Axisymmetric bending of uniformly stressed annular plates with variable rigidity

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Abstract

In this paper, the problem of axisymmetric bending of uniformly stressed annular plates with rigidity varying as the nth power of the radial co-ordinate is discussed. Exact solutions are obtained in terms of Bessel's and Lommel's functions for a general type of loading. The corresponding solutions are deduced for two cases of loading, namely (i) line loading at the inner edge and (ii) uniformly distributed load over the entire annular plate. In the case of line loading a numerical example is given for annuals with a hole size equal to $\frac{1}{4}$ and rigidity varying as r^4.

Key words : Axisymmetric bending, variable rigidity, flexural rigidity, Bessel's function, Lommel's function.

1. Introduction

The problem of symmetrical bending of circular plates of variable thickness was first discussed by H. Holzer[1]. Since then many authors have investigated the problem, the outstanding of which are the investigation of O. Pichler[2] and R. G. Olson[3]. Olson[4] also solved the problem of unsymmetrical bending of circular plates. H. D. Conway[5–6] has solved the axially symmetric plates with linearly varying thickness. Basuli[7] solved the bending of uniformly compressed circular plates of variable thickness. Ghosh[8] solved the problem of the bending of plates under compressive forces in the middle plane of the plate, assuming the flexural rigidity varying as the square of the distance from the centre. The governing differential equation is reduced to second order homogeneous equation of slope[8].

The object of this paper is to extend Ghosh's work[8] for plates with rigidity varying as any power of the distance from the centre. The solution is obtained in closed form.

2. Theory

Fig. 1 represents an element of the deflection surface bounded by two concentric cylindrical surfaces of radii r and $r + dr$ and two radial planes including a small angle $d\theta$ at the centre of the plate.

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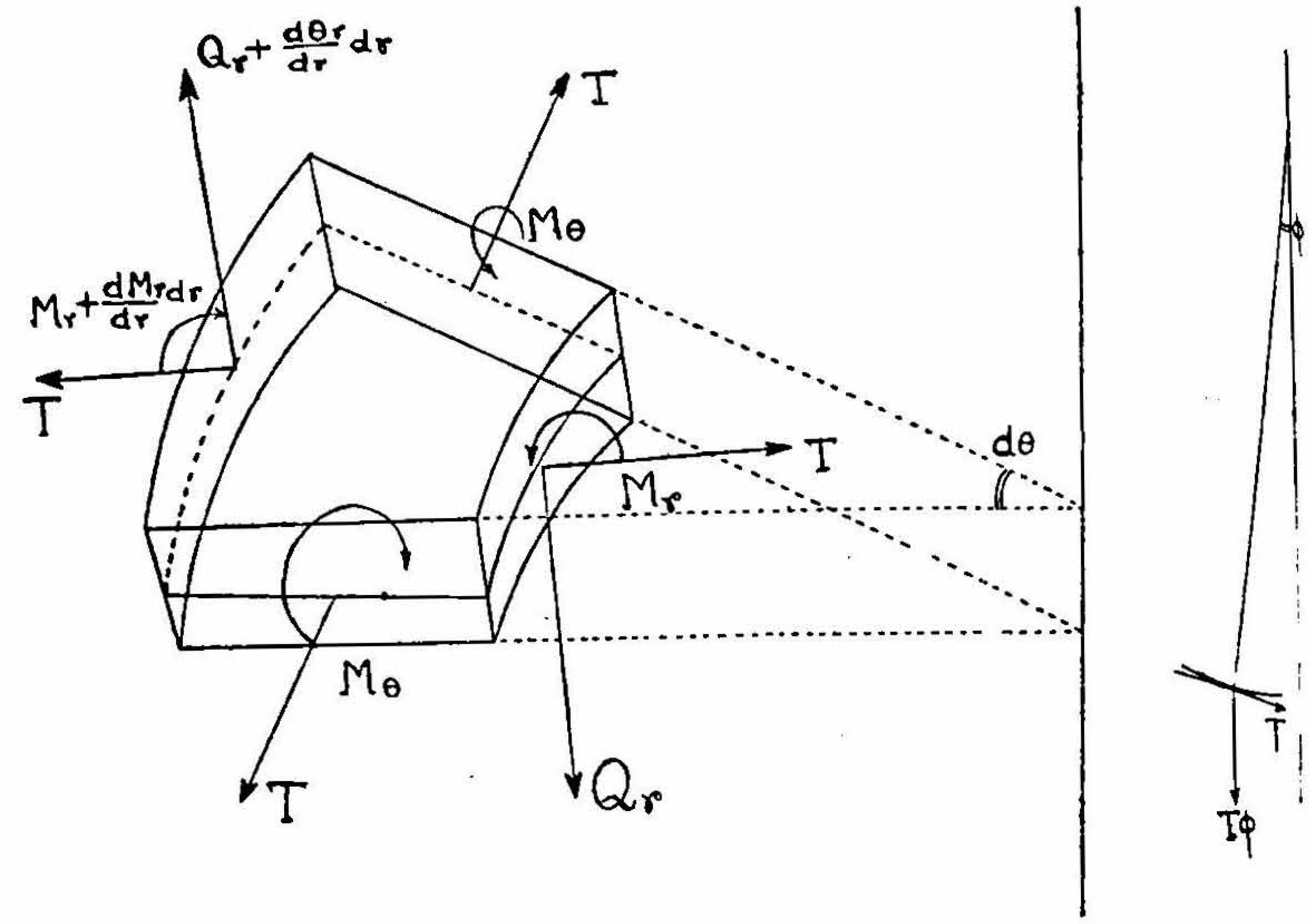

FIG. 1. An element of deflection surface bounded by two Concentric cylindrical surfaces and two radial plates.

Considering equilibrium of the element and taking moments[11] [Timoshenko, Woinowsky-Krieger, 1959], we have,

$$\left(M_r + \frac{dM_r}{dr} \cdot dr\right)(r + dr)\, d\theta - M_r \cdot r d\theta - M_\theta dr d\theta + (Q_r + T\phi) r d\theta dr = 0$$

Neglecting higher order quantities,

$$M_r + \frac{dM_r}{dr} \cdot r - M_\theta + rQ_r + rT\phi = 0 \tag{1}$$

Now using expressions for bending moments M_r, M_θ given by,

$$\left.\begin{aligned} M_r &= D\left[\frac{d\phi}{dr} + \sigma\frac{\phi}{r}\right] \\ M_\theta &= D\left[\frac{\phi}{r} + \sigma\frac{d\phi}{dr}\right] \end{aligned}\right\} \tag{2}$$

and taking into account the flexural rigidity D is a variable quantity, the equation (1) reduces to

$$D\frac{d}{dr}\left(\frac{d\phi}{dr} + \frac{\phi}{r}\right) + \frac{dD}{dr}\left(\frac{d\phi}{dr} + \sigma\frac{\phi}{r}\right) + T\phi = -Q_r \tag{3}$$

3. Problem and its solutions

Let us consider an annular plate whose rigidity D varies as the nth power of the distance from the centre

i.e.,

$$D = D_0 r^n \tag{4}$$

Substituting (4) in (3), we get after simplification

$$r^2 \frac{d^2 \phi}{dr^2} + (n+1) r \frac{d\phi}{dr} + \phi \left[(n\sigma - 1) + \frac{T}{D_0} \cdot \frac{1}{r^{n-2}} \right] = \frac{-Q_r}{D_0 r^{n-2}} \tag{5}$$

Let us assume,

$$Q_r = P_m r^m, \quad (P_m = \text{a real constant}) \tag{6}$$

Then equation (5) reduces to

$$r^2 \frac{d^2 \phi}{dr^2} + (n+1) r \frac{d\phi}{dr} + \phi \left[(n\sigma - 1) + \frac{T}{D_0} \cdot \frac{1}{r^{n-2}} \right] = \frac{-P_m r^{m-n+2}}{D_0} \tag{7}$$

Transforming the variable ϕ to the new variable ϕ_1 by the substitution

$$\phi = r^{-n/2} \phi_1,$$

then substituting $r^{1-n/2} = z$ in the transformed equation and lastly

$\dfrac{2}{n-2} \sqrt{\dfrac{T}{D_0}}\, z = t$, the equation transforms to

$$t^2 \frac{d^2 \phi_1}{dt^2} + t \frac{d\phi_1}{dt} + \phi_1 [t^2 - \nu^2] = R t^{1-2(m+1)/(n-2)} \tag{8}$$

where

$$\left.\begin{aligned} \nu^2 &= \frac{(n-2\sigma)^2 + 4(1-\sigma^2)}{(n-2)^2} \quad (n \neq 2) \\ R &= -\frac{P_m}{D_0} \left(\frac{2}{n-2} \right)^{2(m+1)/(n-2)+1} \left(\frac{T}{D_0} \right)^{m+1/(n-2)-\frac{1}{2}} (n \neq 2) \end{aligned}\right\} \tag{9}$$

The complementary function and the particular integral of the equation (8) are given by

$$C.F. = A J_\nu(t) + B Y_\nu(t)$$

$$P.I. = R S_{-2(m+1)/(n-2),\, \nu}(t)$$

where A, B are arbitrary constants, $J_\nu(t)$, $Y_\nu(t)$ are Bessel's function of first and second kind of order ν and S is the Lomel function*.

* [equation (69) *vide* ref. 10, p. 40].

Thus the complete solution of (7) is given by

$$\phi = r^{-n/2}\Big[AJ_\nu\Big(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\Big) + BY_\nu\Big(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\Big) + RS_{-2(m+1)/n-2,\,\nu}\Big(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\Big)\Big] \tag{10}$$

CASE I. Lateral load distributed uniformly round the radius of the hole, inner boundary clamped and outer boundary clamped and supported.

By outer boundary clamped and supported, we mean $\omega = 0$ and $\phi = 0$ at $\gamma = a$ and the inner boundary clamped means that the slope is zero there but not the deflection. This can be achieved by fixing the inner boundary with the load so that the inner boundary may have downward movement, along with load but the slope remaining zero there. This boundary condition was used by Conway (1948) and since then by other authors.

Let the load P be distributed uniformly round the radius of the hole,

$$\therefore\ Q_r = \frac{P}{2\pi r} = \frac{P}{2\pi}\,r^{-1} \tag{11}$$

Substituting $m = -1$, $P_{-1} = \dfrac{P}{2\pi}$ in the general solution (10) we get

$$\phi = r^{-n/2}\Big[A_1J_\nu\Big(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\Big) + B\,Y_\nu\Big(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\Big) - R_1S_{0,\,\nu}\Big(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\Big)\Big]\ (n \neq 2) \tag{12}$$

where

$$R_1 = \frac{P\sqrt{\dfrac{D_0}{T}}}{\pi D_0(n-2)}\ (n \neq 2) \tag{13}$$

and where A_1, B_1 are arbitrary constants.

For $n = 2$ the equation (7) reduces to homogeneous differential equation of second order[8].

(14)

Boundary conditions are $\phi = 0$ at $\gamma = a$, and $\gamma = b$

Equation (12) and (14) then give

$$A_1 = \frac{-R_1\left[Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,a^{1-n/2}\right)S_{0,\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,b^{1-n/2}\right) - Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,b^{1-n/2}\right)S_{0,\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,a^{1-n/2}\right)\right]}{J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,a^{1-n/2}\right)Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,b^{1-n/2}\right) - J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,b^{1-n/2}\right)Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,a^{1-n/2}\right)} \tag{15}$$

$$B_1 = \frac{R_1\left[J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,a^{1-n/2}\right)S_{0,\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,b^{1-n/2}\right) - J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,b^{1-n/2}\right)S_{0,\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,a^{1-n/2}\right)\right]}{J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,a^{1-n/2}\right)Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,b^{1-n/2}\right) - J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,b^{1-n/2}\right)Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,a^{1-n/2}\right)} \tag{16}$$

Considering equation (12) and

$$M_r = D\left(\frac{d\phi}{dr} + \sigma\frac{\phi}{r}\right)$$

we can find

$$\begin{aligned} M_r = D_0 \Bigg[& r^{n/2-1}(\sigma - n/2)\left\{A_1 J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\;r^{1-n/2}\right)\right. \\ & \left. + B_1 Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\;r^{1-n/2}\right) - R_1 S_{0,\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\;r^{1-n/2}\right)\right\} \\ & - \sqrt{\frac{T}{D_0}}\left\{A_1 J'_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\;r^{1-n/2}\right) + B_1 Y'_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\;r^{1-n/2}\right)\right. \\ & \left. - R_1 S'_{0,\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\;r^{1-n/2}\right)\right\}\Bigg] \end{aligned} \tag{17}$$

and

$$M_\theta = D\left[\sigma\frac{d\phi}{dr} + \frac{\phi}{r}\right]$$

$$= D_0\left[r^{n/2-1}\left(1-\frac{n\sigma}{2}\right)\left\{A_1 J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\right)\right.\right.$$

$$+ B_1 Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\right) - R_1 S_{0,\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\right)\Bigg\}$$

$$-\sigma\sqrt{\frac{T}{D_0}}\left\{A_1 J'_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\right) + B_1 Y'_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\right)\right.$$

$$\left.\left.- R_1 S'_{0,\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\right)\right\}\right] \tag{18}$$

To get the deflection we know that

$$\phi = -\frac{d\omega}{dr}\,. \tag{19}$$

On integrating (12), we get

$$\omega = \frac{2}{n-2}\,r^{1-n/2}\left[A_1\left\{(\nu-1)\,J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\right)\right.\right.$$

$$\times S_{-1,\nu-1}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\right) - J_{\nu-1}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\right)$$

$$\times S_{0,\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\right)\Bigg\} + B_1\left\{(\nu-1)\,Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\right)\right.$$

$$\times S_{-1,\nu-1}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\right) - Y_{\nu-1}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\right)$$

$$\left.\left.\times S_{0,\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\,r^{1-n/2}\right)\right\}\right] + K_1 \text{ (constant)} \tag{20}$$

Using the boundary condition $\omega = 0$ at $r = a$ we get K_1 and so ω is determined. The deflection will be maximum at $r = b$ and the maximum deflection can be obtained from the expression of ω putting $r = b$.

Numerical calculation

The value of ω at different radial distances is calculated (Table I) taking

$$n = 4,\ \sigma = \frac{1}{3},\ \sqrt{\frac{T}{D_0}} = \frac{1}{2}\,,\ a = 16.$$

Table I

Lateral deflection along a radius.

γ	4	6	8	10	12	14	16
$10^6\,\omega \Big/ \dfrac{Pa^2}{\pi D_0}$	4·578	2·931	1·452	·580	·232	·048	·000

CASE II. Uniformly distributed load. Inner boundary clamped and outer boundary clamped and supported.

Let the load be uniformly distributed with intensity q, then

$$Q_r = \frac{1}{2\pi r}\int_b^r q \cdot 2\pi r\,dr = \frac{q}{2r}(r^2 - b^2) = \frac{q}{2}\cdot r - \frac{qb^2}{2} r^{-1} = P_{m1} r^1 + P_{m2} r^{-1} \tag{21}$$

where

$$P_{m1} = \frac{q}{2}\,,\quad P_{m2} = -\frac{qb^2}{2} \tag{22}$$

In this case complementary function will be the same as in the previous case. The particular integral will have two components corresponding to two components of Q_r.

The solution will be of the form

$$\phi = r^{-n/2}\Big[A_2 J_\nu\Big(\frac{2}{n+2}\sqrt{\frac{T}{D_0}}\;r^{1-n/2}\Big) + B_2 Y_\nu\Big(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\;r^{1-n/2}\Big)$$
$$+ R_1' S_{0,\nu}\Big(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\;r^{1-n/2}\Big) - R_1' S_{-4/(n-2),\nu}\Big(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\;r^{1-n/2}\Big)\Big] \tag{23}$$

where

$$R_1' = \frac{qb^2}{D_0(n-2)}\sqrt{\frac{D_0}{T}} \tag{24}$$

$$R_2' = \frac{q}{2D_0}\Big(\frac{2}{n-2}\Big)^{(n\quad 2)/(n-2)}\Big(\frac{D_0}{T}\Big)^{(n-6)/2(n-2)} \tag{25}$$

The boundary conditions are $\phi = 0$ when $r = a$ and $r = b$

Considering the equation (23) and using the boundary conditions (25) and solving we get

$$A_2 = \frac{C_2' Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, a^{1-n/2}\right) - C_1' Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, b^{1-n/2}\right)}{J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, a^{1-n/2}\right) Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, b^{1-n/2}\right) - J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, b^{1-n/2}\right) Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, a^{1-n/2}\right)} \tag{26}$$

$$B_2 = \frac{C_1' J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, b^{1-n/2}\right) - C_2' J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, a^{1-n/2}\right)}{J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, a^{1-n/2}\right) Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, b^{1-n/2}\right) - J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, b^{1-n/2}\right) Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, a^{1-n/2}\right)} \tag{27}$$

where

$$C_1' = R_1' S_{0,\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, a^{1-n/2}\right) - R_2' S_{-4/(n-2),\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, a^{1-n/2}\right) \tag{28}$$

$$C_2' = R_1' S_{0,\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, b^{1-n/2}\right) - R_2' S_{-4/(n-2),\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, b^{1-n/2}\right) \tag{29}$$

Substituting the value of A_2 and B_2 from (26) and (27) in the equation (23) and using (24), (28) and (29) we can find ϕ. Thus ϕ is determined.

To get the deflection ω we know that

$$\phi = -\frac{d\omega}{dr}.$$

Hence

$$\frac{d\omega}{dr} = -r^{-n/2}\left[A_2 J_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, r^{1-n/2}\right) + B_2 Y_\nu\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, r^{1-n/2}\right) + R_1' S_{0,\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, r^{1-n/2}\right) - R_2' S_{-4/(n-2),\nu}\left(\frac{2}{n-2}\sqrt{\frac{T}{D_0}}\, r^{1-n/2}\right)\right] \tag{30}$$

Integrating the equation (30) we get the expression for ω. The boundary condition is $\omega = 0$ at $r = a$ and this determines the constant of integration. Substituting this value in the relevant equation, ω is determined.

The maximum deflection is obtained by putting $r = b$.

4. Acknowledgements

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5. Nomenclature

a = radius of the plate

b = radius of the inner boundary

h = thickness of the plate at a distance γ from the centre

D = flexural rigidity of the plate $= \dfrac{Eh^3}{12(1-6^2)} = D_0 r^n$

E = Young's modulus

σ = Poisson's ratio

M_r, M_θ = bending moments per unit length of the section perpendicular to the radius and tangent.

ϕ = slope at a distance $r = -\dfrac{d\omega}{dr}$.

Q_r = shearing force per unit length acting normally to the middle surface.

ω = displacement at a distance r

T = uniform pressure per unit length of the perimeter in the middle plane of the plate.

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