Forced torsional vibrations of inhomogeneous poroelastic cone

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Abstract

Using Biot's theory, the problem of forced torsional vibrations of an inhomogeneous poroelastic cone with spherical caps twisted by periodic terminal couples is studied. The inhomogeneity is considered only in shear modulus. The effect of imhomogeneity on the stress is displayed graphically.

Key words : Inhomogeneous, poroelastic, Darcy's law, torsional vibrations.

1. In oduction

The governing equations of a poroelastic medium taking solid-fluid aggregate was given by Biot[1,2] under static loads. Based on this, a number of problems were solved by him and others. Later the theory was developed for the case of dynamic loads[3]. In this theory, it is assumed that the mechanical behaviour of solid portion is governed by Hooke's law and flow of fluid produced by deformation is by Darcy's law. A more complete summary was given by Paria[4]. Recently Nowinski and Davis[5] have applied this theory to solve the problem of longitudinal waves in a cylindrical bone element.

In recent years there has been considerable interest given to elasticity of inhomogeneous bodies because of the materials whose elastic coefficients are not same at all points within the body but vary from point to point. This type of work has been previously considered by Awojobi[6,7] in the case of torsional vibrations of a rigid circular body and for planestrain axially symmetric problems of an inhomogeneous elastic half space by Awojobi and Gibson[8]. In this paper, the problem of forced torsional vibrations of an inhomogeneous poroelastic cone with spherical caps applied by periodic terminal couples is solved. The inhomogeneity in shear modulus is assumed to follow some power of the distance from the vertex of the cone. On neglecting fluid effects, results of an elastic medium are obtained as a particular case considered by Mukherjee[9]. The effect of inhomogeneity on the shear stress is presented graphically for three different materials discussed by Biot[3], Nowinski and Davis[5]. The material proposed by Nowinski and Davis is a bone element. These results are compared with a body of homogeneous material.

2. Solution of the problem

The torsional vibrations of an inhomogeneous poroelastic cone with spherical caps twisted by periodic terminal couples is considered. We use spherical polar coordinate system (r, θ, ϕ) with the vertex of the cone as origin of the coordinate system. For torsional vibrations the non-zero displacement components are

$$u_\phi = u(r) \text{ Sin } \theta \, e^{ipt}, \quad U_\phi = U(r) \text{ Sin } \theta \, e^{ipt} \tag{1}$$

where u_ϕ, U_ϕ are the displacement components of solid and liquid media respectively. The non-zero stress equation of motion of a poroelastic body in absence of body forces[3] in this case will be

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r}(3\sigma_{r\phi} + 2\sigma_{\theta\phi}\cot\theta)$$

$$= \frac{\partial^2}{\partial t^2}(\rho_{11}u_\phi + \rho_{12}U_\phi) + b\frac{\partial}{\partial t}(u_\phi - U_\phi)$$

$$\text{grad } (Qe + R\varepsilon) = \frac{\partial^2}{\partial t^2}(\rho_{12}u_\phi + \rho_{22}U_\phi) - b\frac{\partial}{\partial t}(u_\phi - U_\phi) \tag{2}$$

Where σ_{ij} is stress tensor 'ρ's are the mass coefficients, b dissipative coefficient and Q and R elastic constants in Biot's theory.

Because the considered vibrations are shear waves, so dilatations e and ε are zero making excess pore-pressure s to be identically zero. The only non-zero stress component $\sigma_{r\phi}$ in terms of displacement is

$$\sigma_{r\phi} = Nr\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right) \tag{3}$$

where N is shear modulus assumed to be a function of r. It is convenient to introduce a dimensionless parameter x given by

$$x = r/a \tag{4}$$

where a is some reference distance and throughout the work the inhomogeneity considered in shear modulus follows

$$N = N_0 x^n \tag{5}$$

where N_0 is the value of N in homogeneous case and exponent n is a rational number. Substituting Eqns. (1), (3) with (4) and (5) into Eq. (2) gives

$$\frac{d^2u}{dx^2} + \frac{n+2}{x}\frac{du}{dx} - \left(\frac{n+2}{x^2} - \frac{a^2p^2}{x^n} \cdot \frac{\tau_{11}\tau_{22} - \tau_{12}^2}{N_0\tau_{22}}\right)u = 0 \tag{6}$$

where

$$\tau_{11} = \rho_{11} - ib/p, \quad \tau_{12} = \rho_{12} + ib/p, \quad \tau_{22} = \rho_{22} - ib/p \tag{7}$$

When there is no relative motion between solid and liquid media, the shear velocity c_0 is

$$c_0^2 = N_0/\rho \tag{8}$$

Introducing the following non-dimensional variables

$$\sigma_{11} = \rho_{11}/\rho, \quad \sigma_{12} = \rho_{12}/\rho, \quad \sigma_{22} = \rho_{22}/\rho, \quad b_1 = ab/\rho c_0, \quad f = ap_1/c_0$$

with

$$\rho = \rho_{11} + 2\rho_{12} + \rho_{22}, \quad p_1 = -ip$$

into eqn. (6), it will be

$$\frac{d^2 u}{dx^2} + \frac{n+2}{x}\frac{du}{dx} - \left(\frac{n+2}{x^2} - \frac{m^2}{x^n}\right) u = 0 \tag{9}$$

where

$$m^2 = f^2 \left\{\frac{\sigma_{22}(\sigma_{11}\sigma_{22} - \sigma_{12}^2) f^2 - b_1(\sigma_{12} + \sigma_{22})^2 f - b_1^2}{b_1^2 - \sigma_{22}^2 f^2}\right\} \tag{10}$$

Substituting

$$z = \frac{2}{2-n} m x^{(2-n)/2} \quad (n \neq 2)$$

$$u = y(z)\, x^{-(n+1)/2} \tag{11}$$

Eqn. (9) becomes.

$$\frac{d^2 y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) y = 0 \tag{12}$$

where

$$\nu = \frac{3+n}{2-n} \tag{13}$$

The solution of Eqn. (12) is

$$y = c_1 J_\nu(z) + c_2 Y_\nu(z)$$

where c_1 and c_2 are constants; $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions of first and second kind of order ν with argument z. Thus we have

$$u_\phi = \{c_1 J_\nu(z) + c_2 Y_\nu(z)\} x^{-(n+1)/2} \operatorname{Sin}\theta\, e^{ipt} \tag{14}$$

By virtue of eqns. (14) and (3),

$$\sigma_{r\phi} = -\frac{N_0 m}{a} x^{-\frac{1}{2}} [c_1 J_{\nu+1}(z) + c_2 Y_{\nu+1}(z)] \sin\theta\, e^{ipt} \tag{15}$$

The solutions are to be re-examined for $n = 2$. Putting $n = 2$, eqn. (9) becomes

$$\frac{d^2 u}{dx^2} + \frac{4}{x}\frac{du}{dx} - \frac{1}{x^2}(4 - m^2)\, u = 0 \tag{16}$$

The solution of eqn. (16) can be written as

$$u = c_3 x^{(k-3/2)} \times c_4\, x^{-(k+3/2)} \tag{17}$$

where

$$k^2 = \left(\frac{5}{2}\right)^2 - m^2 \tag{18}$$

and c_3 and c_4 are constants. Thus,

$$u_\phi = [c_3 x^{(k-3/2)} + c_4 x^{-(k+3/2)}] \sin\theta\, e^{ipt} \tag{19}$$

The stress component $\sigma_{r\phi}$, in this case will be

$$\sigma_{r\phi} = \frac{N_0}{a}\left[\left(k - \frac{5}{2}\right) c_3 x^{(k-1/2)} - \left(k + \frac{5}{2}\right) c_4 x^{-(k+1/2)}\right] \sin\theta\, e^{ipt} \tag{20}$$

3. Boundary conditions

Consider the cone with semi-vertical angle α with two spherical caps bounded by radii $x = x_1$ and $x = x_2$ and let it be acted by equal and opposite twisting couples. The boundary conditions are

$$\text{At } x = x_1, \quad \int_{\theta=0}^{\alpha} \int_{\phi=0}^{2\pi} \sigma_{r\phi}\, a^3 x^3 \sin^2\theta\, d\theta\, d\phi = Me^{ipt} \tag{21}$$

$$\text{At } x = x_2, \quad \int_{\theta=0}^{\alpha} \int_{\phi=0}^{2\pi} \sigma_{r\phi}\, a^3 x^3 \sin^2\theta\, d\theta\, d\phi = -Me^{ipt}$$

The constants c_1, c_2, c_3, c_4 are evaluated from boundary conditions (21) and eqns. (15) and (20).

Case (i) $n \neq 2$.

From eqns. (15) and (21), after necessary simplifications, the constants c_1 and c_2 are given by

$$c_1 = \frac{I_1}{I_3} MK$$

and

$$c_2 = \frac{I_2}{I_3} MK \tag{22}$$

where

$$I_1 = x_1^{-5/2}\, Y_{\nu+1}\,(m_1 x_2^{(2-n)/2}) + x_2^{-5/2}\, Y_{\nu+1}\,(m_1 x_1^{(2-n)/2})$$

$$I_2 = x_1^{-5/2}\, J_{\nu+1}\,(m_1 x_2^{(2-n)/2}) + x_2^{-5/2}\, J_{\nu+1}\,(m_1 x_1^{(2-n)/2}) \tag{23}$$

$$I_3 = J_{\nu+1}\,(m_1 x_1^{(2-n)/2})\; Y_{\nu+1}\,(m_1 x_2^{2-n/2}) - J_{\nu+1}\,(m_1 x_2^{(2-n)/2})\; Y_{\nu+1}\,(m_1 x_1^{(2-n)/2})$$

and

$$K = \frac{6}{\pi N_0 m a^2\,(9 \cos \alpha - \cos 3\alpha - 8)} \tag{24}$$

$$m_1 = \frac{2}{2-n}\, m$$

Case (ii) $n = 2$.

Proceeding on similar lines, the constants c_3 and c_4 in this case will be

$$c_3 = \frac{Mk_1}{k - 5/2} \cdot \frac{I_4}{I_6}$$

$$c_4 = \frac{Mk_1}{k + 5/2} \cdot \frac{I_5}{I_6} \tag{25}$$

where

$$I_4 = x_1^{-(k+1/2)} + x_2^{-(k-5/2)}$$

$$I_5 = x_1^{k-1/2} + x_2^{k+5/2} \tag{26}$$

$$I_6 = x_1^{-(k+1/2)}\, x_2^{k+5/2} - x_1^{k-1/2}\, x_2^{-(k-5/2)}$$

and

$$k_1 = mk \tag{27}$$

4. Particular cases

Case I :

Homogeneous case : Putting $n = 0$ in eqns. (13), (14), (15), (22), (23) and (24) the results of a homogeneous poroelastic cone follows at once, *Case II* :

Classical theory : On neglecting fluid effects, that is $b_1 \to o$, σ_{12}, $\sigma_{22} \to o$ then from eqn. (10),

$$m^2 \to -f^2\,\sigma_{11}$$

or

$$m^2 \to \frac{a^2 p^2}{c_0^2} \cdot \frac{\rho_{11}}{\rho}$$

and $\rho_{11} = \rho$, in classical theory when fluid effects are neglected, *i.e.*,

$$m^2 \to \frac{a^2 p^2}{c_0^2}.$$

Using this in all the calculations, the results of Mukherjee[9] are obtained.

5. Numerical results and discussions

Due to presence of dissipative nature of the medium, the vibrations are attenuated. For simplicity, to have an idea of effect of inhomogeneity on the stress component $\sigma_{r\phi}$, it is calculated for different materials discussed by Biot[3], Nowinski and Davis[5] for $b_1 = 0$. These are

	σ_{11}	σ_{12}	σ_{22}
(i)	0·50	0·00	0·50
(ii)	0·65	−0·15	0·65
(iii)	0·92	0·00	0·08

The third material corresponds to bone element. In the first two materials the masses of solid and liquid media are equal whereas in the first case mass coupling effect is absent and in the third one these vary considerably.

The shear stress $\sigma_{r\phi}$ is calculated at a given time $t = T$; $x_1 = 1$, $x_2 = 2$, $b_1 = 0$, $f = 1$, $\alpha = 45°$ and $\theta = 30°$ for different cases of homogeneity, that is for (1) $n = o$, (2) $n = 1$, (3) $n = 2$ taking different values of x in $1 \leqslant x \leqslant 2$ · $n = 0$ gives homogeneous case. The numerical values are exhibited graphically in Fig. 1 for the first material. It is observed that the value of stress component for inhomogeneous case ($n = 1, 2$) are less than that of homogeneous case ($n = 0$) in $1 \leqslant x \leqslant 1{\cdot}8$ and is greater in $1{\cdot}8 < x \leqslant 2{\cdot}0$. The values of stress component are calculated for other two materials and from it no considerable difference is noted with the case presented in Fig. 1.

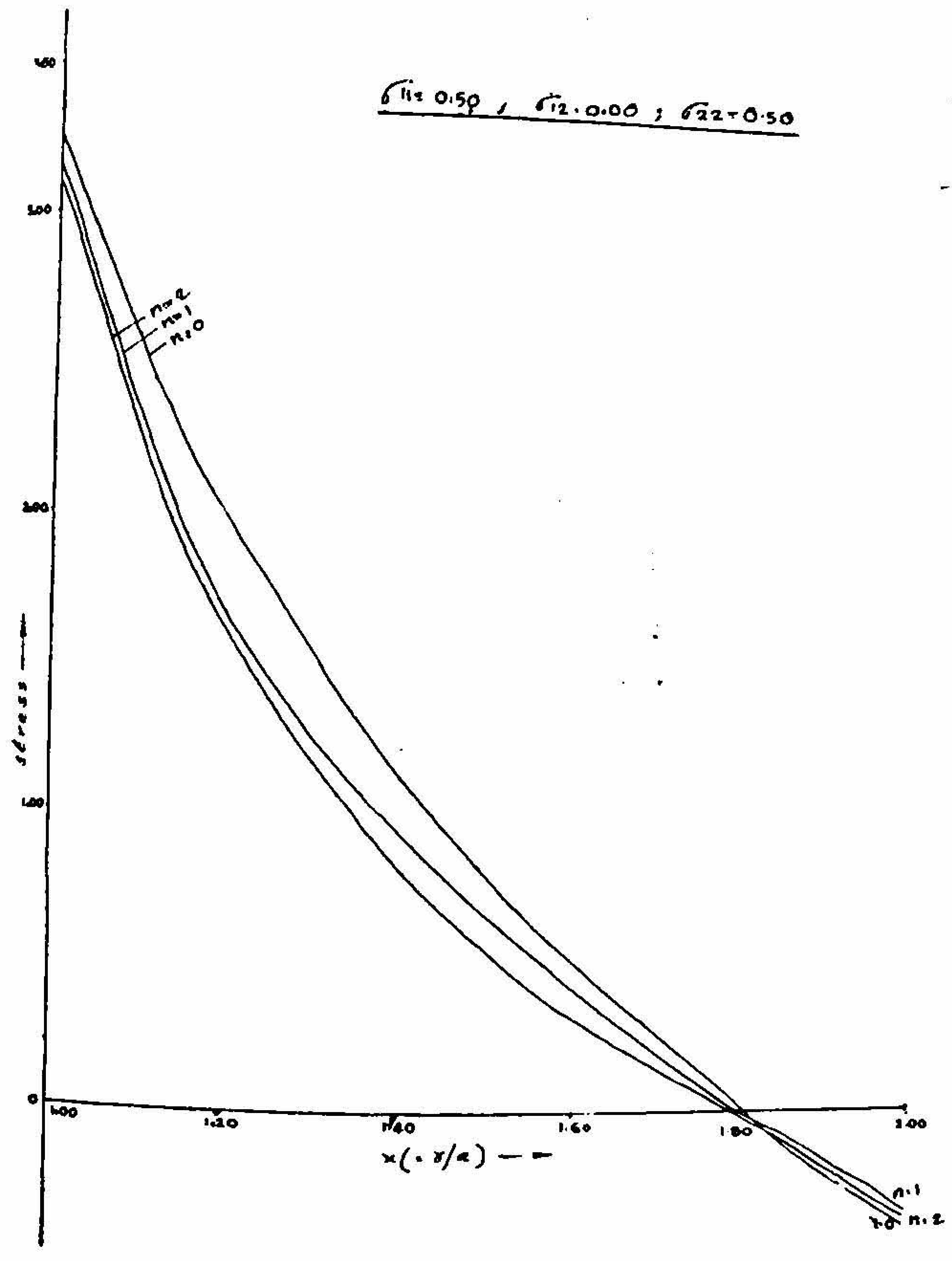

FIG. 1. Shear stress as a function of radius in different cases.

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