USE OF HYPERBOLIC TRIGONOMETRIC SERIES IN RECTANGULAR CARTESIAN CO-ORDINATES FOR PROBLEMS WITH ARBITRARY RECTILINEAR BOUNDARIES

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ABSTRACT

The effectiveness of hyperbolic trigonometric series in rectangular Cartesian co-ordinates for analysing problems with non-rectangular but rectilinear domains is brought out in the present work. Three typical examples—simply supported triangular, rhombic and parallelogrammic plates under uniform pressure—are considered and elegant solutions presented. Fourier expansion, simple collocation and least square collocation techniques are followed for the satisfaction of the boundary conditions. Numerical results for the three examples are obtained for a range of skew angles and side ratios. The convergence of important parameters like maximum deflection and maximum moments with increasing order of approximation is studied. The results show that the method is capable of providing satisfactory results with limited effort in all the cases.

Key words: Solid mechanics, statics, rectilinear domains, arbitrary plates, bending.

1. INTRODUCTION

Series solutions for the Laplace and the biharmonic equations can be written down in various special coordinate systems. Hence there is a natural tendency to match the coordinate system to the geometry of a domain [1], [2], [3]. This is not always the most satisfactory procedure. The simple polar coordinates and rectangular cartesian coordinates are often far superior in application to arbitrary shapes than coordinates specialised to the shape. For example, the use of oblique coordinates for parallelogrammic shapes appears natural. However, Silberstein [3] found that, applying oblique coordinates and developing a series solution ".... was quite a lengthy process, yet the accuracy achieved was sufficient to estimate deflections only; no significant results on bending moments could be obtained using the resulting solution." On the other hand the application of rectangular Cartesian coordinates can often turn out to be effective and superior.

It is the purpose of this paper to demonstrate this useful fact with a few examples in the analysis of thin plate flexure. For simplicity, the analysis is presented for simple supports and uniform transverse pressure. Triangular, rhombic, and parallelogrammic plate configurations are investigated. Depending on the ease of analysis and effectiveness, Fourier expansion, simple collocation and least square collocation methods of satisfaction of boundary conditions are followed for the different shapes considered.

2. Method of Solution

The solution for the governing differential equation for thin plate flexure $\nabla^4 w = q/D$, can be written in the form

$$w = w_p + w_c \tag{1}$$

where w_p is the particular integral satisfying $\nabla^4 w_p = q/D$ and w_c is the complimentary function satisfying $\nabla^4 w_c = 0$. A suitable form for w_c is chosen, in rectangular Cartesian co-ordinates, as

$$\begin{aligned} w_c = & \sum_m [A_{mi} \cosh mx + B_{mi}\, x \sinh mx] \begin{pmatrix}\cos my \\ \sin my\end{pmatrix} \\ & + \sum_m [C_{mi} \sinh mx + D_{mi}\, x \cosh mx] \begin{pmatrix}\cos my \\ \sin my\end{pmatrix} \\ & + \sum_m [E_{mi} \cosh my + F_{mi}\, y \sinh my] \begin{pmatrix}\cos mx \\ \sin mx\end{pmatrix} \\ & + \sum_m [G_{mi} \sinh my + H_{mi}\, y \cosh my] \begin{pmatrix}\cos mx \\ \sin mx\end{pmatrix} \end{aligned}$$

($i = 1, 2$—corresponding to cosine and sine terms respectively) (2l)

When the domain consists of straight edges, two of which are parallel, it is then possible to exactly satisfy the homogeneous boundary conditions along the two edges using a part of the solution chosen from Eqn. (2), with appropriate values for m. When the two parallel edges are simply supported, the m's are real and integral. For other than simple supports, the m's are generally complex. Further, when the domain has one or more axes of symmetry or skew symmetry, by a suitable choice of the terms of the series in Eqn. (2), it is possible to satisfy at least some of the symmetry conditions identically. In the present study, only real and integral values of m are encountered since only simple support conditions, are considered. In addition, $E_{mi} = F_{mi} = G_{mi} = H_{mi} = 0$ ($i = 1, 2$) as the other set of constants is sufficient for our study. We shall now proceed to apply the resulting series to some typical examples.

In each example, the following steps are followed. Depending on the problem, the simple support conditions on one or more of the edges and the symmetry or skew symmetry conditions present, are exactly satisfied by a proper choice of the terms in Eqn. (2). The deflection function $w (= w_p + w_c)$ is obtained by adding a suitable particular solution w_p to w_c. The unknown parameters in the deflection function are determined by approximately satisfying the hitherto unsatisfied boundary conditions on the other edges by applying a suitable technique. The conditions on these edges may, in general, be arbitrary. Here we consider them also to be simply supported.

3. Isosceles Triangular Plates

Locating the origin at the apex '0' of the triangle OAB with the y-axis normal to the edge AB (Fig. 1), and with the distance OC taken as π, w_c is chosen from Eqn. (2) with

$$(a)\ m = 1, 2, 3, \ldots \text{ and } A_{m_1} = B_{m_1} = C_{m_1} = D_{m_1} = 0$$

to exactly satisfy the simple support conditions on AB and (b) $C_{m_2} = D_{m_2} = 0$ so as to satisfy the symmetry conditions on OC Thus

$$w_c = \sum_{m=1,2\ldots} \left[A_m \frac{\cosh mx}{\cosh mt\pi} + B_m \frac{x \sinh mx}{\cosh mt\pi}\right] \sin my \tag{3 a}$$

wherein the second subscript in A_{m_2}, B_{m_2} is conveniently dropped without causing ambiguity.

A suitable w_p, satisfying all the boundary conditions that w_c in Eqn. (3 a) does is

$$w_p = \frac{q}{24\,D}(y^4 - 2\pi y^3 + \pi^3 y) \tag{3 b}$$

so that

$$w = \frac{q}{24\,D}(y^4 - 2\pi y^3 + \pi^3 y) + \sum \left[A_m \frac{\cosh mx}{\cosh mt\pi} + B_m \frac{x \sinh mx}{\cosh mt\pi}\right] \sin my \tag{3}$$

It is now necessary to satisfy the boundary conditions along only one of the edges, say OB the conditions being

$$w = 0,\ \nabla^2 w = 0 \text{ on } x = ty \tag{4}$$

where $t = \tan \alpha$.

Substitution of the deflection function w into the boundary conditions, Eqns. (4), lead to the following pair of boundary error equations

$$\frac{q}{24D}(y^4 - 2\pi y^3 + \pi^3 y) + \sum_m \left[A_m \frac{\cosh mty}{\cosh mt\pi} + B_m \frac{t\, y \sinh mty}{\cosh mt\pi} \right] \sin my = 0 \tag{5}$$

and

$$\frac{q}{2D}(y^2 - \pi y) + \sum_m{}' \; 2m\, B_m \frac{\cosh mty}{\cosh mt\pi} \sin my = 0. \tag{6}$$

In principle, the constants A_m, B_m can be determined by any one of many available methods. The functional form of the error equations (5) and (6) suggests application of Fourier analysis in the range $y = 0$ to π.

Thus

$$\int_0^\pi \sum_m{}' \left[A_m \frac{\cosh mty}{\cosh mt\pi} + B_m \frac{ty \sinh mty}{\cosh mt\pi} \right] \sin my \sin ny \; dy$$

$$= -\frac{q}{24D} \int_0^\pi [y^4 - 2\pi y^3 + \pi^3 y] \sin ny \; dy \tag{7}$$

and

$$\int_0^\pi \sum_m{}' \; 2m\, B_m \frac{\cosh mty}{\cosh mt\pi} \sin my \sin ny \; dy$$

$$= -\frac{q}{2D} \int_0^\pi (y^2 - \pi y) \sin ny \; dy \quad (n = 1, 2, 3, \ldots) \tag{8}$$

Carrying out the integrations of Eqns. (7) and (8) we arrive at the following sets of linear simultaneous equations in A_m, B_m

$$\sum_m A_m (-1)^{m-n} \{S_{mn} - T_{mn}\} (mt/2) \tanh mt\pi + \sum_m B_m (-1)^{mn} (t/2)$$

$$\times \left[\{S_{mn} - T_{mn}\} mt\pi - \{U_{mn} S^2_{mn} - V_{mn} T^2_{mn}\} \tanh mt\pi \right]$$

$$= -(2/n^5)\, q/D \qquad \text{if } n \text{ is odd}$$

$$= 0 \qquad \text{if } n \text{ is even} \tag{9}$$

$$(n = 1, 2, 3, \ldots)$$

$$\sum_m B_m m^2 t \{S_{mn} - T_{mn}\} \tanh mt\pi$$

$$= (2/n^3)\, q/D \qquad \text{if } n \text{ is odd}$$
$$= 0 \qquad \text{if } n \text{ is even} \tag{10}$$
$$(n = 1, 2, 3, \ldots)$$

where

$$S_{mn} = [(mt)^2 + (m-n)^2]^{-1}; \quad T_{mn} = [(mt)^2 + (m+n)^2]^{-1}$$
$$U_{mn} = (mt)^2 - (m-n)^2; \quad V_{mn} = (mt)^2 - (m+n)^2 \tag{11}$$

4. RHOMBIC PLATES

In this case, the origin of coordinates is located at the corner A of the rhombus $ABCD$ (Fig. 2) with the diagonal AC taken to be length π. w_c is given by Eqn. (2) with $m = 1, 3, 5, \ldots$ and we can put

$$A_{m_1} = B_{m_1} = C_{m_1} = D_{m_1} = C_{m_2} = D_{m_2} = 0,\ A_{m_2} = A_m,\ B_{m_2} = B_m$$

so that the symmetry conditions on the diagonals AC and BD are exactly satisfied. w_p is chosen as

$$w_p = qx^4/24D \tag{12}$$

The parameters A_m, B_m are determined by satisfying the simple support conditions on one of the edges radiating from A, say AD. These are given by

$$w = 0,\ \nabla^2 w = 0 \text{ on } x = ty \tag{13}$$

where $t = \tan \alpha$.

Substitution of the deflection function into these equations leads to the two edge error equations,

$$\sum_m \left[A_m \frac{\cosh mty}{\cosh mt\pi} + B_m \frac{ty \sinh mty}{\cosh mt\pi}\right] \sin my$$
$$= -\, qt^4y^4/24D \tag{14}$$

$$\sum_m B_m\, 2m \frac{\cosh mty}{\cosh mt\pi} \sin my = -\, qt^2y^2/2D \tag{15}$$

Again, Fourier analysing Eqns. (14) and (15) in the range $y = 0$ to $\pi/2$ we arrive at the following sets of simultaneous equations in A_m, B_m.

$$\sum_m A_m (-1)^{\frac{m-1}{2}} \{S_{mn} + T_{mn}\} m \tanh mt\pi + \sum_m B_m (-1)^{\frac{m-1}{2}}$$
$$\times [mt\pi \{S_{mn} + T_{mn}\} - \{U_{mn}S^2_{mn} + V_{mn}T^2_{mn}\} \tanh mt\pi]$$

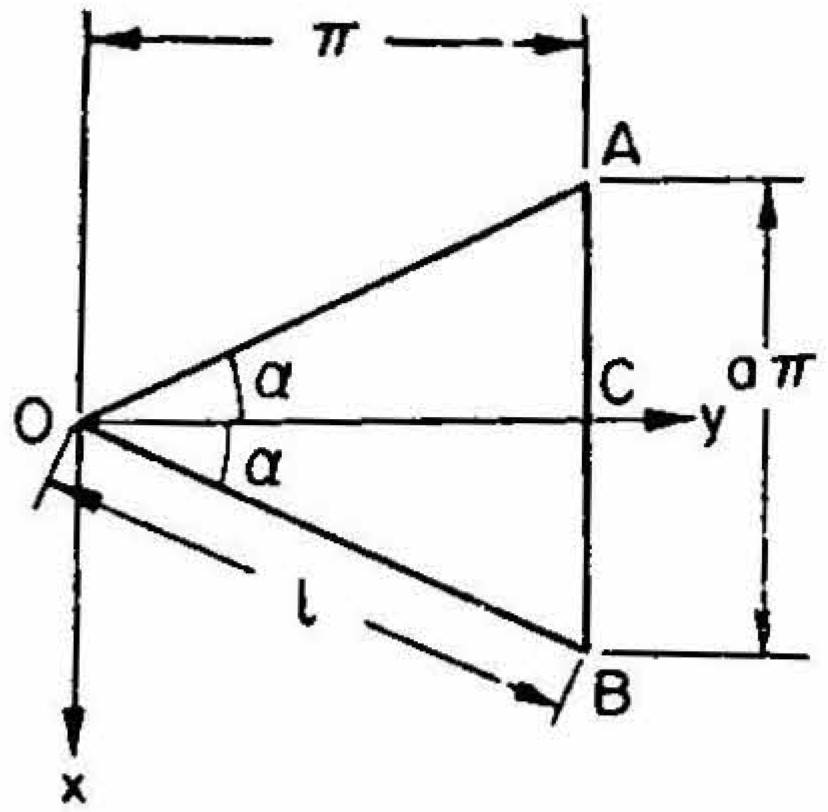

FIG.1. TRIANGULAR PLATE

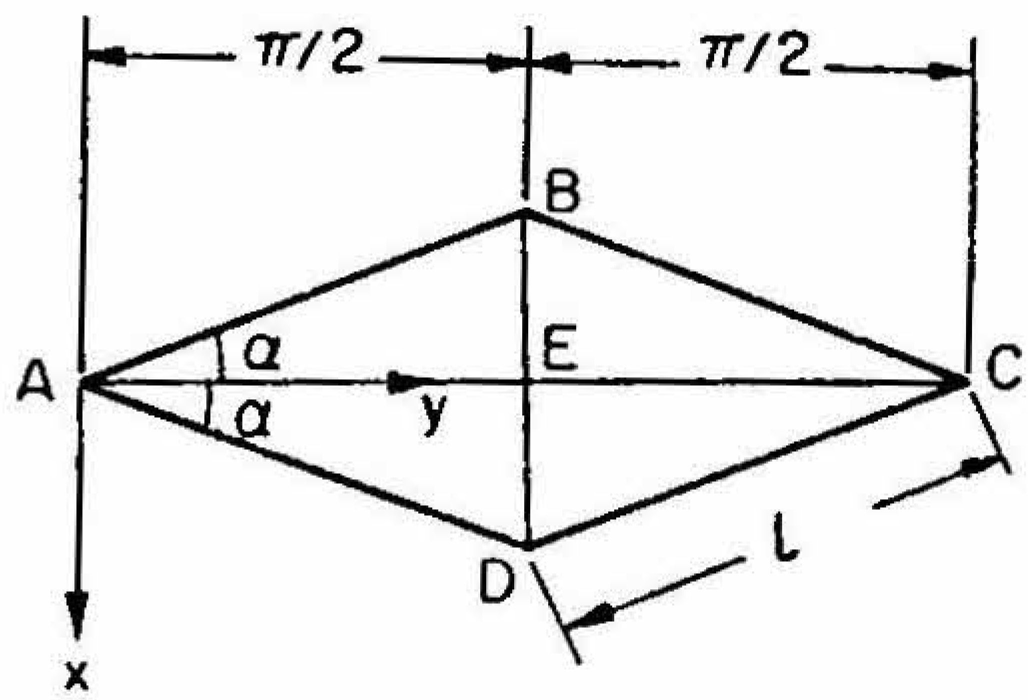

FIG 2 RHOMBIC PLATE

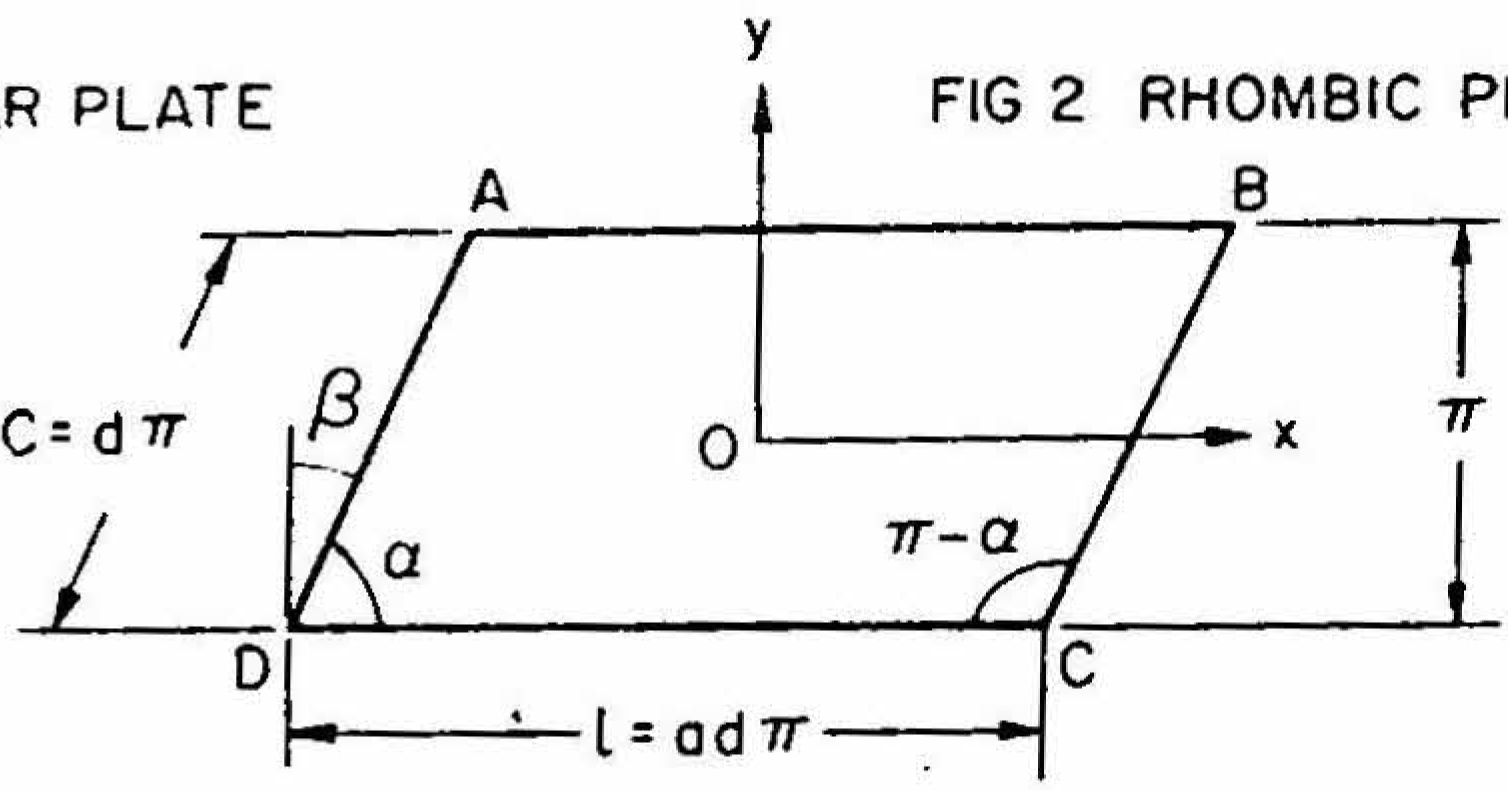

FIG. 3. PARALLELOGRAMMIC PLATE

COORDINATE SYSTEM FOR DIFFERENT CONFIGURATIONS

$$= -\left[\left\{\frac{(n\pi)^3}{2} - 12n\pi\right\} + (-1)^{\frac{n-1}{2}} 24\right] qt^3/12n^5 D \tag{16}$$

and

$$\sum_m B_m (-1)^{\frac{m-1}{2}} \{S_{mn} + T_{mn}\} m^2 \tanh mt\pi$$

$$= -\{n\pi - (-1)^{\frac{n-1}{2}} 2\} \times qt/2n^3 D \tag{17}$$

$$(n = 1, 3, 5, \ldots)$$

where $S_{mn}, \ldots, T_{mn}$ are as defined in Eqn. (11).

5. PARALLELOGRAMMIC PLATES

In this example, the origin is situated at the centre 0 of the plate $ABCD$ (corner angles α and $\pi - \alpha$) as in Fig. 3. w is chosen such that it exactly satisfies the simple support conditions along the two parallel edges AB, DC separated by a distance π and the skew symmetry condition

$$w(x, y) = w(-x, -y).$$

Then in w_c,

(a) $m = 1, 3, 5, \ldots$ with $C_{m_1} = D_{m_1} = 0$, $A_{m_1} = A_m$, $B_{m_1} = B_m$

and

(b) $m = 2, 4, 6, \ldots$ with $A_{m_2} = B_{m_2} = 0$, $C_{m_2} = C_m$, $D_{m_2} = D_m$

w_p is properly chosen as

$$w_p = \frac{q}{384\, D}(16y^4 - 24\,\pi^2\, y^2 - 5\,\pi^4). \tag{18}$$

It now remains to satisfy the simple support conditions along only one edge, say BC, given by

$$w = 0,\ \nabla^2 w = 0 \quad \text{on} \quad x = c + by \tag{19}$$

where a = side ratio $= AB/BC = 2C/d\pi$, $b = \cot \alpha$, $C = ad\pi/2$ and $d = \operatorname{cosec} \alpha$.

The boundary error equations are obtained as

$$\sum_{m=odd} \left[A_m \frac{\cosh m\,(c+by)}{\cosh mc} + B_m \frac{(c+by)\sinh m\,(c+by)}{\cosh mc}\right] \cos my$$

$$+ \sum_{m=even} \left[C_m \frac{\sinh m\,(c+by)}{\cosh mc}\right.$$

$$\left. + D_m \frac{(c+by)\cosh m\,(c+by)}{\cosh mc}\right] \sin my$$

$$= -\frac{q}{384D}(16y^4 - 24\pi^2 y^2 + 5\pi^4) \tag{20}$$

$$\sum_{m=odd} B_m \frac{2m\cosh m\,(c+by)}{\cosh mc} \cos my$$

$$+ \sum_{m=even} D_m \frac{2m \sinh m\,(c+by)}{\cosh mc} \sin my$$

$$= -\frac{q}{8D}(4y^2 - \pi^2) \tag{21}$$

For this problem, we have found the simple collocation and least square collocation methods, to be simpler than Fourier analysis for satisfying the boundary conditions.

In the simple collocation method, we satisfy Eqns. (20) and (21) at certain discrete points along the edge BC. For any choice of p points, one obtains the twin systems of simultaneous equations in A_{m_1}, B_{m_1}, C_{m_2}, D_{m_2} by replacing y by y_i' $i = 1, 2, 3, \ldots p$) in Eqns. (20) and (21). The solution of these equations determines the unknown parameters and hence completes the analysis. For convenience, one chooses a suitable number of equidistant points on the edge BC.

In the least square collocation method, we satisfy the boundary conditions at a larger number of points than the number of unknown parameters employed in the deflection function in a least square sense. Suppose we write m boundary equations for a deflection function with n constants where $m > n$. In matrix notation, this system of equations may be written as

$$a(m \times n) \cdot X(n \times 1) = b(m \times 1).$$

As X cannot be uniquely determined from this equation, we seek a solution $X = c$ such that it minimizes the sum of the square of the residual

$$e = a \cdot c - b.$$

The square of the residual e is in fact the dot product $e \cdot e$, so that

$$e^2 = (ac - b)^{\mathrm{T}} \cdot (ac - b).$$

The minimization process requires that

$$\frac{\delta e^2}{\delta c} = 0$$

which leads to the solution

$$a^{\mathrm{T}} \cdot a \cdot c - a^{\mathrm{T}} \cdot b = 0$$

or

$$c = (a^{\mathrm{T}} \cdot a)^{-1} \cdot (a^{\mathrm{T}} \cdot b)$$

where the superscript T denotes the transpose of the relevant matrix. Thus we reduce the original redundant system of m equations to a determinate set of n simultaneous equations in n unknowns.

6. Numerical Results and Discussion

Numerical studies are carried out for the first three examples and the results are presented in Tables I to III.

Triangular Plates.—Table I contains the results for the isosceles triangular plates. Convergence studies are carried out for four values of the apex angle 2α from 30° to 120°. The order of approximation M is varied from 2 to 10. In each case the maximum deflection value and its location and

TABLE I

Simply supported Isosceles triangular plate under uniform pressure Effect of apex angle on convergence of deflection and moments at maximum deflection location values for various No. of terms (M)

Hyperbolic Trigonometric Functions: Fourier Expansion Method

$\bar{w} = (10^4 D/qa^4)\, w$; $\bar{M}_x = (10^2/qa^2)\, M_x$; $\bar{M}_y = (10^2/qa^2)\, M_y$; $X = x/a$ = Distance of w_{max} from Apex

2α = Apex Angle: $\nu = 0{\cdot}3$

2α	30°				60°			
M	$\bar{w}$	$\bar{M}_x$	$\bar{M}_y$	X	$\bar{w}$	$\bar{M}_x$	$\bar{M}_y$	X
2	1·542	0·989	0·879	0·775	5·252	1·718	1·646	0·675
3	1·472	1·100	0·727	0·743	5·799	1·749	1·860	0·673
4	1·436	1·067	0·708	0·751	5·772	1·798	1·819	0·666
5	1·453	1·066	0·736	0·752	5·798	1·814	1·803	0·666
6	1·448	1·071	0·726	0·750	5·773	1·808	1·797	0·666
7	1·450	1·070	0·726	0·751	5·797	1·806	1·809	0·667
8	1·449	1·068	0·728	0·751	5·775	1·805	1·801	0·666
9	1·450	1·070	0·728	0·751	5·801	1·807	1·809	0·667
10	1·449	1·069	0·727	0·751	5·769	1·805	1·799	0·666
Likely value	1·449^{+}	1·069	0·727	0·751	5·785^{+}	1·806	1·806	0·667
Ref. 5	..	..	..	..	5·787	1·806	1·806	0·667

90°				120°			
$\bar{w}$	$\bar{M}_x$	$\bar{M}_y$	X	$\bar{w}$	$\bar{M}_x$	$\bar{M}_y$	X
6·959	2·186	1·845	0·551	4·761	1·294	2·019	0·503
6·496	1·823	1·979	0·601	3·889	1·297	1·711	0·510
6·521	1·678	2·124	0·612	3·509	1·254	1·582	0·524
6·560	1·639	2·185	0·613	3·328	1·198	1·542	0·538
6·577	1·642	2·194	0·611	3·237	1·144	1·543	0·549
6·581	1·655	2·185	0·610	3·190	1·098	1·560	0·556
6·581	1·664	2·176	0·610	3·164	1·062	1·580	0·560
6·580	1·668	2·171	0·610	3·149	1·035	1·599	0·563
6·579	1·669	2·170	0·610	3·143	1·017	1·615	0·565
6·579^{-}	1·669^{+}	2·170^{-}	0·610	3·140^{-}	1·017^{-}	1·615^{+}	0·567
6·57	..	..	0·60	2·95	..	..	0·57

TABLE II

Simply supported Rhombic plate under uniform pressure
Effect of corner angle on convergence of deflection and moments at centre values for various orders of approximation (M)

Hyperbolic Trigonometric Functions: Fourier Expansion Method

$\bar{w} = (10^3 D/qa^4)\, w$; $\bar{M}_{max} = (10^2/qa^2)\, M_{max}$; $\bar{M}_{min} = (10^2/qa^2)\, M_{min}$
2α = Corner Angle : $\nu = 0{\cdot}3$

2α	30°			45°			60°		
M	$\bar{w}$	$\bar{M}_{max}$	$\bar{M}_{min}$	$\bar{w}$	$\bar{M}_{max}$	$\bar{M}_{min}$	$\bar{w}$	$\bar{M}_{max}$	$\bar{M}_{min}$
2	0·4985	2·044	1·192	1·399	3·147	2·371	2·514	4·173	3·319
3	0·4198	1·877	1·168	1·304	3·160	2·239	2·516	4·177	3·331
4	0·4076	1·874	1·120	1·311	3·214	2·191	2·560	4·258	3·330
5	0·4066	1·891	1·097	1·314	3·225	2·183	2·555	4·253	3·319
6	0·4069	1·900	1·088	1·316	3·228	2·186	2·561	4·254	3·332
7	0·4072	1·904	1·085	1·316	3·227	2·187	2·558	4·252	3·325
8	0·4074	1·906	1·084	1·317	3·227	2·189	2·562	4·255	3·332
9	0·4075	1·906	1·084	1·317	3·227	2·189	2·557	4·251	3·325
10	0·4076	1·906	1·085	1·317	3·227	2·190	2·563	4·256	3·334
Sampath [4]	0·411	1·908	1·099	1·327	3·230	2·210	2·556	4·284	3·322
Morley [2]	0·408	1·91	1·09	—	—	—	2·56	4·25	3·33
Exact [7]	0·4075	1·905	1·086	1·317	3·226	2·190	2·560	4·253	3·329

2α	75°			90°		
M	$\bar{w}$	$\bar{M}_{max}$	$\bar{M}_{min}$	$\bar{w}$	$\bar{M}_{max}$	$\bar{M}_{min}$
2	3·716	5·150	3·950	4·715	5·653	4·409
3	3·532	4·734	4·183	3·934	4·955	4·487
4	3·655	4·763	4·320	4·102	4·723	4·885
5	3·619	4·800	4·234	4·020	4·767	4·768
6	3·652	4·801	4·276	4·117	4·791	4·845
7	3·618	4·791	4·238	3·984	4·792	4·697
8	3·661	4·803	4·289	4·199	4·789	4·942
9	3·601	4·787	4·215	3·811	4·780	4·505
10	3·696	4·814	4·330	3·619	4·796	5·415
Sampath [4]	—	—	—	4·06	4·79	4·79
Morley [2]	—	—	—	—	—	—
Exact [7]	3·637	4·797	4·259	4·062	4·789	4·789

TABLE III

Simply supported Rhombic plate under uniform pressure
Effect of corner angle on convergence of deflection and moments at centre values for various orders of approximation (M)

Hyperbolic Trigonometric Functions : Least Square Collocation Method

$\bar{w} = (10^3 D/qa^4)\, w$; $\bar{M}_{max} = (10^2/qa^2)\, M_{max}$; $\bar{M}_{min} = (10^2/qa^2)\, M_{min}$
2α = Corner Angle: $\nu = 0 \cdot 3$

2α	30°			45°			60°		
M	$\bar{w}$	$\bar{M}_{max}$	$\bar{M}_{min}$	$\bar{w}$	$\bar{M}_{max}$	$\bar{M}_{min}$	$\bar{w}$	$\bar{M}_{max}$	$\bar{M}_{min}$
1	0·4852	2·397	0·836	2·045	4·614	1·922	4·859	6·448	3·389
2	0·4778	2·109	1·091	1·442	3·276	2·318	2·549	4·197	3·337
4	0·4188	1·909	1·124	1·326	3·222	2·212	2·559	4·248	3·332
8	0·4093	1·906	1·091	1·318	3·227	2·192	2·560	4·253	3·329
16	0·4081	1·906	1·086	1·317	3·227	2·190	2·560	4·253	3·329
Sampath (4)	0·411	1·908	1·099	1·327	3·230	2·210	2·256	4·284	3·322
Morley (2)	0·408	1·91	1·09	..	..	..	2·56	4·25	3·33
Exact (7)	0·4075	1·905	1·086	1·317	3·226	2·190	2·560	4·253	3·329

75°			80°			90°		
$\bar{w}$	$\bar{M}_{max}$	$\bar{M}_{min}$	$\bar{w}$	$\bar{M}_{max}$	$\bar{M}_{min}$	$\bar{w}$	$\bar{M}_{max}$	$\bar{M}_{min}$
7·813	7·121	4·923	8·486	7·016	5·352	8·953	6·336	5·921
3·636	5·088	3·947	3·986	5·334	4·096	4·584	5·602	4·363
3·626	4·786	4·251	3·866	4·873	4·460	4·133	4·952	4·683
3·635	4·794	4·258	3·867	4·855	4·486	4·073	4·810	4·777
3·636	4·796	4·259	3·867	4·857	4·485	4·044	4·807	4·779
..	..	..	3·877	4·858	4·497	4·06	4·79	4·79
..	..	..	3·87	4·86	4·48	4·06	4·79	4·79
3·637	4·797	4·259	3·869	4·856	4·489	4·062	4·789	4·789

the principal moment values at that point are presented in Table I for different α and M.

The convergence of w_{max} is, in general, rapid but oscillatory. It deteriorates slightly with increasing corner angle particularly so beyond 90°. The convergence for moments follows the same pattern. From the convergence trend, extrapolations are attempted for the actual values presented in the same table. It is noted that Conway's 5-term values 5 (with polar coordinates) agree with our 5-term results for $2\alpha = 60^\circ$ and 90°, while for $2\alpha = 120^\circ$, the results given by Conway differ from the authors by 6 per cent.

Rhombic Plates.—Convergence studies have been made for a range of corner angles 2α varying from 30° to 90°. The order of approximation M is varied from 2 to 10. The central deflection and principal moment values are presented in Table II for different α and M wherein the exact values obtained in Ref. 7 are also given.

The results exhibit oscillatory convergence for all angles. The convergence is rapid for smaller corner angles and deteriorates slowly as 2α increases. For $2\alpha > 60^\circ$, the solutions with $M = 6$ are within 2% of the exact values for M_x, M_y but the results deteriorate beyond $M = 6$. This is clearly a result of increasing computational errors due to progressive ill-conditioning of the equations. We observe that the 5-term values of Sampath [4] and the 6-term values of Morley [2] are also very accurate. As the Fourier expansion method gives rise to ill-conditioned matrices beyond $M = 6$ and $2\alpha = 60^\circ$, two other methods, the simple collocation and least square collocation, were tried. Preliminary calculations showed that the simple collocation method yielded highly oscillatory, unsatisfactory results even for small corner angles, while the least square collocation gave rapidly converging values. So computations were carried out by the least square method. Equidistant collocation points are chosen on AD with the number of points increased according to $r = 2^s$, $s = 0, 1, 2, 3$ and 4 ($r = 1, 2, 4, 8$ and 16), and the number of boundary equations m is chosen as twice the number of unknown constants n. This ratio of two has been found to be about the optimum to obtain an accurate solution [6]. The central deflection and principal bending moments are determined for corner angles $2\alpha = 30^\circ, 45^\circ, 60^\circ, 75^\circ, 80^\circ$ and 90° and presented in Table III.

The convergence of deflection values is rapid, monotonic upto $2\alpha = 60_\circ$ and oscillatory beyond $2\alpha = 60^\circ$. Convergence of moment values is oscillatory and rapid for all corner angles. The 8-term values compare very well with the exact values for corner angles upto 80°.

Parallelogrammic Plates.—From our experience with the earlier example, we choose the least square collocation method for satisfaction of the error equations. Convergence trends are studied for a range of skew angles

($\beta = 0°$, 15°, 30°, 45°, 60° and 70°) and a few side ratios ($a = 1\cdot0$, 1·5, 2·0, and 3·0). Equidistant collocation points are chosen on BC with the number of points increasing as $p = 2^s - 1$, $s = 2, 3, 4, 5$ (*i.e.*, $p = 3, 7, 15$ and 31). The central deflection and principal bending moments are determined for each combination of β, a and M.

The results in Table IV show that the convergence of solution is monotonic and very rapid for larger side ratios and smaller skew angles and becomes

TABLE IV

Simply supported parallelogrammic plate under uniform pressure
Effect of side ratio and skew angle on convergence of deflection and moments at centre values for various orders of approximation (M)

Hyperbolic Trigonometric Functions: Least Square Collocation Method

$\bar{w} = (10^3\,D/qc^4)\,w$; $\bar{M}_{max} = (10^2/qc^2)\,M_{max}$; $\bar{M}_{min} = (10^2/qc^2)\,M_{min}$
β = Skew Angle; a = Side Ratio; c = Length of Shorter Side
($\nu = 0\cdot3$)

a Side Ratio	β_M	$\bar{w}$ 2	4	8	16	Other Analyes	$\bar{M}_{max}$ 2	4	8	16
	0	4·036	4·063	4·062	4·062	4·06*	4·773	4·798	4·798	4·798
	15	3·825	3·649	3·638	3·627	..	4·776	4·596	4·598	4·598
	30	3·126	2·654	2·578	2·566	2·56 [2]	4·703	4·083	4·028	4·023
1·0	45	1·980	1·578	1·411	1·378	1·33 [4]	4·206	3·458	3·169	3·105
	60	0·697	0·620	0·544	0·436	0·408 [2]	2·771	2·508	2·250	1·802
	70	0·173	0·169	0·166	0·170	..	1·431	1·402	1·377	1·404
	0	7·728	7·724	7·724	7·724	7·72*	8·118	8·116	8·116	8·116
	15	7·042	6·894	6·884	6·884	..	7·879	7·733	7·726	7·725
	30	5·179	4·846	4·786	4·772	..	7·070	6·671	6·602	6·589
1·5	45	2·740	2·533	2·438	2·529	..	5·446	5·091	4·929	5·103
	60	0·785	0·762	0·739	0·736	..	3·040	2·966	2·892	2·880
	70	0·178	0·177	0·176	0·177	0·178*	1·459	1·455	1·452	1·452
	0	10·13	10·13	10·13	10·13	10·1*	10·17	10·17	10·17	10·17
	15	9·061	8·970	8·964	8·963	..	9·710	9·624	9·619	9·618
	30	6·298	6·116	6·081	6·078	..	8·297	8·090	8·051	8·047
2·0	45	3·057	2·967	2·924	2·912	..	5·945	5·798	5·729	5·709
	60	0·907	0·801	0·794	0·795	..	3·105	3·086	3·066	3·066
	70	0·178	0·178	0·178	0·178	0·178*	1·462	1·461	1·461	1·461
	0	12·23	12·23	12·23	12·23	12·2*	11·89	11·89	11·89	11·89
	15	10·75	10·72	10·72	10·72	..	11·18	11·15	11·15	11·15
	30	7·107	7·062	7·054	7·052	..	9·153	9·106	9·096	9·095
3·0	45	3·227	3·213	3·206	3·204	..	6·208	6·186	6·176	6·173
	60	0·813	0·813	0·813	0·813	..	3·124	3·123	3·122	3·122
	70	0·178	0·178	0·178	0·178	0·178*	1·462	1·462	1·462	1·462

slower for smaller side ratios and larger skew angles. This is not surprising because, with increasing skew angle and decreasing side ratio, there is an increase in the ratio between the peripheral length on which the boundary conditions are approximately satisfied and that on which they are exactly satisfied. It is also observed that the convergence of moment values is somewhat slower than that of deflection. Comparison of the present analysis and results with other solutions and procedures reported in literature shows that, for the type of problems considered, the use of rectangular Cartesian coordinates compares favourably with the other methods.

7. Concluding Remarks

It has been demonstrated with examples that hyperbolic trigonometric functions in rectangular Cartesian coordinates can be effectively used for analysing problems when the domain is rectilinear but non-rectangular. It has been found that the use of such functions require a proper choice of

			M_{min}		
Other Analyes	2	4	8	16	Other Analyes
4·79*	4·804	4·789	4·789	4·789	4·79*
..	4·640	4·476	4·460	4·459	..
4·25 [2]	3·998	2·682	3·587	2·567	3·33 [2]
3·23 [4]	2·699	2·577	2·473	3·419	2·21 [4]
1·91 [2]	1·159	1·205	1·244	1·044	1·08 [2]
..	0·464	0·476	0·477	0·466	..
8·12*	4·987	4·984	4·984	4·984	4·92*
..	4·662	4·633	4·630	4·630	..
..	3·690	3·677	3·669	3·668	..
..	2·277	2·336	2·356	2·398	..
..	0·995	1·024	1·052	1·053	..
1·462*	0·442	0·444	0·446	0·445	0·439*
10·17*	4·635	4·635	4·635	4·635	4·64*
..	4·283	4·290	4·290	4·290	..
..	3·313	3·351	3·358	3·358	..
..	2·048	2·097	2·118	2·125	..
..	0·952	0·962	0·972	0·971	..
1·462*	0·439	0·439	0·440	0·440	0·439*
11·89*	4·062	4·063	4·063	4·063	4·06*
..	3·755	3·763	3·764	3·764	..
..	2·940	2·960	2·964	2·964	..
..	1·902	1·914	1·920	1·921	..
..	0·938	0·939	0·940	0·940	..
1·462	0·439	0·439	0·439	0·439	0·439*

* Corresponds to Values from Ref. 1.

the method of satisfaction of boundary conditions in order to obtain satisfactory results with limited effort. With the Fourier expansion method for non-rectangular shapes, convergence is obtained up to a certain stage and then the solution deteriorates due to the ill-conditioning of the equations. On the other hand, the least square collocation procedure has invariably yielded good convergence whenever it has been applied. Comparison of the results by the present analysis with those by other solutions and procedures reported in literature, shows that for the type of problems considered they are highly satisfactory. Thus it may be concluded from the study that, in the direct method of analysis, it is not always necessary to go in for coordinate systems specialised to the shape of the domain.

Further one may conclude that, for some skew domains such as the parallelogram, the rectangular Cartesian coordinates can be superior to oblique coordinates. It would be enlightening to compare the results with rectangular and parallelogrammic elements for the finite element analysis of skew plates.

Even though only simply supported edges and uniform transverse loading have been considered in the present study, other types of homogeneous edge supports and applied loadings can be analysed by choosing proper functional forms.

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