Sector problem with arbitrary loads on the edges

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Received on October 4, 1977; Revised on April 19, 1978

Abstract

The sector problem in plane-elastostatics with arbitrary loads on the edges has been studied in this paper. The loads on each of the edges separately may not be self-equilibrating but all the loads together should keep the sector in equilibrium. By superimposition of suitable self equilibrating stress systems, the problem has been reduced to that of self-equilibrating loads on the circular arc of the sector. An auxiliary problem of sectorial inclusion in a sectorial annulus has also been considered.

Key words : Two-dimensional elasticity, sector, inclusion.

1. Introduction

The sector problem in two-dimensional elasticity has been studied by several authors by taking various types of boundary conditions. A moderate bibliography of sector and related problems with comments on the methods of solutions can be found in refs. 1 and 2. We shall briefly mention the previous relevant works. In ref. 3, Horway and Hanson have studied the sector problem with loads on the edges and an approximate solution has been given by variational methods using self-equilibrating orthogonal polynomials. By expressing the boundary tractions in terms of normalized clamped beam functions, Gopalacharyulu[1], has given the analytical solution of the sector problem with zero loads on the radial edges and self-equilibrating system of loads on the circular edge. The final solution in ref. 1 depends on the solution of two coupled infinite systems of linear algebraic equations in two sets of infinite unknowns (two for symmetric and two for anti-symmetric loadings). Using the method of bi-orthogonality developed by Jhonson and Little[4] for solving a semi-infinite strip problem, Rao, Kale and Shimpi[2] have solved the sector problem with self-equilibrating loads on the circular edge and zero loads on the radial edges. The final solution in ref. 2 depends on the solution of only one infinite system of linear algebraic equations in complex constants (one system each for symmetric and anti-symmetric case). Recently some more papers have appeared[5, 6, 7] using the method of bi-orthogonality developed in ref. 4.

In many physical problems of interest the sector problem occurs as an auxiliary problem with loads on all the three edges. As loads in refs. 1 and 2 are restricted only to the circular edge of the sector, they are of limited use but are important from the solution point of view.

The main theme of the present paper is to superimpose suitable self-equilibrating stress systems which reduce the problem to that of self-equilibrating loads on the circular edge of the sector for which the solution is known[2].

2. Analysis

Consider a sector in a state of generalized plane stress with semi-vertex angle α. The origin is taken at the apex and the X-axis along the axis of angular symmetry. The circular arc is given by $r = 1$ where (r, θ) are the polar coordinates in the plane of the sector. The boundary conditions have been taken as follows:

On the circular edge of the sector

$$\left.\begin{aligned} \tau_{rr} &= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \\ \tau_{r\theta} &= c_0 + \sum_{n=1}^{\infty} (c_n \cos n\theta + d_n \sin n\theta) \end{aligned}\right\} r = 1 \tag{1}$$

where

$$a_1 = d_1 \quad \text{and} \quad c_1 = -b_1. \tag{2}$$

This restriction in (2) on the coefficients will be discussed later.

On the radial edges of the sector

$$\tau_{\theta\theta} = f_1(r), \quad \tau_{r\theta} = f_3(r); \quad \theta = \alpha, \quad 0 \leqslant r < 1 \tag{3}$$

$$\tau_{\theta\theta} = f_2(r), \quad \tau_{r\theta} = f_4(r); \quad \theta = -\alpha, \quad 0 \leqslant r < 1. \tag{4}$$

If the sector is in equilibrium under the loads (1), (3) and (4), then the loads must satisfy certain conditions which can be easily obtained[8].

The solution satisfying the boundary conditions (1)–(4) will be found by the superimposition of three self-equilibrating stress systems.

First stress system

The first stress system is obtained to suit the boundary condition (1). We consider the following analytic functions $\Phi(z)$ and $\psi(z)$ (in the notation of[8])

$$\Phi(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{(a_n + d_n) + i(c_n - b_n)\} \frac{z^n}{2}$$

$$+\frac{c_0}{2\pi}\left\{\log\frac{(e^{i\alpha}-z)}{(e^{-i\alpha}-z)}-\log\frac{(ie^{i\alpha}-z)}{(ie^{-i\alpha}-z)}+\log\frac{(e^{i\alpha}+z)}{(e^{-i\alpha}+z)}\right.$$

$$\left.-\log\frac{(ie^{i\alpha}+z)}{(ie^{-i\alpha}+z)}\right\}+\frac{2c_0}{\pi}z^2\left\{\frac{e^{-2i\alpha}}{e^{-4i\alpha}-z^2}-\frac{e^{2i\alpha}}{e^{4i\alpha}-z^2}\right\} \tag{5}$$

$$\psi(z)=-\sum_{n=2}^{\infty}\frac{\{(a_n-d_n)-i(b_n+c_n)\}}{2(n-1)}z^{n-1}$$

$$-\sum_{n=1}^{\infty}\{(a_n+d_n)+i(c_n-b_n)\}\frac{z^{n-1}}{2}-\frac{c_0}{\pi z}\left\{\log\frac{(e^{-i\alpha}-z)}{(e^{-i\alpha}-z)}\right.$$

$$\left.-\log\frac{(ie^{i\alpha}-z)}{(ie^{-i\alpha}-z)}+\log\frac{(e^{i\alpha}+z)}{(e^{-i\alpha}+z)}-\log\frac{(ie^{i\alpha}+z)}{(ie^{-i\alpha}+z)}\right\}$$

$$-\frac{2c_0z}{\pi}\left\{\frac{e^{-2i\alpha}}{e^{-4i\alpha}-z^4}-\frac{e^{2i\alpha}}{e^{4i\alpha}-z^4}\right\}. \tag{6}$$

The analytic functions $\Phi(z)$ and $\psi(z)$ can be obtained by considering loads equal to the stresses given by (5) and (6) on the boundary ($r=1$, $0\leqslant\theta<2\pi$) of a unit circular disc. Care should be taken to choose appropriate branches of logarithmic functions while calculating stresses (Fig. 1). This unit circular disc with such loads on the whole of the boundary will be in equilibrium only if condition (2) is satisfied.

Stresses can be calculated from $\Phi(z)$ and $\psi(z)$ by using formulas in ref 8. If appropriate branches of logarithmic functions as given below and shown in Fig. 1 are taken, then it can be verified that on the circular edge of the sector ($r=1$, $-\alpha<\theta<\alpha$), $\Phi(z)$ and $\psi(z)$ give the normal and shearing stresses as given in (1).

$$I_m\left\{\log\frac{(e^{i\alpha}-z)}{(e^{-i\alpha}-z)}\right\}=\theta_1$$

$$I_m\left\{\log\frac{(ie^{i\alpha}-z)}{(ie^{-i\alpha}-z)}\right\}=\theta_2$$

$$I_m\left\{\log\frac{(e^{i\alpha}+z)}{(e^{-i\alpha}+z)}\right\}=\theta_3$$

$$I_m\left\{\log\frac{(ie^{i\alpha}+z)}{(ie^{-i\alpha}+z)}\right\}=\theta_4. \tag{7}$$

I_m stands for the imaginary part of the complex quantity.

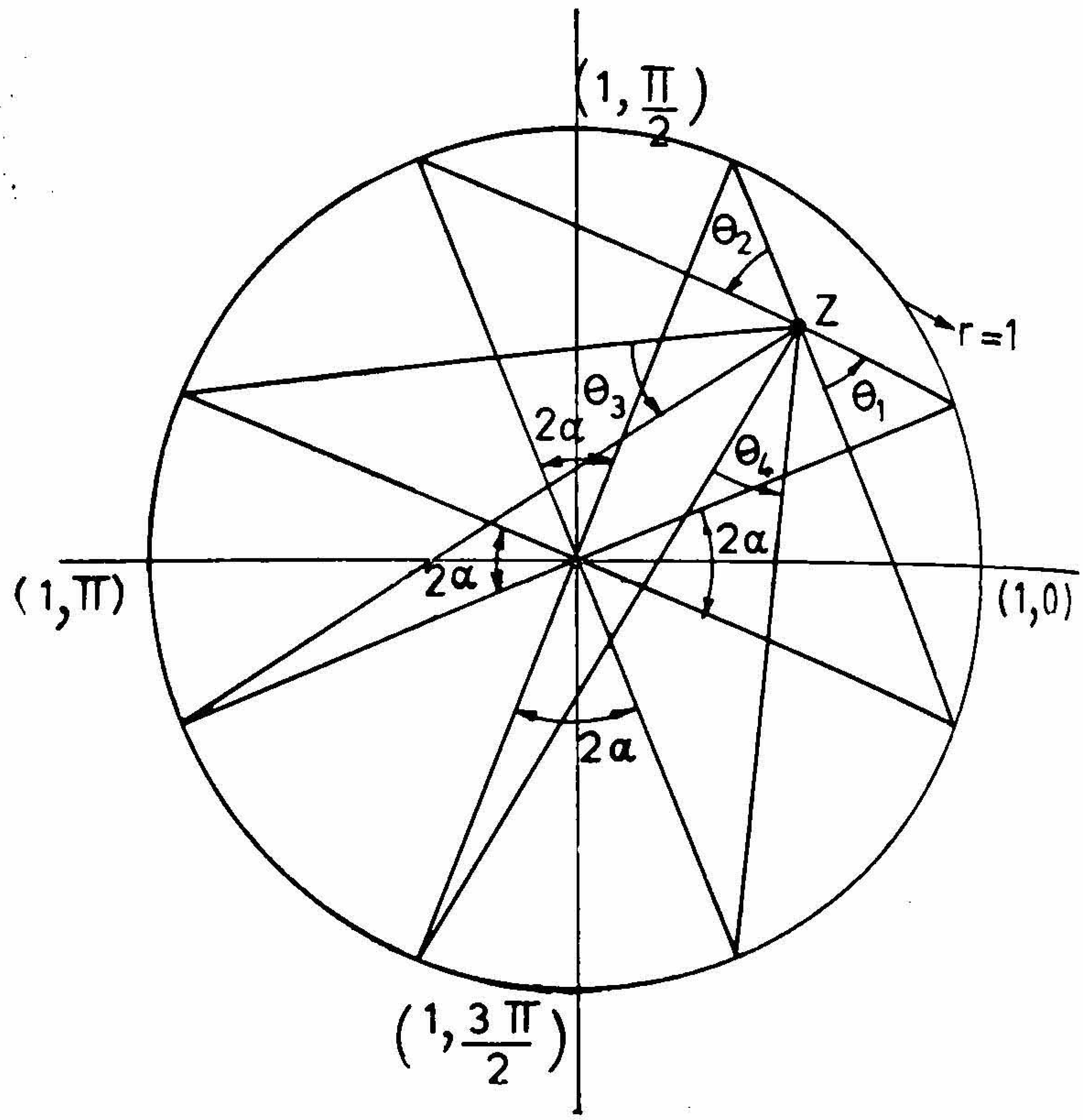

FIG. 1.

On the radial edges $\theta = \pm\,\alpha$, the following normal and shearing stresses are obtained from $\Phi(z)$ and $\psi(z)$.

$$2\,\{(\tau_{\theta\theta})_1\}_{\theta=\pm\alpha} = 2a_0 + \sum_{n=1}^{\infty}(2+n)\,r^n\{(a_n+d_n)\cos n\alpha$$

$$\mp (c_n - b_n)\sin n\alpha\} + \sum_{n=2}^{\infty}(1-n)\,r^{n-2}\{(a_n+d_n)\cos n\alpha$$

$$\mp (c_n - b_n)\sin n\alpha\} - \sum_{n=2}^{\infty} r^{n-2}\{(a_n - d_n)\cos n\alpha \pm (b_n + c_n)\sin n\alpha\}$$

$$+ \frac{2c_0}{\pi} \sum_{n=1}^{\infty} (\cos n\pi - 1)(1 - \cos 4n\alpha)\left(\frac{r^{2n-2}}{n} + \frac{r^{2n}}{n}\right)$$

$$+ \frac{4c_0}{\pi}(r^2 - 1) \sum_{n=0}^{\infty} \{(3 + 2n)\cos 4(2n+1)\alpha - 1\} r^{4n}$$

$$- \frac{8c_0}{\pi}(r^6 - r^4 + r^2 - 1) \sum_{n=0}^{\infty} (1 + n) r^{4n}$$

$$+ \frac{8c_0}{\pi}(r^6 - r^4) \sum_{n=0}^{\infty} (1 + n) r^{4n} \cos 4(2n + 3)\alpha. \qquad (8)$$

$$2\{(\tau_{r\theta})_1\}_{\theta=\pm\alpha} = \sum_{n=2}^{\infty} (1 - n)\, r^{n-2} \{(c_n - b_n)\cos n\alpha \pm (a_n + d_n)\sin n\alpha\}$$

$$+ \sum_{n=2}^{\infty} r^{n-2} \{(b_n + c_n)\cos n\alpha \pm (d_n - a_n)\sin n\alpha\}$$

$$+ \sum_{n=1}^{\infty} n r^n \{(c_n - b_n)\cos n\alpha \pm (a_n + d_n)\sin n\alpha\}$$

$$\pm \frac{8c_0(r^2 - 1)}{\pi} \sum_{n=0}^{\infty} (2 + n) r^{4n} \sin 4(2n + 1)\alpha$$

$$\pm \frac{8c_0}{\pi}(r^6 - r^4) \sum_{n=0}^{\infty} (1 + n) r^{4n} \sin 4(2n + 3)\alpha. \qquad (9)$$

It can be verified that the loads given by (1), (8) and (9) on the edges of the sector form a self-equilibrating system provided the condition (2) is satisfied. The subscript 1 has been added to the stresses as they correspond to the first stress system.

Second stress system

The second stress system is constructed to suit the following boundary conditions on the radial edges of the sector

on $\theta = \alpha, 0 \leqslant r < 1$

$$\tau_{\theta\theta} = f_1(r) - \{(\tau_{\theta\theta})_1\}_{\theta=\alpha}\ ; \qquad \tau_{r\theta} = f_3(r) - \{(\tau_{r\theta})_1\}_{\theta=\alpha} \qquad (10)$$

and on $\theta = -\alpha, \ 0 \leqslant r < 1$

$$\tau_{\theta\theta} = f_2(r) - \{(\tau_{\theta\theta})_1\}_{\theta=-\alpha}\ ; \qquad \tau_{r\theta} = f_4(r) - \{(\tau_{r\theta})_1\}_{\theta=-\alpha}, \qquad (11)$$

We extend the range of the variable r in (10) and (11) to $0 \leqslant r < \infty$ by taking

$$(\tau_{\theta\theta})_{\theta=\pm\alpha} = 0; \quad 1 \leqslant r < \infty \quad \text{and} \quad (\tau_{r\theta})_{\theta=\pm\alpha} = 0; \quad 1 \leqslant r < \infty \tag{12}$$

and thus consider an infinite wedge problem with the boundary conditions (10), (11) and (12) on the radial edges.

It may be noted that the loads given by (10) and (11) on the radial edges of the sector form a self-equilibrating system.

This wedge problem can be solved using Mellin transform method[9]. Stresses can be expressed in terms of a bi-harmonic function U as

$$\tau_{rr} = \frac{1}{r}\frac{U}{r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2}, \qquad \tau_{\theta\theta} = \frac{\partial^2 U}{\partial r^2}, \qquad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial U}{\partial\theta}\right). \tag{13}$$

Using Mellin transform, the solution of $\nabla^4 U = 0$ can be written as:

$$U = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \{A\sin(s-2)\theta + B\cos(s-2)\theta + C\sin s\theta + D\cos s\theta\}\, r^{-s+2}\, ds \tag{14}$$

where the constants A, B, C and D are to be determined with the help of four boundary conditions given in (10), (11) and (12). Substituting these constants in (14), U is known. Stresses may then be determined from (13).

Finally,

$$(\tau_{rr})_2 = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Big[\frac{P_1}{\Delta_1(s,\alpha)}\{(s-2)\sin s\alpha\sin(s-2)\theta - (s+2)\sin s\theta\sin(s-2)\alpha\} - \frac{P_2}{\Delta_1(s,\alpha)}\{s\cos s\alpha\sin(s-2)\theta - (s+2)\sin s\theta\cos(s-2)\alpha\} + \frac{P_3}{\Delta_2(s,\alpha)}\{(s-2)\cos s\alpha\cos(s-2)\theta - (s+2)\cos s\theta\cos(s-2)\alpha\} - \frac{P_4}{\Delta_2(s,\alpha)}\{s\sin s\alpha\cos(s-2)\theta - (s+2)\cos s\theta\sin(s-2)\alpha\}\Big] r^{-s}\, ds, \quad Re(s) > 1 \tag{15}$$

$$(\tau_{r\theta})_2 = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Big[\frac{P_1}{\Delta_1(s,\alpha)}\{(s-2)\sin s\alpha\cos(s-2)\theta$$

$$- s \cos s\theta \sin (s-2)\alpha\} - \frac{P_2 s}{\Delta_1(s,\alpha)} \{\cos s\alpha \cos (s-2)\theta$$

$$- \cos s\theta \cos (s-2)\alpha\} + \frac{P_3}{\Delta_2(s,\alpha)} \{-(s-2)\cos s\alpha \sin (s-2)\theta$$

$$+ s \sin s\theta \cos (s-2)\alpha\} + \frac{P_4 s}{\Delta_2(s,\alpha)} \{\sin s\alpha \sin (s-2)\theta$$

$$- \sin s\theta \sin (s-2)\alpha\} \Big] r^{-s}\, ds, \qquad Re(s) > 1, \tag{16}$$

$$(\tau_{\theta\theta})_2 = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Big[\frac{P_1}{\Delta_1(s,\alpha)} (s-2) \{\sin s\alpha \sin (s-2)\theta$$

$$- \sin (s-2)\alpha \sin s\theta\} - \frac{P_2}{\Delta_1(s,\alpha)} \{s \cos s\alpha \sin (s-2)\theta$$

$$- (s-2)\cos (s-2)\alpha \sin s\theta\} + \frac{P_3(s-2)}{\Delta_2(s,\alpha)} \{\cos s\alpha \cos (s-2)\theta$$

$$- \cos s\theta \cos (s-2)\alpha\} - \frac{P_4}{\Delta_2(s,\alpha)} \{s \sin s\alpha \cos (s-2)\theta$$

$$- (s-2)\cos s\theta \sin (s-2)\alpha\} \Big] r^{-s}\, ds, \quad Re(s) > 1, \tag{17}$$

here

$$\Delta_1(s,\alpha) = 2\{s \sin 2\alpha - 2 \sin s\alpha \cos (s-2)\alpha\}$$

$$\Delta_2(s,\alpha) = -2\{s \sin 2\alpha + 2 \cos s\alpha \sin (s-2)\alpha\}$$

$$P_1 = \int_0^1 r^{s-1}(f_3 + f_4)\, dr - \frac{1}{(s+1)}(c_1 - b_1)\cos\alpha$$

$$- \sum_{n=2}^{\infty} \frac{2(sc_n + nb_n)}{(s+n)(s+n-2)} \cos n\alpha,$$

$$P_2 = \int_0^1 r^{s-1}(f_1 - f_2)\, dr + \frac{3}{(1+s)}(c_1 - b_1)\sin\alpha$$

$$+ \sum_{n=2}^{\infty} \frac{2\{(2s+n-2)c_n - (s-2)b_n\}}{(s+n)(s+n-2)} \sin n\alpha,$$

$$P_3 = \int_0^1 r^{s-1}(f_4 - f_3)\,dr + \frac{1}{(s+1)}(a_1 + d_1)\sin\alpha$$

$$- \sum_{n=2}^{\infty} \frac{2(na_n - sd_n)}{(s+n)(s+n-2)} \cos n\alpha$$

$$+ \frac{8c_0(r^2-1)}{\pi} \sum_{n=0}^{\infty} (2n+n)\, r^{4n} \sin 4(2n+1)\alpha$$

$$+ \frac{8c_0}{\pi} r^4 (r^2 - 1) \sum_{n=0}^{\infty} (1+n)\, r^{4n} \sin 4(2n+3)\alpha,$$

$$P_4 = \int_0^1 (f_1 + f_2)\, r^{s-1}\,dr - \frac{2a_0}{s} - \frac{3}{(s+1)}(a_1 + d_1)\cos\alpha$$

$$- \sum_{n=2}^{\infty} \frac{2\{(s-2)a_n + (2s+n-2)d_n\}}{(s+n)(s+n-2)} \cos n\alpha$$

$$- \frac{8c_0}{\pi} r^4 (r^2 - 1) \sum_{n=0}^{\infty} (1+n)\, r^{4n} \cos 4(2n+3)\alpha$$

$$+ \frac{8c_0}{\pi} (r^6 - r^4 + r^2 - 1) \sum_{n=0}^{\infty} (1+n)\, r^{4n}$$

$$- \frac{4c_0}{\pi} (r^2 - 1) \sum_{n=0}^{\infty} \{(3+2n)\cos 4\alpha(2n+1) - 1\}\, r^{4n}.$$

It may be noted that stresses in (15)–(17) can be easily written as the sum of sym[illegible] tric and anti-symmetric stresses.

The integrals in (15)–(17) are not affected by the particular choice of γ, as long [illegible] $\gamma = Re(s) > 1$. The choice of $\gamma = 2$ which in turn implies

$$s = 2 + it, \quad -\infty < t < \infty,$$

is suggested by the fact that the integrals in (15)–(17) are simplified to a great exte[illegible] It may be noted that the integrals in (15)–(17) considered as functions of s, s given [illegible]

18) have no singularities for $-\infty < t < \infty$. This establishes the validity of the choice of γ. On putting $\gamma = 2$ and $s = 2 + it$ in (15)–(17), the integrals become real line integrals with complex integrands. Although these integrands can be easily separated out into real and imaginary parts but they are too lengthy to be reported here. Moreover, the numerical evaluation of real line integrals with complex integrands presents no difficulty and they can be evaluated with the help of well-known methods[10].

At $r = 1$, normal and shearing stresses can be easily calculated from (15)–(16). It may be noted that these normal and shearing stresses at $r = 1$ form a self-equilibrating system of stresses by virtue of construction of the second stress system.

The third stress system is obtained to suit the following boundary conditions on the edges of the sector

$$\tau_{\theta\theta} = \tau_{r\theta} = 0 \quad \text{on} \quad = \pm\alpha, \quad \left.\begin{aligned} \tau_{rr} &= -(\tau_{rr})_2 \\ \tau_{r\theta} &= -(\tau_{r\theta})_2 \end{aligned}\right\} r = 1. \tag{19}$$

This third stress system can be determined by following either the approach in ref. 1 or in ref. 2. As pointed out earlier that only one infinite system of linear algebraic equations is to be solved in ref. 2 (one for symmetric and one for anti-symmetric stresses), we shall follow the approach given in ref. 2. To avoid repetition, details in ref. 2 are not being given in the present paper.

An auxiliary problem

Consider a sectorial inclusion (refer ref. 11 for the definition of inclusion) in a sectorial annulus. The sectorial annulus is bounded by the circular arcs $r = r'$ ($r' < 1$) and $r = 1$ and the radii $\theta = \alpha$ and $\theta = -\alpha$; $\theta = 0$ being the line of axial symmetry. The inclusion occupies the region bounded by the lines $\theta \pm \alpha$ and the arc $r = r'$. The origin of coordinates is denoted by 0. Let the inclusion in the absence of matrix (sectorial annulus) undergo a displacement $(\varepsilon_1 x, \varepsilon_2 y)$ with respect to the origin at 0. The following boundary conditions should hold

$$\left.\begin{aligned} u^+ - u^- &= -\varepsilon_1 x, \quad v^+ - v^- = -\varepsilon_2 y \\ \tau_{rr}^+ + i\tau_{rr}^+ &= \tau_{rr}^- + i\tau_{rr}^- \end{aligned}\right\} r = r' \tag{20}$$

$$\tau_{rr} = \tau_{r\theta} = 0, \ r = 1; \ \tau_{\theta\theta} = \tau_{r\theta} = 0, \ \theta \pm \alpha \tag{21}$$

where u and v are the displacement components in Cartesian coordinates, τ_{rr}, $\tau_{r\theta}$, etc., are the stresses in polar coordinates and $+$ and $-$ superscripts refer to the quantities belonging to inclusion and matrix respectively.

The solution of this sectorial inclusion problem with boundary conditions (20) and (21) has been obtained by the superposition of three stress systems.

The first system is obtained to suit the condition (20). We consider a circular inclusion in a circular annulus by extending the range of θ in the above sectorial inclusion problem to $0 \leq \theta < 2\pi$. The solution of this circular inclusion problem is well known[11]. The following normal and shearing stresses are obtained on $\theta = \pm \alpha$:

$$\{(\tau_{\theta\theta})_1\}_{\theta=\pm\alpha} = -2B_1(1 - r'^2) - A_1(12r^2 - 4 + 1/r'^2 - 12r^2 r'^2 + 3r'^2)\cos 2\alpha, \quad 0 \leq r < r' \tag{22}$$

$$\{(\tau_{\theta\theta})_1\}_{\theta=\pm\alpha} = 2B_1 r'^2(1 + 1/r^2) + A_1(4 - 12r^2 + 12r^2 r'^2 - 3r'^2 + 3r'^2/r^4)\cos 2\alpha, \quad r' < r \leq 1 \tag{23}$$

$$\{(\tau_{r\theta})_1\}_{\theta=\pm\alpha} = \pm A_1(4 - 6r^2 - 1/r'^2 + 6r^2 r'^2 - 3r'^2)\sin 2\alpha, \quad 0 \leq r < r' \tag{24}$$

$$\{(\tau_{r\theta})_1\}_{\theta=\pm\alpha} = \pm A_1(4 - 6r^2 + 2/r^2 - 3r'^2/r^4 + 6r^2 r'^2 - 3r'^2)\sin 2\alpha, \quad r' < r \leq 1 \tag{25}$$

where

$$A_1 = -\mu r'^2(\varepsilon_1 - \varepsilon_2)/(1 + K), \quad B_1 = -\mu(\varepsilon_1 + \varepsilon_2)/(1 + K),$$

μ is the shear modulus of elasticity and $K = 3 - 4\nu$ for the plane strain and $K = (3-\nu)/(1 + \nu)$ for the generalized plane stress, ν being Poisson ratio. These stresses in (22)-(25) form a self-equilibrating system on $\theta = \pm \alpha$.

The second stress system is given by (15)–(17) where $P_1 = P_2 = 0$ and P_3 and P_4 can be easily calculated.

The third stress system is obtained by considering a sector problem with zero loads on the radial edges and stresses equal and opposite to those given in (15) and (16) at $r = 1$. As pointed out earlier for finding the third stress system, the approach given in ref. 2 may be followed.

It may be noted that although in boundary condition (21), normal and shearing stresses have been taken to be zero but even if they are non-zero the problem can be handled without difficulty.

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