Short Communication

A note on the thermal buckling of a thin annular circular plate of variable thickness

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Abstract

In this note, the buckling of a heated annular circular plate of thickness varying as an exponential function of radial distance is considered. The general stability criterion has been derived.

Key words : Thermal buckling, Whittaker functions, annular plate, stability criterion.

1. Introduction

The literature on thermal buckling of plates is rare. Nowacki[4] has studied the buckling of a heated rectangular plate under various boundary conditions. The buckling and curling of a heated thin circular plate of constant thickness has been investigated by Mansfield[3] when the temperature varies through the thickness and the edges are restrained. A similar problem for the buckling of simply supported plate under symmetrical temperature distribution has been described by Klosner and Forry[2] by using Rayleigh-Ritz method. Sarkar[5] has discussed the thermal buckling of a circular plate taking its thickness to be constant.

This paper records the discussion of a buckled annular circular plate when its thickness varies as the exponential function of radial distance under stationary temperature distribution. The edge of the plate is restrained. The general stability criterion has been obtained from which the critical compression and critical temperature can be determined for different values of parameters involved.

2. Statement of the problem and fundamental equations

Let us consider an isotropic thin annular circular plate of variable thickness $h = h_0 e^{-qr}$, h_0 and q being constants and r the radial distance. The plate is supposed to occupy the space $-h/2 \leq z \leq h/2$ and is subjected to stationary temperature distribution. We further assume that the edge of the plate is restrained so that the displacement in the plane of the plate is zero. The deflections are supposed to be governed by the linear theory of thermo-elasticity. The compression in the middle plane of the plate is assumed to be $N_r = N_0 e^{-2qr}$.

The equilibrium equation relating the components of moment for the problem at hand is given as[7]

$$\left(M_r + \frac{dM_r}{dr} \cdot dr\right)(r + dr)\, d\theta - M_r \cdot r d\theta - M_\theta \cdot dr d\theta$$
$$+ hN_r \phi r d\theta dr = 0 \tag{2.1}$$

where $d\theta$ is the angle subtended between the two radial planes and

$$M_r = D\left(\frac{d\phi}{dr} + \frac{v}{r}\phi\right) - \frac{M_T}{1 - v}$$

$$M_\theta = D\left(\frac{vd\phi}{dr} + \frac{\phi}{r}\right) - \frac{M_T}{1 - v}$$

$$M_T = \alpha E \int_{-h/2}^{+h/2} Tzdz$$

$$N_r = K\left(\frac{dU}{dr} - \frac{v}{r} U\right) - \frac{N_T}{1 - v} = N_0 e^{-2qr}$$

$$N_T = \alpha E \int_{-h/2}^{+h/2} Tdz$$

$$K = \frac{Eh}{(1 - v^2)}, \quad h = h_0 e^{-qr}, \quad D = D_0 e^{-3qr}$$

where

$$D_0 = \frac{Eh_0^3}{12(1 - v^2)} \tag{2.2}$$

Neglecting higher order quantities, equation (2.1) becomes

$$M_r + r\frac{dM_r}{dr} - M_\theta + h N_r \phi r = 0. \tag{2.3}$$

Substituting values of M_r, M_θ, h, D and N_r in (2.3) we get

$$\frac{r^2 d^2\phi}{dr^2} + (1 - 3qr)\frac{rd\phi}{dr} + \left(\frac{h_0 N_0}{D_0} r^2 - 3vq^{r-1}\right)\phi$$
$$= \frac{r^2 e^{3qr} \frac{dM_T}{dr}}{D_0 (1 - v)} \tag{2.4}$$

Putting $\rho = r/a$, $\beta = b/a$ the outer and inner boundaries correspond to $\rho = 1$ and $\rho = \beta$. Then (2.4) becomes

$$\rho^2 \frac{d^2 \phi}{d\rho^2} + (1 - k\rho)\, \rho \frac{d\phi}{d\rho} + (\lambda\rho^2 - \nu k \rho - 1)\, \phi = \frac{a\, \rho^2 e^{k\rho} \dfrac{dM_T}{d\rho}}{D_0 (1 - \nu)} \tag{2.5}$$

where

$$k = 3aq, \quad \lambda = \frac{h_0 N_0 a^2}{D_0} \quad \text{are constants.}$$

We further suppose that the outer boundary of the plate is clamped and supported and the inner boundary is clamped. Then the appropriate boundary conditions of the problem are[1]

$$\begin{aligned} \phi &= 0 \quad \text{when} \quad \rho = 1, \quad \rho = \beta \\ \omega &= 0 \quad \text{when} \quad \rho = 1. \end{aligned} \tag{2.6}$$

3. Solution of the problem

Let the plate be subjected to the temperature field

$$T(r, z) = \tau_0 (r) + z\, \tau (r). \tag{3.1}$$

For a plate of medium thickness buckling occurs when $\tau = 0$, $\tau_0 \neq 0$. Therefore $M_T = 0$ and from (2.5) we get

$$\rho^2 \frac{d^2 \phi}{d\rho^2} + (1 - k\rho)\, \rho \frac{d\phi}{d\rho} + (\lambda\rho^2 - \nu k\rho - 1)\, \phi = 0. \tag{3.2}$$

Putting $X = k\rho$, $\phi = \psi\, e^{X/2}$ the above equation becomes

$$X^2 \frac{d^2 \psi}{dX^2} + X \frac{d\psi}{dX} + \left\{ \left(\frac{1}{2} - \nu\right) X + \left(\frac{\lambda}{k^2} - \frac{1}{4}\right) X^2 - 1 \right\} \psi = 0. \tag{3.3}$$

Now writing $\psi = X^{-1/2} \xi$ in equation (3.3) we obtain

$$4X^2 \frac{d^2 \xi}{dX^2} + \left\{ -3 + 2(1 - 2\nu) X - \left(1 - \frac{4\lambda}{k^2}\right) X^2 \right\} \xi = 0 \tag{3.4}$$

Again, putting

$$\sqrt{1 - \frac{4\lambda}{k^2}}\, X = \eta, \qquad \frac{1 - 2\nu}{2\sqrt{-1 \frac{4\lambda}{k^2}}} = \epsilon$$

in (3.4) we get

$$\frac{d^2 \xi}{d\eta^2} + \left\{ -\frac{1}{4} + \frac{\epsilon}{\eta} + \frac{\frac{1}{4} - 1}{\eta^2} \right\} \xi = 0. \tag{3.5}$$

The solution of (3.5) is given by

$$\xi = A W_{\epsilon,1}(\eta) + B\, W_{-\epsilon,1}(-\eta) \tag{3.6}$$

where $W(\eta)$ are Whittaker functions[6] and A, B are constants. Hence

$$\phi = (k\rho)^{-1/2} e^{k\rho/2} [A W_{\epsilon,1}(y\rho) + B W_{-\epsilon,1}(-y\rho)]$$

where

$$y = \sqrt{1 - \frac{4\lambda}{k^2}} \tag{3.7}$$

since

$$a\phi = -\frac{d\omega}{d\rho}$$

we obtain

$$\omega = A F_1(\rho) + B F_2(\rho) + C \tag{3.8}$$

where $F_1(\rho)$ and $F_2(\rho)$ are functions of ρ obtained after integration and C is a constant of integration. Introducing the boundary conditions (2.6) in (3.8) and eliminating constants A, B, C we get

$$\begin{vmatrix} W_{\epsilon,1}(y) & W_{-\epsilon,1}(-y) & 0 \\ W_{\epsilon,1}(y\beta) & W_{-\epsilon,1}(-y\beta) & 0 \\ F_1(1) & F_2(1) & 1 \end{vmatrix} = 0 \tag{3.9}$$

The determinant on expansion yields

$$W_{\epsilon,1}(y)\, W_{-\epsilon,1}(-y\beta) - W_{-\epsilon,1}(-y)\, W_{\epsilon,1}(y\beta) = 0. \tag{3.10}$$

This is the general stability criterion from which the critical compression and critical temperature can be determined for different values of the parameters if the roots of equation (3.10) are known.

4. Nomenclature

ω = Deflection of the middle surface

$\phi = -\frac{d\omega}{dr}$ = Slope of the middle surface

h = Thickness of the plate, a variable quantity

$D \quad = \dfrac{Eh^3}{12(1-\nu^2)}$ = Flexural rigidity

α = Coefficient of thermal expansion

E, ν = Young's modulus and Poisson's ratio

M_r, M_θ = Components of moment

N_r = Compression in the middle plane of the plate, a variable quantity

a, b = Radii of the outer and inner boundaries of the plate

$\beta \quad = \dfrac{a}{b}$

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