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2} - k \pm m)} H^{2,1}_{2,2}\left[Y^{-\delta} \middle| \begin{matrix} (k+2, \delta), (k + 2v + 1, \delta) \\ (\frac{3}{2} \pm m, \delta) \end{matrix} \right]. \; (-\tfrac{3}{2} < m < \tfrac{3}{2}) \tag{13}$$

$\theta X^{6-4v} [I_v(X) I_\mu(X) - I_{-\mu}(X) I_{-v}(X)]$

$$\doteqdot \frac{-\sin(\mu + v)\pi}{\pi^2 y^{\frac{1}{2}}} Y^{(4v-3)/2} H^{2,2}_{3,4}\left[Y^{-\delta} \middle| \begin{matrix} (\frac{1}{2}, \delta), (1, \delta), (2v - \frac{1}{2}, \delta) \\ \left(\frac{\mu \mp v + 1}{2}, \delta\right), \left(\frac{1-\mu \mp v}{2}, \delta\right) \end{matrix} \right]. \tag{14}$$

$\theta X^{5-4v} e^{-\frac{1}{2}X^2} I_v(\frac{1}{2}X^2)$

$$\doteqdot \frac{Y^{(4v-1)/2}}{\pi y^{\frac{1}{2}}} H^{1,1}_{2,2}\left[Y^{-\delta} \middle| \begin{matrix} (\frac{3}{2}, \delta), (2v + \frac{1}{2}, \delta) \\ (v+1, \delta), (1 - v, \delta) \end{matrix} \right]. \tag{15}$$

$\theta X^{5-v} [I_{-v}(2X) - L_v(2X)]$

$$\doteqdot \frac{\cos(v\pi)}{\pi^{3/2} y^{\frac{1}{2}}} Y^{(5v-2)/2} H^{2,1}_{2,3}\left[Y^{-\delta} \middle| \begin{matrix} \left(\frac{v}{2} + 1, \delta\right), \left(\frac{5}{2}v, \delta\right) \\ \left(1 + \frac{v}{2}, \delta\right), \left(\frac{1}{2} \pm \frac{v}{2}, \delta\right) \end{matrix} \right]. \tag{16}$$

$\theta X^{5-4v} e^{\frac{1}{2}X^2} K_v(\frac{1}{2}X^2)$

$$\doteqdot \frac{\sec(v\pi)}{y^{\frac{1}{2}}} Y^{(4v-1)/2} H^{2,1}_{2,2}\left[Y^{-\delta} \middle| \begin{matrix} (\frac{3}{2}, \delta), (2v + \frac{1}{2}, \delta) \\ (1 \pm v, \delta) \end{matrix} \right]. \tag{17}$$

$\theta X^{4-4v-2k} W_{k,m}(2iX) W_{k,m}(-2iX)$

$$\doteqdot \frac{Y^{(4v+2k-1)/2}}{\pi y^{\frac{1}{2}} \Gamma(\frac{1}{2} - k \pm m)} H^{4,1}_{3,4}\left[Y^{-\delta} \middle| \begin{matrix} (\frac{3}{2} \pm k, \delta), (2v + k + \frac{1}{2}, \delta) \\ (1 \pm m, \delta), (\frac{3}{2}, \delta), (1, \delta) \end{matrix} \right]. \tag{18}$$

$$\theta X^{5(1-v)}\,[H_v(2X) - Y_v(2X)]$$

$$\doteqdot \frac{\cos(v\pi)}{\pi^{5/2}\,y^{\frac{1}{2}}}\,Y^{(5v-2)/2}\,H^{2,1}_{2,3}\left[Y^{-\delta}\left|\begin{matrix}\left(\frac{v}{2}+1,\delta\right),\ \left(\frac{5}{2}v,\delta\right)\\ \left(\frac{v}{2}+1,\delta\right),\ \left(\frac{1}{2}\pm\frac{v}{2},\delta\right)\end{matrix}\right.\right]. \tag{19}$$

$$\theta X^{5-4v-\mu}\,S_{\mu,v}(2X)$$

$$\doteqdot -\frac{Y^{(4v+\mu-2)/2}\,(\pi y)^{-\frac{1}{2}}}{2^{-\mu+1}\,\Gamma\left(\frac{1}{2}-\frac{\mu}{2}\pm\frac{v}{2}\right)}\,H^{3,1}_{2,3}\left[Y^{-\delta}\left|\begin{matrix}\left(1+\frac{1}{2}\mu,\delta\right),\ \left(2v+\frac{\mu}{2},\delta\right)\\ \left(1+\frac{\mu}{2},\delta\right),\ \left(\frac{1}{2}\pm\frac{v}{2},\delta\right)\end{matrix}\right.\right]. \tag{20}$$

(B) *The H-function expressed as a named original function*

For the sake of brevity we shall use the following abbreviations in this section :

$$Z=(4xy)^{-\frac{1}{4}},\quad U=(pq)^{\frac{1}{2}},\quad \phi=\left(\frac{p}{\delta^2}\right)^{-\frac{1}{2}}.$$

$$\phi U^{(4-w)/2}\,H^{2,1}_{1,4}\left[U^{\delta}\left|\begin{matrix}\left(\frac{1}{2}+\frac{1}{2}w,\delta\right)\\ \left(\frac{w}{2}+\frac{\mu}{2}+\frac{v}{2},\delta\right),\ \left(\frac{w}{2}-\frac{\mu}{2}-\frac{v}{2},\delta\right),\ \left(\frac{w}{2}-\frac{\mu}{2}+\frac{v}{2},\delta\right),\\ \left(\frac{w}{2}+\frac{\mu}{2}-\frac{v}{2},\delta\right)\end{matrix}\right.\right]$$

$$\doteqdot Z^2\,[J_\mu(Z)\,J_v(Z) - J_{-v}(Z)\,J_{-\mu}(Z)]\,y^{-\frac{1}{2}}\,\frac{1}{2}\sec\left(\frac{\mu+v}{2}\right)\pi. \tag{21}$$

$$\phi U^{(4-w)/2}\,H^{4,1}_{1,4}\left[U^{\delta}\left|\begin{matrix}\left(\frac{1}{2}+\frac{1}{2}w,\delta\right)\\ \left(\frac{w}{2}\pm\frac{\mu}{2}+\frac{v}{2},\delta\right),\ \left(\frac{w}{2}\pm\frac{\mu}{2}-\frac{v}{2},\delta\right)\end{matrix}\right.\right]$$

$$\doteqdot \frac{(\pi Z)^2\,y^{-\frac{1}{2}}}{2}\,[(\cos\mu\pi+\cos v\pi)]^{-1}$$
$$\times\,[e^{i\pi(v-\mu)/2}\,H^{(1)}_v(Z)\,H^{(2)}_\mu(Z) + e^{i\pi(\mu-v)/2}\,H^{(1)}_\mu(Z)\,H^{(2)}_v(Z)]. \tag{22}$$

$$\phi U^{(2k-a-2)/2}\,H^{4,1}_{1,4}\left[\left(\frac{U}{4}\right)^{\delta}\left|\begin{matrix}\left(k+1+\frac{1}{2}a,\delta\right)\\ \left(\frac{a}{2}+\frac{1}{2}\pm m,\delta\right),\ \left(1+\frac{a}{2},\delta\right),\ \left(\frac{a}{2}+\frac{1}{2},\delta\right)\end{matrix}\right.\right]$$

$$\doteqdot 2^{4-a}\,Z^{2k-4}\,y^{-\frac{1}{2}}\,\Gamma\left(\tfrac{1}{2}-k\pm m\right)\,W_{k,m}(iZ)\,W_{k,m}(-iZ). \tag{23}$$

$$\phi U^{(4l+7)/4} H^{3,1}_{1,4}\left[U^{\delta} \middle| \begin{matrix} (\frac{1}{4}+k,\delta) \\ (m-\frac{1}{4},\delta),(\pm\frac{1}{4},\delta),(-m-\frac{1}{4},\delta) \end{matrix} \right]$$

$$\Gamma(m-k+\tfrac{1}{2}) Z^{2k} [\Gamma(2m+1)]^{-1} y^{-\frac{1}{2}} W_{k,m}(2Z) M_{-k,m}(2Z). \quad \ldots \quad (24)$$

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References

1. BRAAKSMA, B.L.J. Asymptotic expansions and analytic continuations for a class of Barnes integrals, *Comp. Math.*, 1963, 15, 239–341.

2. DAHIYA, R. S. Computation of two-dimensional Laplace integrals for G-functions, *Jnanabha Sect. A*, 1973, 3, 15–21.

3. DITKIN, V. A. AND PRUDNIKOV, A. P. *Operational Calculus in two Variables and its Applications*, Pergamon Press, New York, 1962.

4. ERDÉLYI, A. *et al.* *Tables of Integral Transforms*, McGraw-Hill, New York 1954, Vol. 1.

5. FOX, C. The *G*- and *H*-functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.*, 1961, 98, 395–429.

6. GUPTA, K. C. AND JAIN, U. C. On the derivative of the *H*-function, *Proc. Nat. Acad. Sci. India Part A*, 1968, 38, 189–192.

7. MATHAI, A. M. AND SAXENA, R. K. *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences*, Springer-Verlag, 1973.