Stability of nonlinear systems *via* Bellman-Gronwall Lemma

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Abstract

Governing equation of a system with a linear element and a nonlinear time varying gain can be expressed in an integral equation form. Bellman-Gronwall Lemma can be applied to this integral equation to derive bounded-input-bounded-output stability and Liapunov stability. The criterion derived leads to less conservative results, compared with other methods, such as circle criterion, for cases where the fluctuations in the gain are large and rapid. This is so because in the new criterion the gain appears in the argument of a time integral, which significantly reduces the effect of fluctuations

Key words: Stability, control system, time varying system.

1. Introduction

Sufficient conditions for stability for a Lure problem having a single memoryless nonlinear gain with large and rapid fluctuations, when derived with the help of available methods, are generally very conservative [1, 2, 3]. This is so, because, the criteria are derived for the maximum value and rate of change of the gain. If the governing equation is expressed as an integral equation instead of a differential equation, Bellman-Gronwall Lemma can be used to derive sufficient conditions for stability. In this condition, the gain appears in the argument of a time integral which significantly reduces the effect of fluctuations. As such, this method is specially suitable for cases where the fluctuations in the gain are large and rapid. For example, if the fluctuations are in the form of delta functions, in general, other available methods lead to very conservative results, or fail to be applicable.

2. The type of systems considered

Systems described by equations of the following type are considered;

$$Y(s) = G(s)\ (R(s) - U(s)) \tag{1}$$

$$u(t) = \phi(t,\ y(t)) \tag{2}$$

where R, U and Y are Laplace transforms of r, u and y, $r(t)$ is an external input, and ϕ is such that for all $t \geq 0$

$$-B_0 \leq A - b(t) \leq \frac{\phi(t, y)}{y} \leq A + b(t) \leq B_0 < \infty \tag{3}$$

$$\phi(t, o) = 0 \tag{4}$$

where A, B_0 and b are non-negative.

Let H, W and ψ be introduced such that

$$H(s) = \frac{G(s)}{1 + AG(s)} \tag{5}$$

$$Y(s) = -H(s)[W(s) - R(s)] \tag{6}$$

$$w(t) = \psi(t, y(t)) = \phi(t, y(t)) - Ay \tag{7}$$

where

$$|\psi(t, y(t))| \leq b\,y(t) \text{ for all } t \geq 0 \tag{8}$$

The output, y, can now be expressed as,

$$y(t) = y_i(t) - \int_0^t h(t - t')[\psi(t', y(t')) - r(t')]\,dt' \tag{9}$$

where $y_i(t)$ represents initial condition response. Let $X(t)$ represent the state of the system at time t. If $G(s)$ is n^{th} order, then $X : R^+ \to c^n$. In this case, $\|X(t)\|$ represents Euclidean norm of X. If G represents an infinite dimensional system, such as distributed parameter or a time delay system, then a suitable normed space has to be chosen.

3. Theorem

(*a*) If there exist positive constants M_0, M_1, a and h_0 such that for all $t \geq 0$

$$|y_i(t)| \leq \|X(o)\| \, M_0 \, e^{at} \tag{10}$$

$$|h(t - t')| \leq h_0 \, e^{-a(t-t')} \tag{11}$$

and

$$\int_0^t h_0 b(t') \operatorname{Exp}\left[-at + \int_{t'}^t h_0 b(p)\,dp\right] dt' \leq M_1 < \infty \tag{12}$$

then the trivial solution of equations (1) and (2), for $r(t) = 0$, is globally Liapunov-stable with respect to $(|u(t)| + |y(t)|)$, that is, given any $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$, such that,

$$(|u(t)| + |y(t)|) \leq \varepsilon \text{ for all } t \geq 0, \text{ if } \| X(0) \| \leq \delta.$$

(*b*) The trivial solution is globally asymptotically stable with respect to $(|u| + |y|)$ if, in addition to conditions (10)–(12), we have

$$\int_0^t h_0 b(t') \operatorname{Exp}\left[-at + \int_{t'}^t h_0 b(p)\, dp\right] dt \to 0 \tag{13}$$

as $t \to \infty$.

(*c*) The system described by equations (1) and (2) is bounded-input-bounded-output stable if, in addition to conditions (10) and (11), $|r(t)|$ is bounded and

$$\int_0^t h_0 b(t') \operatorname{Exp}\left[a(t'-t) + \int_{t'}^t h_0 b(p)\, dp\right] dt' \leq M_2 < \infty \tag{14}$$

for all $t \geq 0$

Proof: Suppose,

$$|r(t)| \leq R \text{ for all } t \geq 0 \tag{15}$$

It follows from relations (8), (9), (11) and (15) that

$$|y(t)| \leq g(t) + \int_0^t h_0 b(t')\, |y(t')|\, e^{-a(t-t')}\, dt' \tag{16}$$

for all $t \geq 0$

where g is defined as

$$g(t) = \left(\| X(o) \|\ M_0 - \frac{Rh_0}{a}\right) e^{-at} + \frac{Rh_0}{a} \tag{17}$$

Introduce $v(t)$ such that

$$v(t) = y(t)\, e^{at} \tag{18}$$

Inequality (16) can be rewritten as

$$v(t) \leq g(t)\, e^{at} + \int_0^t h_0 b(p)\, v(p)\, dp \;\forall\; t \geq 0 \tag{19}$$

Application of Bellman-Gronwall Lemma to inequality (19) leads to[4]

$$v(t) \leq g(t)\, e^{at} + \int_0^t h_0 b(t')\, g(t') \operatorname{Exp}\left[at' + \int_{t'}^t h_0 b(p)\, dp\right] dt' \text{ for all } t \geq 0 \tag{20}$$

or

$$|y(t)| \leq g(t) + \int_0^t h_0 b(t') \, g(t') \text{ Exp } [a(t' - t) + \int_{t'}^t h_0 b(p) \, dp] \, dt', \text{ for all } t \geq 0 \tag{21}$$

Consider the case where there is no external input, that is, $R = 0$. It can be seen from relations (12), (17) and (21) that

$$|y(t)| \leq \| X(0) \| M_0 e^{-at} (1 + \int_0^t h_0 b(t') \, dt' \text{ Exp } \int_{t'}^t h_0 b dp)$$
$$\leq \| X(0) \| M_0 (1 + M_1) \text{ for all } t \geq 0 \tag{22}$$

Hence, from inequalities (3) and (22)

$$|u(t)| + |y(t)| \leq \| X(0) \| M_0 (1 + M_1)(1 + B_0) \tag{23}$$

for all $t \geq 0$

Part (*a*) of the theorem is proved.

If condition (13) is satisfied, it is seen from inequality (22) that $y(t) \to 0$, as $t \to \infty$ Hence, $u(t)$ also tends to zero. Part (*b*) is proved.

Consider the case where $\| X(0) \| = 0$. From inequalities (14), (15), (17) and (21), it follows that

$$|y(t)| \leq \frac{Rh_0}{a} (1 + M_2) \text{ for all } t \geq 0 \tag{24}$$

part (*c*) is proved.

Example : Consider the following Hill's equation ;

$$\ddot{y}(t) + 2a\dot{y} + (A + f(t)) \, y = 0 \tag{25}$$

where $f(t)$ is bounded and periodic with zero mean value, and $A > a^2$. Circle criterion predicts stability if[2]

$$|f(t)| \leq 2a\sqrt{A - a^2} \text{ for all } t \geq 0 \tag{26}$$

In order to apply the criterion developed in this paper, let us choose

$$H(s) = \frac{1}{s^2 + 2as + A} \tag{27}$$

$$\psi(t, y) = f(t) \, y \tag{28}$$

then

$$b(t) = |f(t)| \leq b_{max} \tag{29}$$

$$h_0 = \frac{1}{\sqrt{A - a^2}} \tag{30}$$

Since, $f(t)$ is periodic, $b(t)$ is also periodic. Let

$$b(t) = |f(t)| = f_0 + f_1(t) \tag{31}$$

where f_0 is constant and f_1 is periodic with zero mean.

Conditions (10) and (11) of the theorem are satisfied in this case. Condition (12) requires that there be an M_1 such that

$$\int_0^t h_0 b(t') \operatorname{Exp}\left[-at + h_0 f_0 (t - t') + \int_{t'}^t f_1(p)\, dp\right] dt' < M_1 < \infty \tag{32}$$

Since, f_1 is periodic and bounded, an m can be found such that

$$\left| \int_{t'}^t f_1(p)\, dp \right| \leq m \text{ for all } t,\ t',\ \text{such that } 0 \leq t' \leq t \tag{33}$$

Inequality (32) is satisfied, if for all $t \geq 0$

$$b_{max}\, e^{(h_0 f_0 - a)t} \quad (1 - e^{-h_0 f_0 t}) \leq M_1\, e^{-m} f_0 \tag{34}$$

It can be seen that if

$$f_0 \leq \frac{a}{h_0} = a\sqrt{A - a^2} \tag{35}$$

inequality (34), and hence condition (12), is satisfied for sufficiently large M_1. So that inequality (35) ensures stability. It can be shown that if

$$f_0 < a\sqrt{A - a^2} \tag{36}$$

Condition (13) is also satisfied. In which case global asymptotic stability is ensured.

If

$$f(t) = b_0 \cos^3 \omega t \tag{37}$$

then circle criterion predicts stability for

$$|b_0| < a\sqrt{A - a^2} \tag{38}$$

whereas, the above theorem predicts stability for

$$|b_0| < \frac{3\pi}{4} a\sqrt{A - a^2} \tag{39}$$

which is less conservative.

We can also find an example where other methods do not give any meaningful results. Consider the equation

$$\ddot{y}\,(t) + 2a\dot{y} + \left(A + \sum_{-\infty}^{\infty} \delta\,(t - n)\,(-1)^n\right) y = 0 \tag{40}$$

Sufficient condition for global stability, as given by condition (36) is

$$a\sqrt{A - a^2} > 1 \tag{41}$$

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