On the theory of transforms associated with eigenvectors (II)

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Abstract

In this paper the author studies some applications of the transform theory developed in ref. 1 and based on the solutions of the differential system

$$(L-\lambda I)\phi = 0,$$

where

$$L = \begin{pmatrix} -d^2/dx^2 + p(x) & r(x) \\ r(x) & -d^2/dx^2 + q(x) \end{pmatrix}$$

and ϕ is a two component column vector.

Transforms of some suitable vectors are first evaluated which in turn lead to some useful results under the conditions of uniqueness of the Green's matrix. Some theorems concerning spectrum, transforms and partial derivatives of the Green's matrix are then proved which ultimately lead to the following :

Theorem : If $Tf(x) = F(t)$ and $TLf(x) = tF(t)$, then a necessary and sufficient condition that $F(t)$, $tF(t) \in \mathcal{L}^2$ is that $f(x)$, $L_f(x) \in L^2$.

Some of the results obtained in this paper are generalisations of those of Sears[8, 9].

Key words : Transform, Reverse transform, Green's matrix, Parseval formula, Spectrum, L^2-solution.

1. Introduction

The object of this paper is to study some applications of the transform theory developed in ref. 1 and based on the solutions of the differential system

$$(L-\lambda I)\phi = 0, \tag{1.1}$$

where

$$L = \begin{pmatrix} -d^2/dx^2 + p(x) & r(x) \\ r(x) & -d^2/dx^2 + q(x) \end{pmatrix}$$

and

$$\phi = \phi(x) = \{u(x), v(x)\}$$

is a two component column vector function of x.

In order to avoid repetition of the preliminaries, we have written this paper as an addendum to ref. 1 and consequently we make free use of symbols, notations, and results contained therein.

We denote our transform as

$$Tf = Tf(x) = \{T_1 f, T_2 f\} = F(t),$$

where

$$T_r f = \langle \phi_r(0 \mid x, t), \; f(x) \rangle_{0,\infty} = F_r(t)$$

and reverse transform as

$$\mathcal{J}F = \mathcal{J}F(t) = \sum_{r=1}^{2} \int_{-\infty}^{\infty} \phi_r(0 \mid x, t) (F(t), d\rho_v(t))$$

$$= \{\langle U, F, d\rho \rangle, \langle V, F, d\rho \rangle\} = f(x),$$

where

$$U = U(x,t) = \{u_1(0 \mid x, t), u_2(0 \mid x, t)\}$$

$$V = V(x,t) = \{v_1(0 \mid x, t), \; v_2(0 \mid x, t)\},$$

t is real; and $f(x) = \{f_1(x), f_2(x)\} \in L^2$, $F(t) \in \mathcal{L}^2$.

(*cf.* definition in ref 1. § 4, §9)

2. Some transforms

(i) Let $f(x) = \{1, 0\}$ $(c \leqslant x \leqslant \xi)$ and $f(x) = \{0, 0\}$ otherwise, then

$$F(t) = Tf = \int_c^{\xi} U(x,t)\, dx.$$

(ii) Let $f(x) = \{0, 1\}$ $(c \leqslant x \leqslant \xi)$ and $f(x) = \{0, 0\}$ otherwise, then

$$F(t) = Tf = \int_c^{\xi} V(x,t)\, dx.$$

(iii) Let $f(x) = \psi_r(x, \lambda)$ $(r = 1, 2)$, $\quad \operatorname{Im}(\lambda) \neq 0$.

Then

$$F(t) = T\psi_1(x, \lambda) = \{1/(\lambda - t), 0\}$$

and

$$F(t) = T\psi_2(x, \lambda) = \{0, 1/(\lambda - t)\},$$

t real. [*cf.* ref. 1 § 7 and Lemma (9.1)]

(iv) Let $f(x) = G_r(x, y, \lambda)$, $(r = 1, 2)$, $\operatorname{Im}(\lambda) \neq 0$, where $G_r(x, y, \lambda)$ is the r-th column of the Green's matrix

$$G(x, y, \lambda) = \begin{pmatrix} G_{11} & G_{21} \\ G_{12} & G_{22} \end{pmatrix} = \begin{pmatrix} \psi_{*1}(x, \lambda)\, U(y, \lambda) & \psi_{*1}(x, \lambda)\, V(y, \lambda) \\ \psi_{*2}(x, \lambda)\, U(y, \lambda) & \psi_{*2}(x, \lambda)\, V(y, \lambda) \end{pmatrix} \quad (y < x)$$

$$= \begin{pmatrix} U^T(x, \lambda)\, \psi^T_{*1}(y, \lambda) & U^T(x, \lambda)\, \psi^T_{*2}(y, \lambda) \\ V^T(x, \lambda)\, \psi^T_{*1}(y, \lambda) & V^T(x, \lambda)\, \psi^T_{*2}(y, \lambda) \end{pmatrix}, \quad (y > x)$$

$(x, \lambda) \in L^2$ in x and ψ_{*r} is the r-th row of the matrix

$$(\psi_{rs}(\lambda)) = \begin{pmatrix} \psi_{11}(x, \lambda) & \psi_{21}(x, \lambda) \\ \psi_{12}(x, \lambda) & \psi_{22}(x, \lambda) \end{pmatrix}.$$

[*cf.* Chakravarty[3, 4]]

Let

$$\Phi(x, \lambda, f) = \int_0^\infty G^T(y, x, \lambda) f(y)\, dy \tag{2.5}$$

then it follows in the usual manner that $\Phi(x, \lambda, f)$ satisfies the non-homogeneous system

$$(L - \lambda I)\, \phi = -f$$

and that

$$\Phi(x, \lambda_2, f) = \frac{1}{\lambda}\left(f(x) + \Phi(x, \lambda, \tilde{f})\right), \tag{2.6}$$

where $f(x) = \{f_1, f_2\} \in L^2$ has continuous derivatives up to the second order in $[0, \infty)$ and satisfies the boundary conditions in the b-case;

$$\tilde{f}(x) = Lf(x) \in L^2[0, \infty),$$

and λ is not an eigenvalue. [*cf.* Chakravarty[3, 4]].

Putting $f(x) = \phi_1(0 \mid x, t)$, t real, in (2.6) we obtain

$$\int_0^\infty G^T(y, x, \lambda)\, \phi_1(0 \mid y, t)\, dy = \phi_1(0 \mid x, t)/(\lambda - t)$$

so that

$$\langle \phi_1(0 \mid y, t), G_1(y, x, \lambda) \rangle_{0, \infty} = u_1(0 \mid x, t)/(\lambda - t)$$

and

$$\langle \phi_1(0 \mid y, t), G_2(y, x, \lambda) \rangle_{0, \infty} = v_1(0 \mid x, t)/(\lambda - t).$$

Similarly

$$\langle \phi_2(0 \mid y, t), G_1(y, x, \lambda) \qquad = u_2(0 \mid x, t)/(\lambda - t)$$

and

$$\langle \phi_2(0 \mid y, t), G_2(y, x, \lambda) \rangle = v_2(0 \mid x, t)/(\lambda - t).$$

Thus

$$TG_1(x, y, \lambda) = U(y, t)/(\lambda - t)$$

and

$$TG_2(x, y, \lambda) = V(y, t)/(\lambda - t)$$

(v) Let

$$f(x) = \Phi(x, \lambda, f), \qquad \text{Im}(\lambda) \neq 0.$$

Since

$$\Phi(y, \lambda, f) = \int_0^\infty G^T(x, y, \lambda) f(x)\, dx$$

$$= \{\langle G_1(x, y, \lambda), f(x) \rangle_{0,\infty}, \ \langle G_2(x, y, \lambda), f(x) \rangle_{0,\infty}\}$$

it follows by using (2.7) and (2.8) in the formula (4.5) of ref. 1 that

$$\Phi(y, \lambda, f) = \{\langle U/(\lambda - t), \ F, d\rho \rangle, \ \langle V/(\lambda - t), F, d\rho \rangle\}$$

$$= \{\langle U, F/(\lambda - t), d\rho \rangle, \ \langle V, F/(\lambda - t), d\rho \rangle\}$$

$$= \mathcal{J}\,(F(t)/(\lambda - t))$$

by (1.3). Hence

$$T\Phi(y, \lambda, f) = F(t)/(\lambda - t)$$

almost everywhere.

(vi) Let

$$f(x) = Lf(x) = \tilde{f}(x).$$

From (2.6)

$$T\Phi(x, \lambda, f) = \frac{1}{\lambda} Tf(x) + \frac{1}{\lambda} T\Phi(x, \lambda, \tilde{f})$$

whence

$$F(t)/(\lambda - t) = \frac{1}{\lambda} F(t) + T\tilde{f}/(\lambda - t)$$

by (2.9). Therefore

$$T\tilde{f} = tF(t).$$

3. Some useful results

(*I*) $[m_{rs}(\lambda_1) - m_{rs}(\lambda_2)]/(\lambda_2 - \lambda_1) = \int_{-\infty}^{\infty} d\rho_{rs}(t)/(\lambda_1 - t)(\lambda_2 - t), \quad (r, s = 1, 2).$ (3.1)

Equation (2.8) of ref. 1 with the relevant properties contained in § 6 of the said paper, yields

$$[m_{rs}(\lambda_1) - m_{rs}(\lambda_2)]/(\lambda_2 - \lambda_1) = \langle \psi_r(x, \lambda_1), \quad \psi_s(x, \lambda_2)\rangle_{0, \infty}. \tag{3.2}$$

Applying the formula (4.5) of ref. 1 to the transforms given by (2.3) and (2.4), the desired result follows.

(*II*) $(G(y, \xi, \lambda_1) - G(y, \xi, \lambda_2))/(\lambda_1 - \lambda_2)$

$$= \begin{pmatrix} \langle U(y,t)/(\lambda_1 - t), U(\xi, t)/(\lambda_2 - t), d\rho\rangle & \langle U(y,t)/(\lambda_1 - t), V(\xi, t)/(\lambda_2 - t), d\rho\rangle \\ \langle V(y,t)/(\lambda_1 - t), U(\xi, t)/(\lambda_2 - t), d\rho\rangle & \langle V(y,t)/(\lambda_1 - t), V(\xi, t)/(\lambda_2 - t), d\rho\rangle \end{pmatrix} \tag{3.3}$$

for any non-real $\lambda_1 \neq \lambda_2$.

It follows in usual manner under the conditions of uniqueness of the Green's matrix that for any non-real $\lambda_1 \neq \lambda_2$

$$\int_0^{\infty} G(y, x, \lambda_1)\, G(x, \xi, \lambda_2)\, dx = (G(y, \xi\, \lambda_1) - G(y, \xi, \lambda_2))/(\lambda_1 - \lambda_2). \tag{3.4}$$

[*cf.* Tiwari and Jaiswal[5]].

Making use of the Parseval formula (4.5) of ref. 1 for the transforms given by (2.7) and (2.8) on the left-hand side of (3.4), the required result follows.

(*III*) Let

$$Tg = G(t),$$

where

$$g = g(x) = \{g_1, g_2\} \in L^2.$$

Then

$$\langle \Phi(x, \lambda_1, f), \Phi(x, \lambda_2, g\rangle_{0,\infty} = \langle F(t)/(\lambda_1 - t), G(t)/(\lambda_2 - t), d\rho\rangle. \tag{3.5}$$

Applying the Parseval formula (4.5) of ref. 1 to the transform of g and the transform given by (2.9), we obtain

$$\langle \Phi(x, \lambda, f), g(x)\rangle_{0, \infty} = \langle F(t) / (\lambda - t), G(t), d\rho\rangle. \tag{3.6}$$

Putting $\lambda = \lambda_1$ and $g(x) = \Phi(x, \lambda_2, f)$ in (3.6), the desired result follows.

4. Spectrum

Following Titchmarsh[6] (pp. 66–67), we define the 'spectrum' as the complement of the set of points in the neighbourhood of which the matrix $(\rho_{rs}(\lambda))$ is constant. Any point of discontinuity of $(\rho_{rs}(\lambda))$ clearly belongs to the spectrum. The set of such points is the 'point spectrum'. The derived set of this set also belongs to the spectrum.

Theorem (4.1) : Let $\lambda = \mu + i\nu$, where μ is not in the spectrum. Then

(i) The results of § 2 and § 3 hold with λ replaced by μ ;

(ii) $\psi_r(x, \lambda) \in L^2$ in x $(r = 1, 2)$ and satisfies (1.1) with $\lambda = \mu$,

(iii) $\Phi(x, \mu, f) \in L^2$ in x if $f(x) \in L^2$; and satisfies the non-homogeneous system with $\lambda = \mu$ at all points of continuity of $f(x)$.

PROOF : The integrals with respect to $\rho_{rs}(t)$ $(r, s = 1, 2)$ over $(-\infty, \infty)$ in this case are actually the integrals over $(-\infty, \mu - \delta)$, $(\mu + \delta, \infty)$ for some $\delta > 0$.

Further, the arguments contained in § 3 of ref. 1 yield

$$\left.\begin{array}{l} \int_{-\infty}^{\infty} d\rho_{rs}(u)/(u^2 + 1) \leq k \\ \text{and} \\ \rho_{rs}(u) \leq k(1 + u^2) \end{array}\right\}. \tag{4.1}$$

Hence by (4.1) and (3.1)

$$m_{rs}(\mu) = \lim_{\nu \to 0} m_{rs}(\lambda) \qquad (r, s = 1, 2)$$

exists and

$$[m_{rs}(\lambda_1) - m_{rs}(\mu)]/(\mu - \lambda_1) = \int_{-\infty}^{\infty} d\rho_{rs}(t)/(\lambda_1 - t)(\mu - t), \operatorname{Im}(\lambda_1) = 0.$$

Let us now define

$$\psi_r(x, \mu) = \lim_{\nu \to 0} \psi_r(x, \lambda)$$

$$= \sum_{s=1}^{2} m_{rs}(\mu)\, \phi_r(0 \mid x, \mu) + \theta_r(0 \mid x, \mu) \qquad (r = 1, 2).$$

Then, it follows, from (3.1) and (3.2) with $\lambda_1 = \lambda = \bar{\lambda}_2$, that $\psi_r(x, \mu) \in L^2$, since in this case

$$\operatorname{Im} m_{rs}(\lambda) = -\nu \int_{-\infty}^{\infty} d\rho_{rs}(t)/\{(\mu - t)^2 + \nu^2\}$$

$$= 0(\nu) \qquad \text{as} \qquad \nu \to 0$$

by (4.1).

Also, by the Parseval formula and the relations (4.1) with $\nu, \nu' > 0$, we obtain

$$\| \psi_r(x, \mu + i\nu) - \psi_r(x, \mu + i\nu') \|_{0,\infty}$$

$$= \int_{-\infty}^{\infty} \left| \frac{1}{\mu + i\nu - t} - \frac{1}{\mu + i\nu' - t} \right|^2 d\rho_{rs}(t)$$

$$= 0(|\nu - \nu'|^2).$$

Hence making $\nu' \to 0$, it follows easily by Fatou's theorem that $\psi_r(x, \lambda)$ converges to (x, μ) as $\nu \to 0$ and that $\psi_r(x, \mu) \in L^2$ in x.

The transforms of $\psi_1(x, \mu)$ and $\psi_2(x, \mu)$ are, therefore, given by

$$T\psi_1(x, \mu) = \{1/(\mu - t), 0\}$$

and

$$T\psi_2(x, \mu) = \{0, 1/(\mu - t)\}.$$

By similar arguments $G(x, y, \lambda)$ converges to $G(x, y, \mu)$ as $\nu \to 0$ and $G_r(x, y, \mu) \in L^2$ in x (or y) $(r = 1, 2)$. The transforms of $G_1(x, y, \mu)$ and $G_2(x, y, \mu)$ are given by

$$TG_1(x, y, \mu) = U(y, t)/(\mu - t)$$

and

$$TG_2(x, y, \mu) = V(y, t)/(\mu - t).$$

Finally, let

$$\Phi(x, \mu, f) = \int_0^{\infty} G^T(y, x, \mu) f(y)\, dy = \lim_{\nu \to 0} \int_0^{\infty} G^T(y, x, \lambda) f(y)\, dy.$$

Then by (3.5) with $f = g$ and $\lambda = \lambda_1 = \bar{\lambda}_2$, it follows that

$$\Phi(x, \mu, f) \in L^2.$$

Hence, the theorem.

5. Transform theorems

Theorem (5.1): Let $F(t)$, $tF(t) \in \mathcal{L}^2$ and let

$$\mathcal{J}F(t) = f(x), \mathcal{J}(tF(t)) = h(x) \in L^2.$$

Let

$$\begin{aligned} g(x) &= \sum_{r=1}^{2} \int_{-\infty}^{\infty} \phi_r(0 \mid x, t)(F(t), d\rho_r(t)) \\ &= \{\langle U(x, t), F(t). d\rho(t)\rangle, \quad \langle V(x, t), F(t), d\rho(t)\rangle\} \end{aligned} \tag{5.1}$$

and

$$M(x) = \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}.$$

Then $g'(x)$ is absolutely continuous over any finite internal $[0, b]$ with $b > 0$; and $g(x) = f(x)$ almost everywhere $(x > 0)$. Also, $g(x)$ satisfies the boundary conditions of our boundary value problem at $x = 0$, and

$$g''(x) = M(x)\,g(x) - h(x). \tag{5.2}$$

PROOF : We have

$$\begin{aligned} \langle |U(x,t)|, |F(t)|, d\rho\rangle &= \langle |U(x,t)/(\lambda - t)|, |(\lambda - t)F(t)|, d\rho\rangle \\ &\le [\| |U(x,t)/(\lambda - t)|, d\rho \| \, \| |(\lambda - t)F(t)|, d\rho \|]^{\frac12} \\ &\le C(\lambda)\,[\operatorname{Im} G_{11}(x, x, \lambda)]^{\frac12} \end{aligned} \tag{5.3}$$

by (3.3), (3.4) and (3.5) with $f = g$ and $\lambda = \lambda_1 = \bar{\lambda}_2$, $G(x, y, \lambda)$ being continuous. Similarly

$$\langle |V(x,t)|, |F(t)|, d\rho\rangle \le C(\lambda)\,[\operatorname{Im} G_{22}(x, x, \lambda)]^{\frac12}. \tag{5.4}$$

Hence $g(x)$ is defined for all $x > 0$. Also

$$\begin{aligned} h(y) &= \sum_{r=1}^{2} \int_{-\infty}^{\infty} \phi_r(0\,|\,y, t)\,(tF(t), d\rho_r(t)) \\ &= \{\langle U(y,t), tF(t), d\rho(t)\rangle, \ \langle V(y,t), tF(t), d\rho(t)\rangle\} \end{aligned}$$

and for any $c > 0$, we obtain

$$\int_c^x (x - y)\,h(y)\,dy = \sum_{r=1}^{2} \int_{-\infty}^{\infty} (F(t), d\rho_r(t)) \int_c^x (x - y)\,t\phi_r(0\,|\,y, t)\,dy, \tag{5.5}$$

on changing the order of integration which is justified as follows:

$\phi_r(0\,|\,y, t)$ $(r = 1, 2)$ are continuous functions of y and t, and

$$|\langle tF(t), \int_c^x U(y,t)\,dy, d\rho\rangle| \le [\, \| tF(t), d\rho \| \, \| \int_c^{\xi} U(y,t)\,dy, d\rho \| \,]^{\frac12}$$

$$\le k\,[\int_c^{\xi} dy]^{\frac12} < \infty$$

for, defining $f(x)$ as in § 2 (i) and making use of the Parseval formula (3.4) of [1], we obtain

$$\| \int_c^{\xi} U(x,t)\,dx, d\rho \| = \| f(x) \|_{c,\xi} = \int_c^{\xi} dx.$$

Similarly, defining $f(x)$ as in § 2 (ii), it follows that

$$|\langle tF(t), \int_c^x V(y,t)\,dy, d\rho\rangle| < k\,[\int_c^{\xi} dx]^{\frac{1}{2}} < \infty.$$

From (5.5)

$$\int_c^x (x-y)\,h(y)\,dy$$

$$= \sum_{r=1}^{2} \int_{-\infty}^{\infty} (F(t), d\rho_r(t)) \int_c^x (x-y)\,L\phi_r(0\,|\,y,t)\,dy$$

$$= \sum_{r=1}^{2} \int_{-\infty}^{\infty} (F(t), d\rho_r(t)) \int_c^x (x-y)\;M(y)\,\phi_r(0\,|\,y,t)\,dy$$

$$- \sum_{r=1}^{2} \int_{-\infty}^{\infty} (F(t), d\rho_r(t)) \int_c^x (x-y)\,\phi_r''(0\,|\,y,t)\,dy$$

$$= \sum_{r=1}^{2} [\int_c^x (x-y)\,M(y)\,dy \int_{-\infty}^{\infty} \phi_r(0\,|\,y,t)\,(F(t), d\rho_r(t))$$

$$- \int_{-\infty}^{\infty} (F(t), d\rho_r(t))\,([(x-y)\,\phi_r'(0\,|\,y,t)]_c^x + [\phi_r(0\,|\,y,t)]_c^x)] \tag{5.6}$$

on changing the order of integration in the first integral and intergrating by parts twice the second. To justify the change in the order of integration we note that the integral involved is, by (5.3) and (5.4), dominated by

$$C(\lambda) \int_c^x (x-y) \begin{pmatrix} |p(y)| & |r(y)| \\ |r(y)| & |q(y)| \end{pmatrix} \begin{pmatrix} [\operatorname{Im} G_{11}(y,y,\lambda)]^{\frac{1}{2}} \\ [\operatorname{Im} G_{22}(y,y,\lambda)]^{\frac{1}{2}} \end{pmatrix} dy$$

which is finite, $G_{rs}\,(x,y,\lambda)$ being continuous.

Finally from (5.6), we obtain

$$\int_c^x (x-y)\,h(y)\,dy$$

$$= \sum_{r=1}^{2} [\int_c^x (x-y)\,M(y)\,dy \int_{-\infty}^{\infty} \phi_r(0\,|\,y,t)\,(F(t), d\rho_r(t))$$

$$+ \int_{-\infty}^{\infty} (x-c)\,\phi_r'(0\,|\,c,t)\,(F(t), d\rho_r(t)) - \int_{-\infty}^{\infty} \phi_r(0\,|\,x,t)\,(F(t), d\rho_r(t))$$

$$+ \int_{-\infty}^{\infty} \phi_r(0\,|\,c,t)\,(F(t), d\rho_r(t))]$$

$$= \int_c^x (x-y)\,M(y)\,g(y)\,dy + \sum_{r=1}^{2} [\int_{-\infty}^{\infty} (x-c)\,\phi_r'(0\,|\,c,t)\,(F(t), d\rho_r(t))$$

$$+ \int_{-\infty}^{\infty} \phi_r(0\,|\,c,t)\,(F(t), d\rho_r(t))] - g(x).$$

Hence

$$g(x) = \int_c^x (x - y)\,(M(y)\,g(y) - h(y))\,dy + \sum_{r=1}^{2} \Big[(x - c) \int_{-\infty}^{\infty} \phi_r'(0 \mid c, t) \times (F(t), d\rho_r(t)) + \int_{-\infty}^{\infty} \phi_r(0 \mid c, t)\,(F(t), d\rho_r(t))\Big], \tag{5.7}$$

The results stated in the theorem are now easy consequences of the integral equation (5.7)

Theorem (5.2): For given y and non-real λ

$$T \frac{\partial}{\partial y} G_1(x, y, \lambda) = U'(y, t)/(\lambda - t)\,; \quad T \frac{\partial}{\partial y} G_2(x, y, \lambda) = V'(y, t)/(\lambda - t).$$

PROOF : For any $f(x) \in L^2$, we get from § 2 (v)

$$\Phi(y, \lambda, f) = \sum_{r=1}^{2} \int_{-\infty}^{\infty} \phi_r(0 \mid y, t)\,(F(t)/(\lambda - t), d\rho_r(t)).$$

Also $F(t) \in \mathcal{L}^2$ and $F(t)/(\lambda - t)$ satisfies the conditions of Theorem (5.1). Hence

$$\Phi'(y, \lambda, f) = \sum_{r=1}^{2} \int_{-\infty}^{\infty} (\phi_r'(0 \mid y, t)/(\lambda - t))\,(F(t), d\rho_r(t)) \tag{5.8}$$

which implies that for every $F(t) \in \mathcal{L}^2$

$$\langle | U'(y, t)/(\lambda - t) |, | F(t) |, d\rho(t) \rangle < \infty$$

and

$$\langle | V_2'(y, t)/(\lambda - t) | \,.\, | F(t) |, d\rho(t) \rangle < \infty.$$

Hence

$$U'(y, t)/(\lambda - t) \in \mathcal{L}^2$$

and

$$V'(y, t)/(\lambda - t) \in \mathcal{L}^2.$$

Also,

$$\frac{\partial}{\partial y} G(x, y, \lambda) = \begin{pmatrix} \psi_{*1}(x, \lambda)\,U'(y, \lambda) & \psi_{*1}(x, \lambda)\,V'(y, \lambda) \\ \psi_{*2}(x, \lambda)\,U'(y, \lambda) & \psi_{*2}(x, \lambda)\,V'(y, \lambda) \end{pmatrix} \quad (y < x) \tag{5.9}$$

$$= \begin{pmatrix} U^T(x, \lambda)\,\psi_{*1}'^T(y, \lambda) & U^T(x, \lambda)\,\psi_{*2}'^T(y, \lambda) \\ V^T(x, \lambda)\,\psi_{*1}'^T(y, \lambda) & V^T(x, \lambda)\,\psi_{*2}'^T(y, \lambda) \end{pmatrix} \quad (y > x) \tag{5.10}$$

from the definition of the Green's matrix.

Hence $\frac{\partial}{\partial y} G(x, y, \lambda) \in L^2$ in x for fixed y. Now

$$\Phi(y, \lambda, f) = \int_0^\infty G^T(x, y, \lambda) f(x)$$

$$= \sum_{r=1}^{2} [\psi_r(y, \lambda) \int_0^y (\phi_r(0 \mid x, \lambda), f(x))\, dx$$

$$+ \phi_r(0 \mid y, \lambda) \int_y^\infty (\psi_r(x, \lambda), f(x))\, dx].$$

Therefore

$$\Phi'(y, \lambda, f) = \sum_{r=1}^{2} [\psi_r'(y, \lambda) \int_0^y (\phi_r(0 \mid x, \lambda), f(x))\, dx$$

$$+ \phi_r'(0 \mid y, \lambda) \int_y^\infty (\psi_r(x, \lambda), f(x))\, dx)]$$

$$= \int_0^\infty G_y^T(x, y, \lambda) f(x)\, dx. \tag{5.11}$$

Let

$$\mathcal{G}(x, y) = \begin{pmatrix} \mathcal{G}_{11}(x, y) & \mathcal{G}_{21}(x, y) \\ \mathcal{G}_{12}(x, y) & \mathcal{G}_{22}(x, y) \end{pmatrix}$$

where $\mathcal{G}_1(x, y) = \{\mathcal{G}_{11}(x, y), \mathcal{G}_{12}(x, y)\}$ and $\mathcal{G}_2(x, y) = \{\mathcal{G}_{21}(x, y), \mathcal{G}_{22}(x, y)\}$ denote the reverse transforms of $U'(y, t)/(\lambda - t)$ and $V'(y, t)/(\lambda - t)$ respectively. Then the Parseval formula for these transforms yields

$$\int_0^\infty \mathcal{G}^T(x, y) f(x)\, dx$$

$$= \{\langle \mathcal{G}_1(x, y), f(x)\rangle_{0, \infty}, \ \langle \mathcal{G}_2(x, y), f(x)\rangle_{0, \infty}\}$$

$$= \{\langle U'(y, t)/(\lambda - t), F(t), d\rho(t)\rangle, \ \langle V'(y, t)/(\lambda - t) F(t), d\rho(t)\rangle\}.$$

Hence

$$\int_0^\infty \mathcal{G}^T(x, y) f(x)\, dx = \sum_{r=1}^{2} \int_{-\infty}^{\infty} (\phi_r'(0 \mid y, t)/(\lambda - t))\, (F(t), d\rho_r(t))$$
$$= \Phi'(y, \lambda, f) \tag{5.12}$$

by (5.8). Thus from (5.11) and (5.12) we obtain

$$\int_0^\infty \left(\mathcal{G}(x, y) - \frac{\partial}{\partial y} G(x, y, \lambda)\right)^T f(x) = 0$$

for every $f(x) \in L^2$, which proves the theorem.

Now, it follows that the relations (3.3) and (3.4) may be differentiated partially with respect to y and ξ so as to yield

$$\int_0^\infty G_Y^T(x, y, \lambda_1)\, G_\xi(x, \xi, \lambda_2)\, dx$$

$$= (G_{y\xi}(y, \xi, \lambda_1) - G_{y\xi}(y, \xi, \lambda_2))/(\lambda_1 - \lambda_2)$$

$$= \begin{pmatrix} \langle U'(y,t)/(\lambda_1 - t), U'(\xi,t)/(\lambda_2 - t), d\rho(t)\rangle \\ \langle U'(y,t)/(\lambda_1 - t), V'(\xi,t)/(\lambda_2 - t), d\rho(t)\rangle \\ \langle V'(y,t)/(\lambda_1 - t), U'(\xi,t)/(\lambda_2 - t), d\rho(t)\rangle \\ \langle V'(y,t)/(\lambda_1 - t), V'(\xi,t)/(\lambda_2 - t), d\rho(t)\rangle \end{pmatrix}.$$

$(y \neq \xi)$.

Finally, combining the transforms (2.10) with Theorem (5.1), we obtain the following:

Theorem (5.3): If $Tf(x) = F(t)$ and $TLf(x) = tF(t)$, then a necessary and sufficient condition that $F(t)$, $tF(t) \in \mathcal{L}^2$ is that $f(x)$ and $Lf(x) \in L^2$.

This theorem is a generalisation of theorem 68 § 3.14 of Titchmarsh[7] on Fourier transforms.

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