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The turbulence problem : A survey

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Abstract

This paper is a survey of the present state of understanding, prediction and control of turbulent flows, intended to be intelligible also to workers not directly involved in research in turbulence. 'Complex' turbulent flows, as affected by such factors as rotation, additional rates of strain, compressibility, etc., are not covered by the survey, but an attempt has been made to include all aspects of simpler flows in an incompressible fluid. Inevitably, the survey is strongly influenced by the work done by the author himself and his colleagues.

A major conclusion of the survey is that there are still wide gaps between the experimental discoveries being made on the structure of turbulent flows, the mathematical tools necessary for describing such structure, the numerical models invented for predicting turbulent shear flow behaviour, and the devices being tried for turbulence management. A great deal of work is necessary in all areas before these gaps can be bridged.

Key words : Turbulent flows, incompressible fluid, turbulence management, irregular eddying motions, Reynolds number, chaotic fluid motions, boundary layer.

1. Introduction

"*I had less difficulty in the discovery of the motion of heavenly bodies in spite of their astonishing distances, than in the investigation of the movement of flowing water before our very eyes.*"

Galileo

Although Galileo, Leonardo and many others devoted some thought to the irregular eddying motions that are so familiar in air and water, the scientific study of the problem of turbulent fluid flow may be said to have begun only about a hundred years ago, with Reynolds's famous paper of 1883[1]. Reynolds made the important

distinction between laminar and turbulent flow—or between 'direct' and 'sinuous' motion, as he called them—and showed that the criterion governing transition from one state of flow to the other is the non-dimensional group that has since come to be known as the Reynolds number. (Incidentally, Reynolds's use of the word 'sinuous', for what later came to be called 'turbulent' following Lord Kelvin (1887), appears now to be not without some virtue, in the light of recent work that indicates a far greater degree of spatial order in the turbulent flow than had been generally thought for a long time : see section 7 below.) In spite of much inspired and ingenious work by experimenters and theoreticians, and by engineers, mathematicians and physicists, however, it is to this day not possible to predict such simple gross parameters as *e.g.* the pressure loss experienced by a fluid in turbulent flow through a pipe : the loss is of course *known*, but has not yet been *deduced* from first principles (say the Navier-Stokes equations), without having to appeal to pipe-flow experiments at some stage. Although analyses of various kinds enable one to reduce the amount of testing required, it cannot yet be eliminated.

A more complicated problem is that presented by the flow past a simple circular cylinder. Figure 1 shows the data collected by Roshko[2] on the base pressure coefficient, measured at the rear of the cylinder, as a function of the Reynolds number. The function shows at least three maxima and two minima ; and it is still a matter of debate what the limiting value of the pressure coefficient will be as Reynolds number tends to infinity. There is no satisfactory theory that can predict the pressure variations shown in fig. 1 (but a fascinating account of the phenomena responsible has been given by Roshko and Fiszdon[3]).

The difficulty encountered by Galileo therefore still persists. Feynman has called turbulence the greatest puzzle in classical physics. Turbulence is so often the natural state of fluid motion—in technological applications, and in the atmosphere and the oceans—that the potential rewards for being able to 'manage' it (see section 8 below) would be immense : Reynolds himself seems to have realized this, and listed the factors conducive to direct and sinuous motion (see Table I).

The defining characteristics of turbulent motion are that it is :

— unsteady,
— chaotic,
— rotational.

The field variables in turbulent flow are time-dependent ; they may of course be steady in the mean, *i.e.*, statistically stationary. If a turbulent field is sought to be expressed as a superposition of sine waves, it will be found that all wave numbers would have to be included. The element of 'chaos' is essential, but all chaotic fluid motions are not necessarily turbulent ; *e.g.*, acoustic noise and random gravity

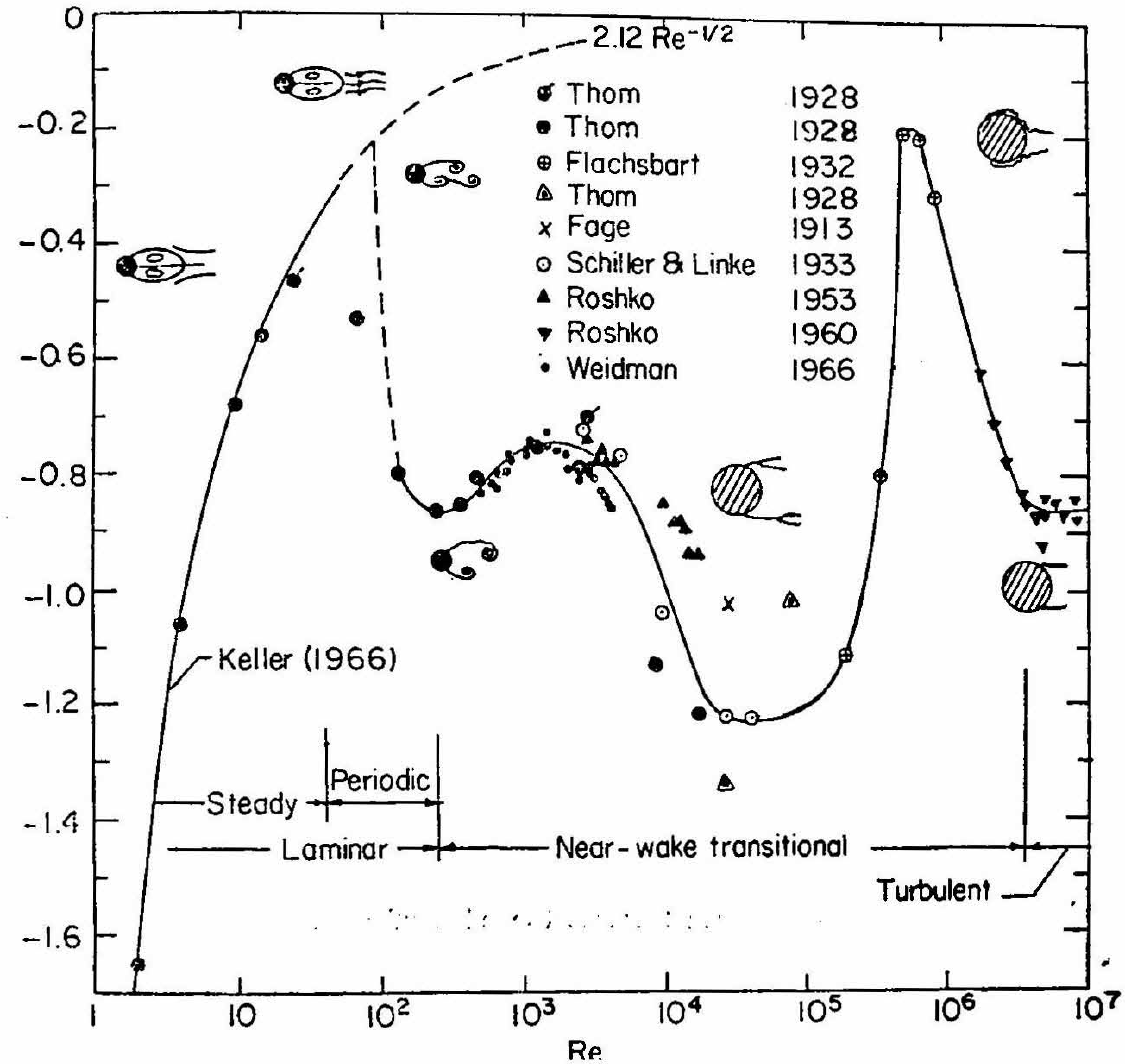


FIG. 1. Experimental data on pressure at the rear stagnation point in flow past a circular cylinder (Roshko[2]) ; the pressure p_b is expressed as the coefficient $(p_b - p_\infty)/\frac{1}{2}\rho U^2$, where p_∞ and $\frac{1}{2}\rho U^2$ are respectively the static and dynamic pressure at upstream infinity. Seven different flow regimes, corresponding to different Reynolds number ranges, are also sketched in the diagram, and indicate (in order of increasing Re) : (i) steady laminar wake with recirculating flow, (ii) unsteady far wake, (iii) vortex shedding, (iv) transition in wake, (v) transition in free shear layer following separation, (vi) laminar separation followed by transition and turbulent reattachment, comprising a separation bubble, followed by turbulent separation, (vii) transition to turbulent boundary layer on cylinder, followed directly by turbulent separation.

waves both involve chaotic motion, but are distinguished from turbulence in that they are irrotational, whereas turbulence is basically rotational. Another kind of chaotic potential flow occurs, *e.g.*, just outside a turbulent jet, being induced by the (rotational) turbulence within the jet[4] ; but this 'potential turbulence' decays as it cannot sustain

Table I*

Circumstances conducive to *Direct or steady motion*	*sinuous or unsteady motion*
Viscosity or fluid friction which continually destroys disturbances (Treacle is steadier than water).	Particular variation of velocity across the stream, as when a stream flows through still water.
A free surface.	Solid bounding walls.
Converging solid boundaries.	Diverging solid boundaries.
Curvature with the velocity greatest on the outisde.	Curvature with the velocity greatest on the inside.

* From Reynolds[128].

itself*, and is therefore of no great interest. Finally, turbulent motions are generally three-dimensional, but two-dimensional turbulence is neither trivial nor academic : a form of it occurs in large scale motions in the atmosphere[6], and in certain magneto-hydrodynamic flows[7].

We may in summary say that turbulence is chaotic vorticity, although there may be associated irrotational motions as well.

A striking feature of turbulence is that it is intermittent—in many rather different senses. First of all, turbulent shear flows have a surprisingly sharp although convoluted and fluctuating boundary (fig. 2); wakes, jets and boundary layers all provide examples[8–10]. Thus, a probe placed towards the edge of such a flow shows a trace of the kind illustrated in fig. 3, revealing distinct periods of activity when a turbulent patch passes the probe. Secondly, if the velocity signal at any point in a turbulent flow (not necessarily sheared) is filtered at a sufficiently high frequency, it is again found that there are periods of intense activity followed by relative lulls ; thus, the dissipation which occurs at these high wave numbers also has a certain spotty character[11,12] (see Kraichnan[13] for a simple argument suggesting why the fine-scale strutucre of turbulence should be intermittent). Thirdly, during transition from laminar to turbulent flow, turbulence often first appears in relatively well-defined 'spots', entirely surrounded (both upstream and downstream) by laminar flow. For example, in the boundary layer on a flat plate, turbulent and laminar flow are not separated from each other across a jagged line spanning the flow, as had once been thought ; instead, there are many 'islands' of turbulence, which grow downstream and eventually cover the whole flow[14,15]. Although these intermittencies have been measured, and sometimes there are satisfactory models for them[16], no convincing *dynamical* theories of the phenomenon exist.

* Phillips's theory, predicting a decay of energy in the potential turbulence as the inverse fourth power of distance away from the jet, is in surprisingly good agreement with measurements (Bradshaw)[5].

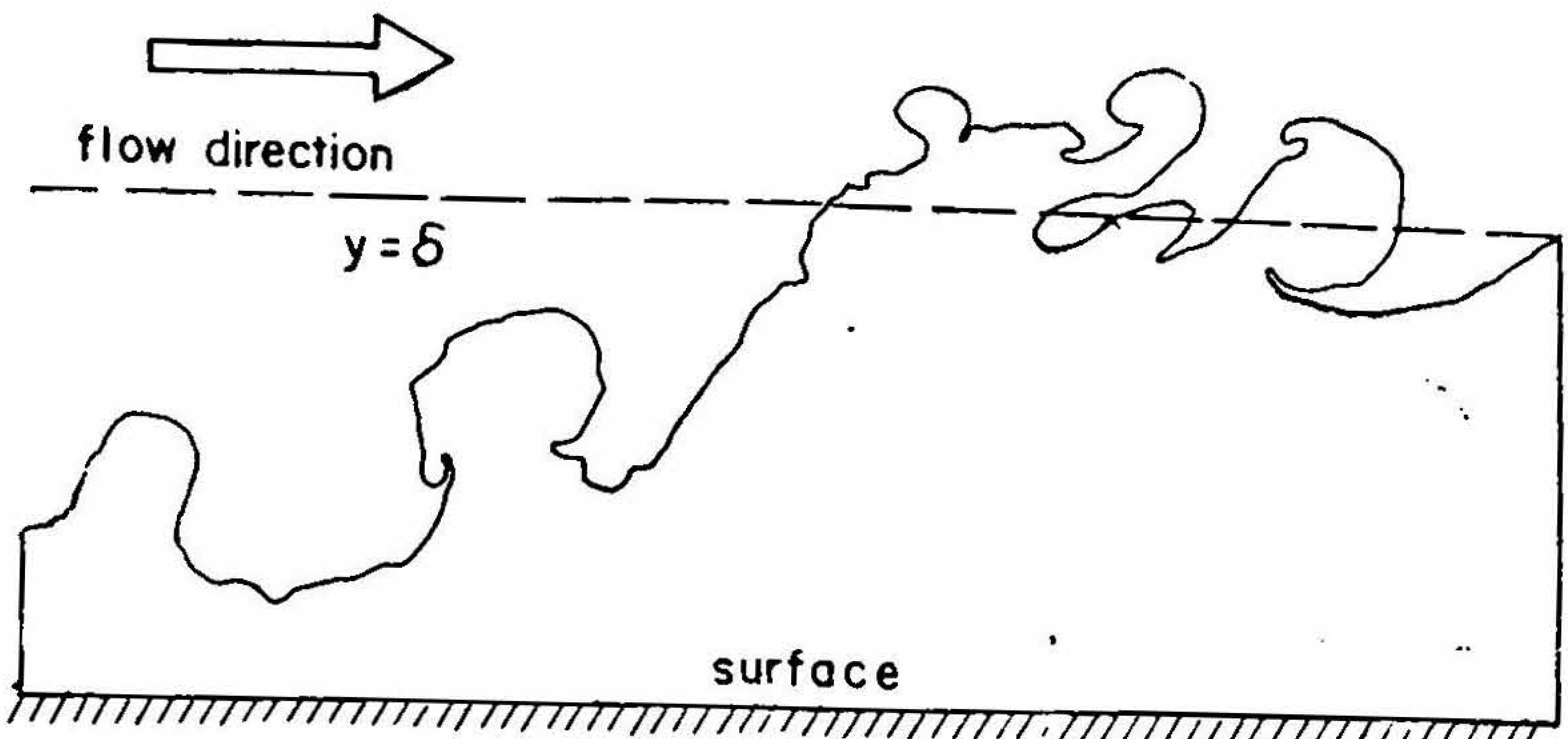


FIG. 2. The edge of a turbulent boundary layer, as seen in smoke-flow visualizations.

Finally, the production of turbulent energy in many flows, including in particular the turbulent boundary layer, is very ' bursty ', in the sense that production does not occur uniformly in time but in short, intense bursts of activity[17].

There is no evidence whatsoever to suggest that the phenomenon of turbulence is ' beyond ' the Navier–Stokes equations ; certainly, turbulence is entirely consistent with them, whose consequences—in so far as they can be deduced at all—are borne out by all measurements made to-date. It is sometimes argued that the phenomenon of turbulence must be traced to the motion of molecules. All phenomena must of course be capable of explanation eventually in terms of molecules (or even more fundamental particles), but they cannot hold the key to turbulence as fluids with vastly different *molecular* structure (*e.g.*, air and water) exhibit the same *flow* structure in equivalent flow conditions (*e.g.*, at the same Reynolds number). In any case, at normal temperature and pressure the mean free path air is about 10^{-7} m, which is far smaller than in the smallest eddies of interest in turbulent flow, which in typical laboratory or atmospheric situations would be of the order of 10^{-4} m. Thus, except in cases where giant molecules are involved, we may safely assume that it is unnecessary to invoke molecules in the study of turbulence.

The century of studies of turbulent flow is marked by certain well-defined advances ; Table II lists some of these.

The present paper is a general survey of the present status of understanding and research on the turbulence problem. It is not claimed to be complete : effects of compressibility, body force, rotation, etc., are not considered at all. On the other hand, all aspects of low speed turbulent flow in the absence of the complicating factors mentioned are sought to be covered, in the hope that the reader can get a

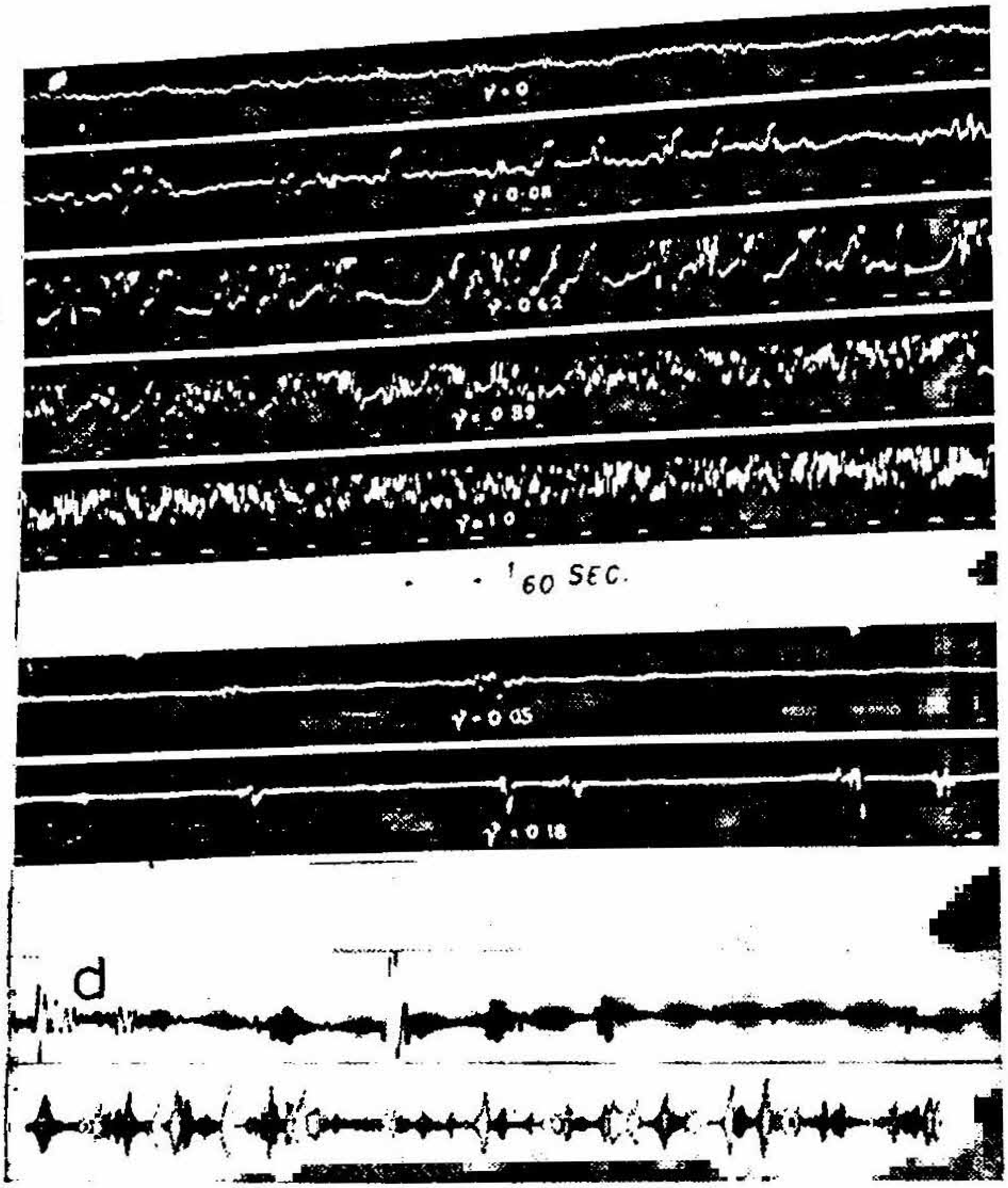


FIG. 3. Hot wire traces of longitudinal velocity in boundary layers ; values of the intermittency γ are also marked on most traces.

A1-A5, constant pressure boundary layer undergoing transition from laminar to turbulent flow, after being tripped by wire on surface. The traces were all taken at a distance of about 0·3 mm above the surface, but at different streamwise stations. Tollmien-Schlichting instability waves can be observed in traces A1, A2. Note the striking changes in velocity in traces A2, A3, corresponding to the passage of turbulent spots past the probe : there is clearly an appreciable change in both the fluctuating and mean velocity levels within the spot. A5, fully turbulent flow. The slight flattening of the traces near the top is due to overloading of the instrumentation.

B1, B2 show intermittency towards the outer edge of the boundary layer ; there are strong fluctuations as each turbulent tongue (see fig. 3c) crosses the probe, but clearly no appreciable difference in mean velocities.

C1, C2 : filtered longitudinal velocity signals in fully turbulent boundary layer, revealing bursts or pulses of high activity. C1 is band-passed through 800–5000 Hz, at $Re_\theta = 1110$. C2 is filtered at 6 kHz, $Re_\theta = 9450$. Note the richer high-frequency structure at the higher Reynolds number[16,105].

feel for the present position (as seen by the author !). For a more detailed account of some of the topics dealt with here, the reader is referred to Liepmann[18,19], Batchelor[20], Bradshaw[21,22], Townsend[23], Durst *et al*[24] and the papers on turbulence in Narasimha and Deshpande[25].

Table II

History of studies of turbulent flow

Event	*Reference*
Formal recognition of two states of fluid motion ; discovery of the Reynolds number as the criterion governing transition, 'Reynolds' averaging.	Hagen (1839) Reynolds (1883) Reynolds (1894)
Use of the word 'turbulence'	Kelvin (1887)
Introduction of eddy viscosity	Boussinesq (1897)
Study of eddies in atmosphere ; diffusion	Taylor (1915, 1921)
Mixing length theory	Taylor (1915), Prandtl (1925)
Hierarchy of eddies ; cascade process	Richardson (1922)
Statistical theory, use of generalized harmonic analysis	Taylor (1935)
Discovery of 'log law' in flow near surface, and of outer 'defect' law	Prandtl (1925) Karman (1930)
Theory of spectrum at high wave numbers	Kolmogorov (1941)
Exploration of possibilities of electronic computation	Von Neumann (1940s)
Development of statistical theory for homogeneous turbulence	Batchelor (1953)
Role of instability in transition to turbulence	Schubauer and Skramstad (1947)
The 'turbulent spot'	Emmons (1951), Schubauer and Klebanoff (1955)
Concepts of equilibrium and self-preservation	Clauser (1954), Townsend (1956)
The bursting phenomenon in boundary layers	Kline *et al* (1967)
Evidence for coherent structures	Brown and Roshko (1971, 1974)
Use of 'big computing' for calculation of turbulent flows	Now a minor industry !

2. The Reynolds equations

As just mentioned, we shall consider here only the simple incompressible fluid with a constant viscosity and no body force. The Navier–Stokes equations for such a fluid are

$$\text{div } u = 0, \tag{2.1}$$

$$\frac{du}{dt} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u, \tag{2.2}$$

representing respectively conservation of mass and momentum. Here $u = u(x, t)$ is the instantaneous velocity vector, p is the pressure, ρ the density and ν the kinematic viscosity.

A consequence of (2.2) is that the vorticity $\omega = \text{curl } u$ is governed by the equation

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + (u.\ \text{grad})\ \omega = (\omega.\ \text{grad})\ u + \nu \nabla^2 u\ ; \tag{2.3}$$

advection vortex-stretching viscous diffusion

this says that the rate of change of vorticity following a fluid particle is determined by viscous diffusion and vortex stretching (see *e.g.*, Batchelor[26], p. 267). In two-dimensional flow ω is normal to the plane of the flow, so $(\omega \cdot \text{grad})\ u$ is zero; hence there is no vortex stretching. In three-dimensional flow, if the distance between two neighbouring fluid particles on a vortex-line increases because of the flow, the vorticity increases as well ; there is thus an important mechanism available for vorticity amplification.

One line of approach that is due to Reynolds is to split the flow into mean and fluctuating components,

$$u = \bar{u} + u'. \tag{2.4}$$

The mean $\bar{u}$ here can be defined in several ways. It can be (i) an average over time if the flow is statistically stationary, *i.e.*, if the statistics are time-independent ; (ii) an average over a spatial dimension if the flow is statistically homogeneous over that dimension, *e.g.*, the spanwise coordinate in two-dimensional flow ; (iii) an average over an ensemble of identical copies of the flow in question.

With a decomposition similar to (2.4) for p as well, the equations for the *mean flow* become

$$\text{div } u = 0$$

$$\frac{du}{dt} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u + \text{div } \tau,$$

$$\tau_{ij} \equiv -\overline{u'_i u'_j} \tag{2.5}$$

where we have omitted bars on u and p, and $\underset{\sim}{\tau}$ is the (kinematic) Reynolds stress tensor. The Cartesian component τ_{ij} represents an effective stress acting on the fluid because of momentum transfer due to velocity fluctuations in direction j on an elementary surface perpendicular to direction i.

We shall call (2.5) the Reynolds (or the Reynolds-averaged) equation, and shall always make a distinction between this and the Navier-Stokes equations (2.1) and (2.2).

The Reynolds equations basically contain new unknowns τ_{ij}, and so cannot be solved as they stand. A dynamical equation for τ_{ij} can however be obtained by multiplying the equation for u_i' (itself obtained by subtracting (2.5) from (2.1) and (2.2)) by u_j' and averaging ; the result is[27]

$$\frac{d}{dt}\overline{u_i' u_j'} = -\overline{u_i' u_k'}\frac{\partial u_i}{\partial x_k} - \overline{u_j' u_k'}\frac{\partial u_i}{\partial x_k} - 2\nu\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}\right)$$

(generation) (viscous destruction)

$$+ \nu\frac{\partial^2}{\partial x_k \partial x_k}\overline{u_i' u_j'} + \frac{1}{\rho}\overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}$$

(viscous diffusion) (redistribution)

$$+ \frac{1}{\rho}\frac{\partial}{\partial x_k}(\overline{p' u_i'}\,\delta_{jk} + \overline{p' u_j'}\,\delta_{jk}) \quad \text{(pressure transport)}$$

$$-\frac{\partial}{\partial x_k}(\overline{u_i' u_j' u_k'}) \quad \text{(third order correlation)} \qquad (2.6)$$

The significance of each of the terms is indicated in brackets above, and is discussed in greater detail by, *e.g.*, Bradshaw[22] (also see fig. 4).

The major feature of (2.6) that we wish to highlight here is that the dynamical equation for the second order product $\overline{u_i' u_j'}$ involves the third order product $\overline{u_i' u_j' u_k'}$. A dynamical equation for the latter can be derived along lines similar to those followed above, but such an equation will be found to involve fourth order products ; and so on. Thus, the hierarchy of equations for mean products of different orders, derived from the Navier-Stokes equations, does not form a closed set for any finite number of such products. This is the problem of closure ; it arises because of the nonlinear term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ in the Navier-Stokes equations.

The above problem arises even in homogeneous flow (*i.e.*, one in which all mean quantities are constant in space; in particular therefore there is no shear). Thus, if in (2.6) we put $u_i = 0$, and consider the dynamical equation for $\overline{u_i'^2}$, for example*,

* In homogeneous flow $\overline{u_i' u_j'} = 0$ if $i \neq j$.

we will find on the right hand side the triple product $\overline{u_i'^2 u_k'}$ —one order higher than the moment being considered.

It follows that, to solve for the mean quantities for turbulent flow using the Reynolds equations, closure must be 'forced', by making suitable assumptions on the Reynolds stresses. None of the assumptions in use today can be called 'rational' in the sense of Van Dyke[28], *i.e.*, they cannot be considered approximations that become more nearly valid in some limit ; they are for this reason incapable of being improved upon (even in principle) in any *systematic* way.

This leads us naturally to consider turbulence 'modelling', which is the name given for constructions of closure schemes involving mean products. Although these models are essentially empirical, some of them seem to be quite successful in handling certain classes of flows. We will consider modelling in Section 5, but before doing so shall briefly describe certain basic considerations in the statistical theory of turbulence.

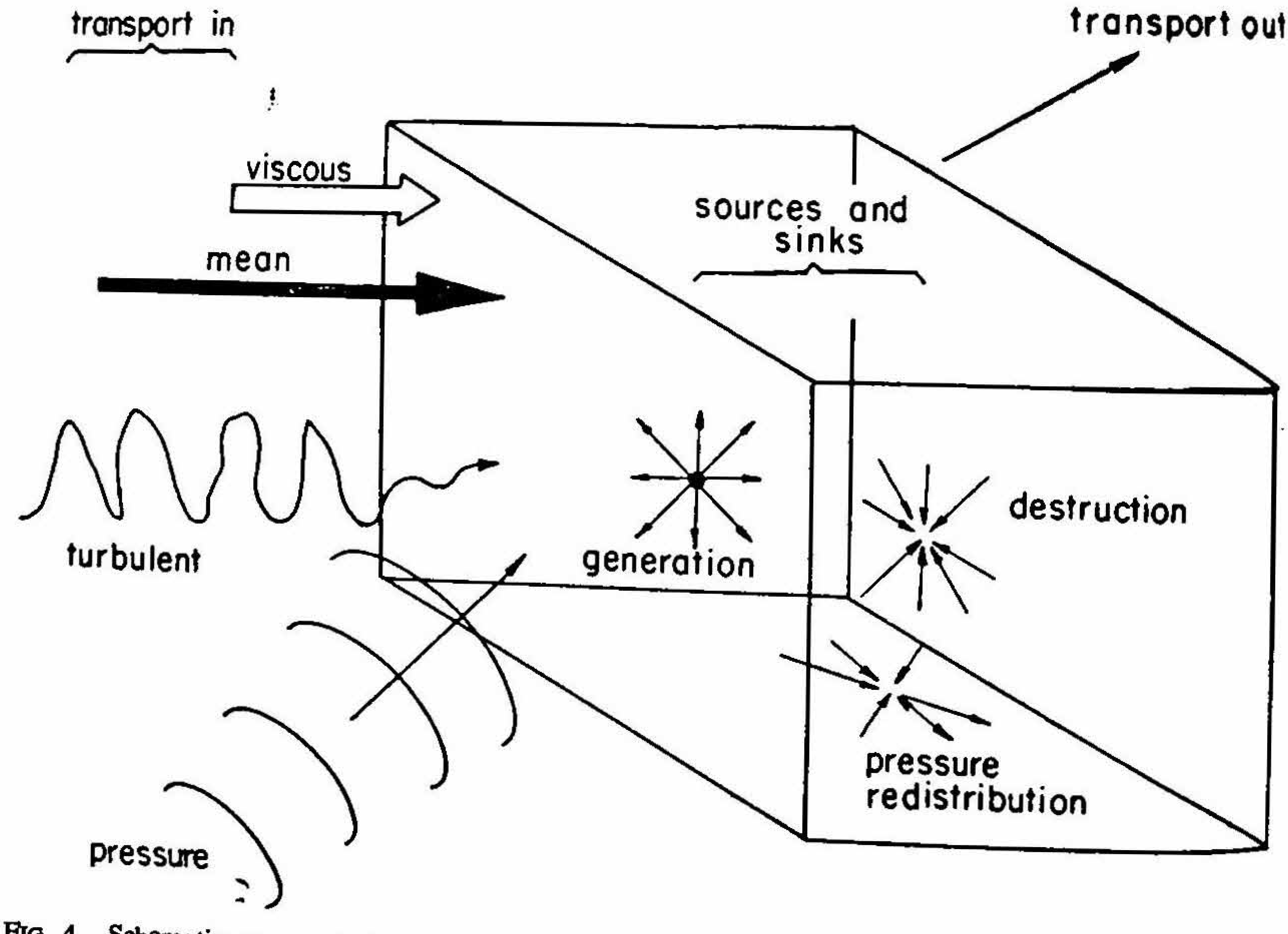


FIG. 4. Schematic representation of terms in the turbulent energy balance equation[22]. Note that production need not necessarily be always positive. In a similar equation written for the Reynolds shear stresses, viscosity may also contribute to generation, in addition to its role in transport and dissipation.

3. The statistical theory

3.1. *Description of random fields*

Any dynamical theory of turbulent flow first needs a suitable method of description of the flow field. Several methods are available (see *e.g.*, Lumley[29]), but one of the earliest and still the most common, going back to G. I. Taylor's pioneering studies (see *e.g.*, Taylor[30]), utilizes what has become known as generalized harmonic analysis. Among the quantities of major interest in this description are the spectrum functions $E(k)$ and $\Phi(k)$, defined by

$$\tfrac{1}{2}\langle |u'|^2 \rangle == K = \int_0^\infty E(k)\, dk = \int \Phi\ (k)\ Dk, \tag{3.1}$$

where k is the magnitude of a wave number vector $\boldsymbol{k}$; the function $E(k)$ is the contribution to the turbulent kinetic energy K (per unit mass of fluid) from the wave number range dk at k, and $\Phi(\boldsymbol{k})$ is similarly the contribution from the elementary volume $D\boldsymbol{k}$ at $\boldsymbol{k}$ in wave number space.

We could instead have asked for the wave-number contribution to the velocity $\boldsymbol{u}'$ (rather than the energy) ; this may be expressed through a Fourier coefficient $dZ(\boldsymbol{k})$, writing

$$\boldsymbol{u}'(\boldsymbol{x}, t) = \int \exp\ (i\, \boldsymbol{k}.\boldsymbol{x})\, d\boldsymbol{Z}(\boldsymbol{k}, t) \tag{3.2}$$

(the integral being used as in general *all* wavenumbers $\boldsymbol{k}$ may be present). It can be shown that, for a turbulent trace of the kind shown in fig. 2, $|\boldsymbol{u}'|$ is not integrable, and consequently that a Fourier *transform* of u' does not exist, *i.e.*, $Z(\boldsymbol{k}, t)$ is not a function differentiable in $\boldsymbol{k}$. A Fourier *series* for $\boldsymbol{u}'$ does not exist, of course, as there is no basic period or 'fundamental' in a turbulent signal. The representation (3.2) must therefore be interpreted as a Fourier–Stieltjes integral.

We will for brevity suppress the time dependence below. The representation (3.2) implies that

$$\begin{aligned}\tfrac{1}{2}\langle \boldsymbol{u}'^2 \rangle &= \langle \int (\exp i\, \boldsymbol{k}\, .\, \boldsymbol{x})\, d\boldsymbol{Z}(\boldsymbol{k})\ [\int (\exp\ i\, \boldsymbol{k}'\, .\, \boldsymbol{x})\, d\boldsymbol{Z}(\boldsymbol{k}')]^* \rangle \\ &= \int \exp\ i\,(\boldsymbol{k} - \boldsymbol{k}').\boldsymbol{x}]\, \langle d\boldsymbol{Z}(\boldsymbol{k})\ d\boldsymbol{Z}^*(\boldsymbol{k}') \rangle\end{aligned}$$

(angular brackets denote a suitable mean : see Section 1), and comparison with (3.1) suggests that[18] we must have

$$\langle d\boldsymbol{Z}(\boldsymbol{k})\ d\boldsymbol{Z}^*(\boldsymbol{k}') \rangle = \delta(\boldsymbol{k} - \boldsymbol{k}')\, \Phi(\boldsymbol{k})\, D\boldsymbol{k}. \tag{3.3}$$

Thus, if $\boldsymbol{k} \neq \boldsymbol{k}'$, the coefficients $d\boldsymbol{Z}(\boldsymbol{k})$, $d\boldsymbol{Z}(\boldsymbol{k}')$ are uncorrelated or statistically orthogonal to each other ; and each of them is (loosely speaking) of order $(D\boldsymbol{k})^{1/2}$!

A generalization of (3.1) puts

$$\langle u'_i\ (\boldsymbol{x})\ u'_j(\boldsymbol{x}) \rangle = \int \Phi_{ij}(\boldsymbol{k})\ D\boldsymbol{k},$$

so that $\frac{1}{2}\Phi_{ii} = \Phi$. A related quantity of great interest is the correlation tensor

$$R_{ij}(x, r) = \langle u_i(x)\, u_j(x + r) \rangle.$$

If turbulence is homogeneous, *i.e.*, its statistics are independent of space, R_{ij} depends only on the separation vector r, and not on the location x; we will then have $R_{ij} = R_{ij}(r$ only). According to the Wiener–Khintchine theorem R_{ij} and Φ_{ij} are Fourier transforms of each other :

$$R_{ij}(r) = \int \Phi_{ij}(k) \exp(i\,k\,.\,r)\, Dk,$$

$$\Phi_{ij}(k) = \frac{1}{8\pi^3} \int R_{ij}(r) \exp(-i\,k\,.\,r)\; Dr.$$

The correlations R_{ij} clearly tell us how velocity components at different places are related to each other 'on an average'. Measurements of some of the R_{ij} have been made in a very large number of flows, and have sometimes led to interesting inferences on the structure of such flows[31].

As an example, we may cite the correlation measurements of Grant[32] in a turbulent boundary layer, on the basis of which Townsend[33] proposed that the flow near the wall was in the form of nearly two-dimensional, upward jets marking the boundaries of counter-rotating, streamwise vortices—a picture not unlike that found by flow visualization in the elaborate studies of Kline *et al*[17].

We may generalize the definition of R_{ij} to include a separation in time (or 'lag' or 'delay') as well as in space : a general 'space-time correlation' in statistatically stationary turbulence would be*

$$R_{ij}(x, r, \tau) = \langle u_i(x, t)\, u_j(x + r,\; t + \tau) \rangle.$$

Measurements of such correlations in a turbulent boundary layer, reported by Favre *et al*[34], show graphically (fig. 5) the time and space scales involved in the flow.

Some quantitative measures of such scales are in common use. If $R(s)$ is the correlation of a variable v with separation s (which may be space or time), we define a micro-scale λ (often called after G. I. Taylor) and a macro-scale L by the relations

* If $i = j$ and $r = 0$, R_{ij} reduces to the 'auto-correlation' $R_i(\tau) = \langle u_i(t)\, u_i(t + \tau) \rangle$ (no sum over i) of the component u_i at a given place. The Wiener–Khintchine relations now give

$$\Phi_i(\omega) = \frac{1}{2\pi} \int R_i(\tau)\, e^{-i\omega\tau}\, d\tau,$$

$$R_i(\tau) = \int \Phi_i(\omega)\, e^{i\omega\tau}\, d\omega,$$

where ω is the frequency, and $\Phi_i(\omega)$ is the spectrum—or, as it is more elaborately called, the power spectral density of u_i.

$$\lambda = [-\langle v^2 \rangle / R''(0)]^{1/2},$$

$$L = \frac{1}{\langle v^2 \rangle} \int_0^\infty R(s)\, ds.$$

[Note that $\lambda > 0$ as $R''(0) = d^2 R/ds^2 |_{s=0} < 0$.] We shall encounter these scales again in the following : in general L represents the large scales in the turbulent flow, and λ appears to be characteristic of the *separation* between the regions of high vorticity that occur in the turbulent velocity field*.

Other descriptions of turbulence could be in terms of eddies or of a series of stochastic processes ; these are discussed by Lumley[29].

3.2. *Behaviour of the spectrum*

The typical form of $E(k)$ is shown in fig. 5. The viscous dissipation is proportional to the viscosity and to the square of the velocity derivatives ; hence in terms of E the dissipation is

$$\epsilon = (\ )\, \nu \int k^2 E(k)\, dk. \tag{3.4}$$

The maximum contribution to ϵ occurs therefore at higher wave numbers. Correspondingly, the low wave numbers do not suffer dissipation, and tend to remain permanent to a greater or lesser degree.

If a bottle of liquid is shaken, turbulence is produced in the liquid ; it seems clear in this case that energy is put in at length scales corresponding to the size of the bottle, and transferred to higher wave numbers, (presumably by nonlinearity); energy must be dissipated at the highest wave numbers by viscosity. One may therefore visualize a 'cascade process', in which energy is transferred from low to high wave numbers and finally dissipated as heat (see fig. 5). For a given energy and viscosity, it is also clear that the scales contributing most to energy become larger as the Reynolds number increases ; thus the separation in wave number space between the energy-bearing and dissipating wave numbers increases.

3.3. *The spectrum at high wave numbers*

Consider a turbulent flow with velocity and length scales $\hat{u}$ and L respectively, the Reynolds number $\hat{u}L/\nu$ being large ($\hat{u}$ could be the r.m.s. value of the fluctuating velocity u', and L could be the macroscale). If there is no external input of energy, the kinetic energy in the turbulent motion will eventually decay because of viscous dissipation. At the scales corresponding to L dissipation must be negligible, as the Reynolds number is large. It follows that the dissipation must occur at very small scales, and that this drain must be reflected in the decrease in energy of the large scales.

* Tennekes[35] has shown that the total dissipation can be accounted for if the vorticity is confined to tubes with a characteristic separation of length λ.

It further follows, by dimensional arguments, that the dissipation per unit mass ϵ must be given by

$$\epsilon = (\ )\, \hat{u}^3/L\ ;$$

i.e., the dissipation is actually viscous, but it must be capable of being expressed in large-eddy variables, *not* including the viscosity. The dependence on viscosity is shown in terms of the 'Taylor' microscale (*e.g.*, Batchelor[20], p. 100)

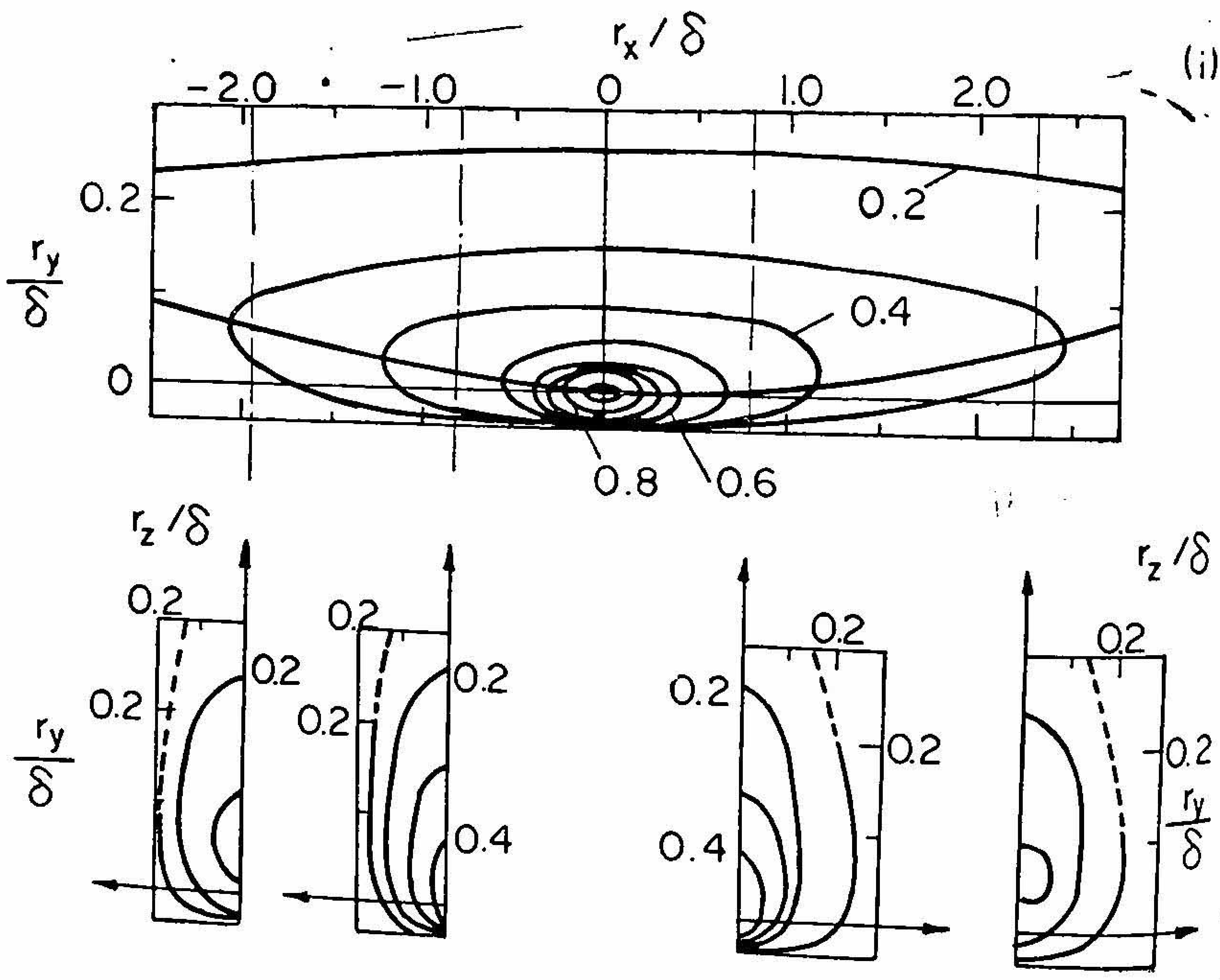
$$\epsilon = (\ )\, \nu \hat{u}^2/\lambda^2\ ;$$

clearly

$$\frac{\hat{u}\lambda}{\nu} = (\ )\left(\frac{\hat{u}L}{\nu}\right)^{1/2}.$$

The actual dissipation occurs at even smaller length scales, because the velocity characteristic of the scale λ is not necessarily $\hat{u}$. Kolmogorov[36], in his famous theory, argued that at very high wave numbers turbulence must be isotropic, and must depend only on ϵ and ν ; this suggests the length and velocity scales

$$l_k = (\nu\epsilon)^{1/4},\ u_k = \nu^{3/4}\,\epsilon^{-1/4}\ ;$$



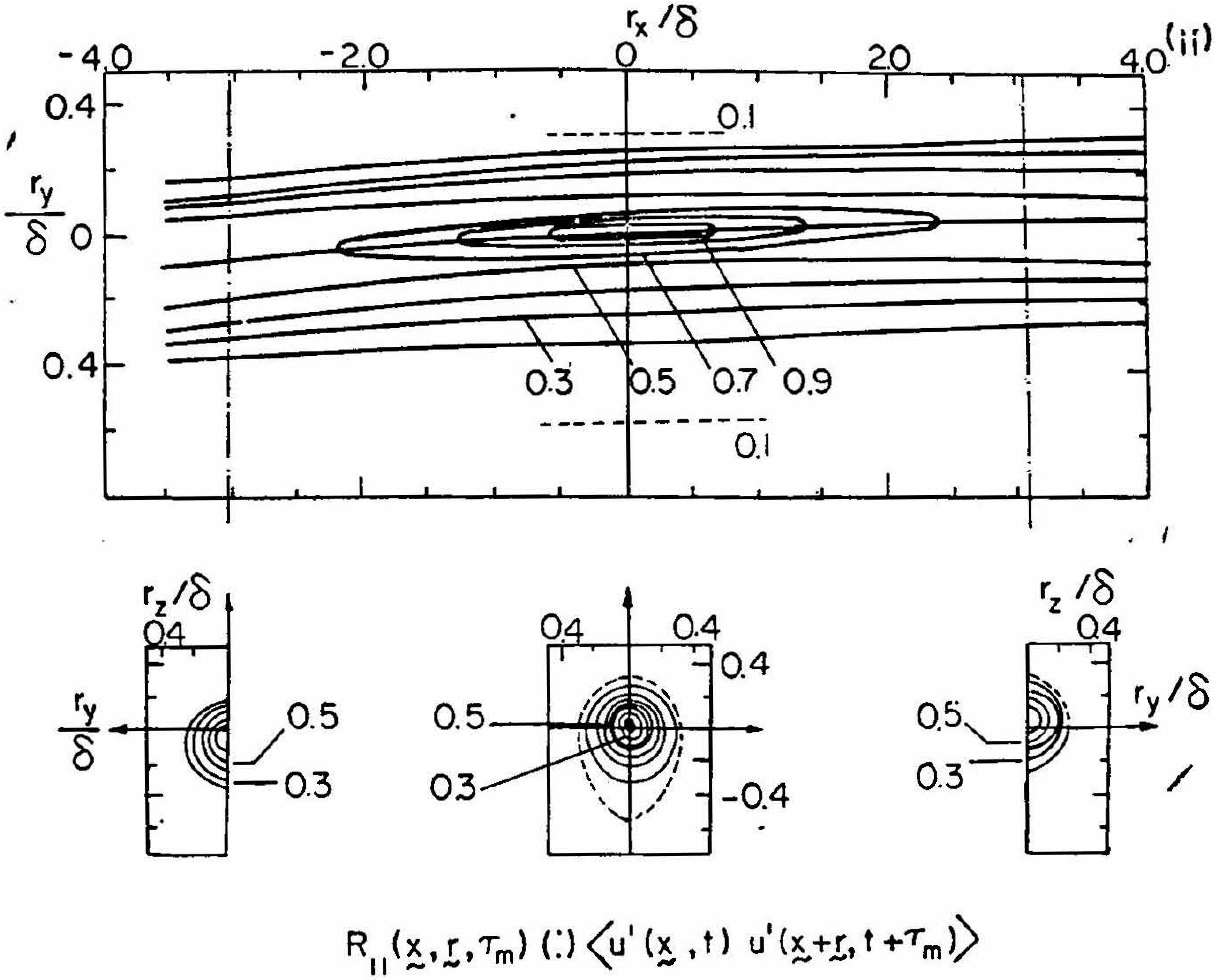


FIG. 5. Space-time iso-correlation surfaces in a flat plate turbulent boundary layer, after Favre *et al* [34]. The correlation coefficient shown at any point is with that value of the time lag that produces the highest correlation at that point. Two sets of curves are shown for two different points x in the boundary layer. Boundary layer Reynolds number is $Re_\delta = 27900$.

(i) Fixed probe at $y = 0{\cdot}03\,\delta$.

(ii) Fixed probe at $y = 0{\cdot}77\,\delta$.

Note how the maximum correlation surfaces extend over several δ for separations (r_x) in the streamwise direction, but only over a small fraction of δ in the transverse direction (r_z), suggesting the presence of relatively slender streamwise structures in the turbulence.

If the maximum correlation at separation r_m occurs at a time delay τ_m, then the quantity $dr_m/d\tau_m$ represents the velocity of propagation of maximum correlation, often called the 'celerity'. In turbulent boundary layers, experiments suggest that the celerity in the outer region of the boundary layer is about $0{\cdot}8$ times the free stream velocity.

note that the Reynolds number

$$l_k\, u_k/\nu = 1,$$

as we should expect when viscosity is dominant ; and also that

$$l_k/L = (\ )\ Re_L^{-3/4}.$$

The universal spectrum at high wave numbers must therefore have the form

$$E(k, t) = u_k^2\, l_k\, g(k l_k) \tag{3.5}$$

where g is a universal function. A principle that is often used in turbulence is that a viscous limit like the above must match the inviscid limit valid at lower wave numbers (see Section 4). Thus, as k decreases, E in (3.5) should become independent of viscosity. Using (3.4), this requires that

$$E(k) = \alpha \epsilon^{2/3} k^{-5/3}, \tag{3.6}$$

where α (according to Kolmogorov) should be a universal constant, *i.e.*, the same in *all* high Re flows. (3.6) should be valid in the so-called 'inertial' sub-range. A detailed discussion is presented by Batchelor[20].

Much effort has been expended to determine whether the Kolmogorov spectrum is valid. The most impressive evidence for the $k^{-5/3}$ law is provided by the experiments of Grant *et al*[37] in Discovery Passage, off the coast of Vancouver (fig. 6). Although the

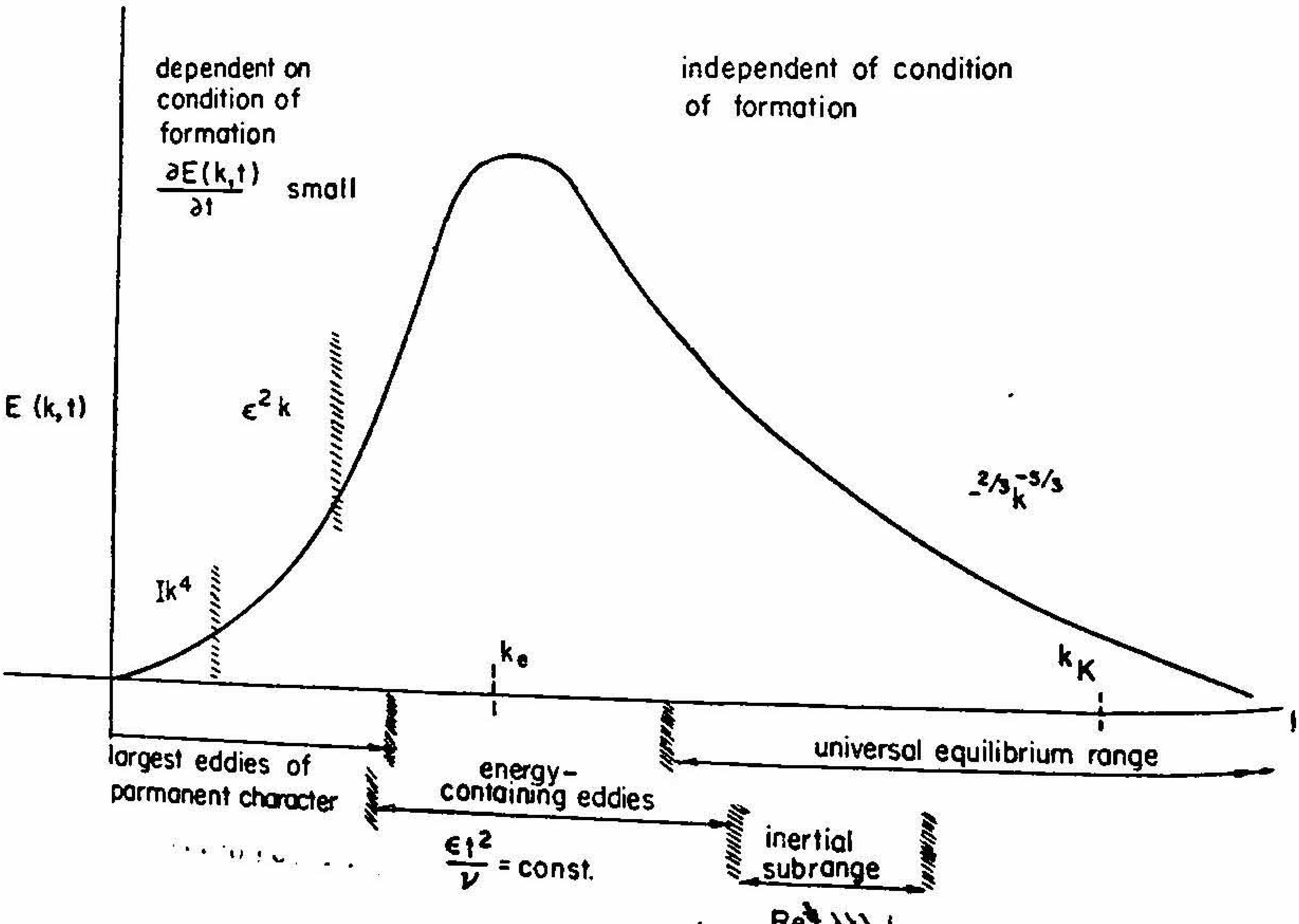


FIG. 6. The spectrum $E(k, t)$ in decaying homogeneous turbulence (*i.e.*, turbulence which is statistically uniform over all space, but decays because there is no forcing or turbulent energy production). For 'largest eddies of permanent character', it has been proposed by Loitsianskii that $E(k) = (\ )k^4$, and by Saffman that $E(k) = (\ )k^2$. In the diagram, k_K is the Kolmogorov wave number, and λ is the Taylor microscale.

exponent of k in the inertial sub-range is evidently very close to the Kolmogorov value, there is no agreement on the precise value of α (—it is around $1\cdot5$)*, and often the turbulence even in the dissipation range is not isotropic.

In 1962, Kolmogorov published a modification of the theory, to account for the spottiness of dissipation that we have noted in Section 1[38]. Many attempts, too numerous to mention here, have been made to demonstrate by more elaborate dynamical theories the correctness or otherwise of the Kolmogorov spectrum ; and, when it is inferred to be correct, to estimate the value of α. In particular, Kraichnan has over the last three decades used perturbation methods developed in quantum mechanics for attacking the problem ; a full account of this kind of theory will be found in Leslie[39]. However, these calculations make a variety of assumptions whose validity is hard to assess ; furthermore, all calculations of this kind do not agree among themselves.

Yakhot[40] has recently developed a dynamic renormalization group method to study the spectrum of turbulence in a randomly stirred fluid. He assumes that the stirring force decreases as an inverse power of k for large wave numbers. He concludes that the spectrum

$$E(k) = (\ )\,\frac{k^{-5/3}}{\ln kL}$$

at large k ; the Kolmogorov spectrum without corrections never exists in the high wave number limit. It would appear that some of the conclusions of Yakhot's theory, in particular for the moments, are in qualitative agreement with recent observations by Klebanoff and his colleagues.

4. Some general principles in turbulent shear flows

Before examining the more elaborate numerical models that are now being constructed in such profusion to calculate turbulent shear flows, it is necessary to ask whether any general features or principles governing such flows can be identified. We attempt in this section to describe such principles in the hope that they will give an overall appreciation of the qualitative behaviour of turbulent shear flows.

4.1. *Rapid distortion*

There are many situations in which a turbulent field is strained rapidly. Examples are provided by flow through a wind tunnel contraction or rocket nozzle, round a building or hill, past a Prandtl–Meyer corner at supersonic speeds or in a highly accelerated shear layer. If the final position of a fluid particle in such a flow is determined by the

* Townsend[23] (p. 98) summarises and critically examines available experimental data on $C' = 18\alpha/55$ and concludes $C' \doteq 0\cdot50 \pm 0\cdot03$.

imposed strain to within a distance less than the (macro-) scale of turbulence, it can be shown that both inertial and viscous forces can be neglected[41], and the problem becomes linear : vortex-stretching is the dominant mechanism. The condition as stated is very severe, but can be considerably relaxed if interest is limited to the energy in the turbulence (or to the Reynolds stresses). After a detailed re-examination of the matter Hunt[42] has suggested that the appropriate conditions are that

$$\frac{\hat{u}_0}{u_0} \ll 1, \ \frac{\hat{u}_0}{u_0}\frac{D}{L} \ll 1, \tag{4.1}$$

where subscript 0 denotes the initial state, D characterises the distance over which distortion takes place, and L is the macro-scale of the turbulence ; there are two other conditions similar to (4.1) on irrotational fluctuations that may be present in the flow. Conditions (4.1) are not infrequently satisfied in practice, as $\hat{u}_0/u_0$ is generally small.

Batchelor and Proudman[41] provide a complete solution of the rapid distortion problem for initially isotropic, homogeneous turbulence ; Sreenivasan and Narasimha[43] have done the same for several forms of axisymmetric turbulence. An interesting feature of these solutions is that the change in the turbulence intensities due to rapid distortion depends on the total strain experienced by the fluid, and not on the strain *rate*. As the intensities are proportional to the normal Reynolds stresses, turbulence in these situations resembles an elastic (but not necessarily linear) solid more than a fluid !

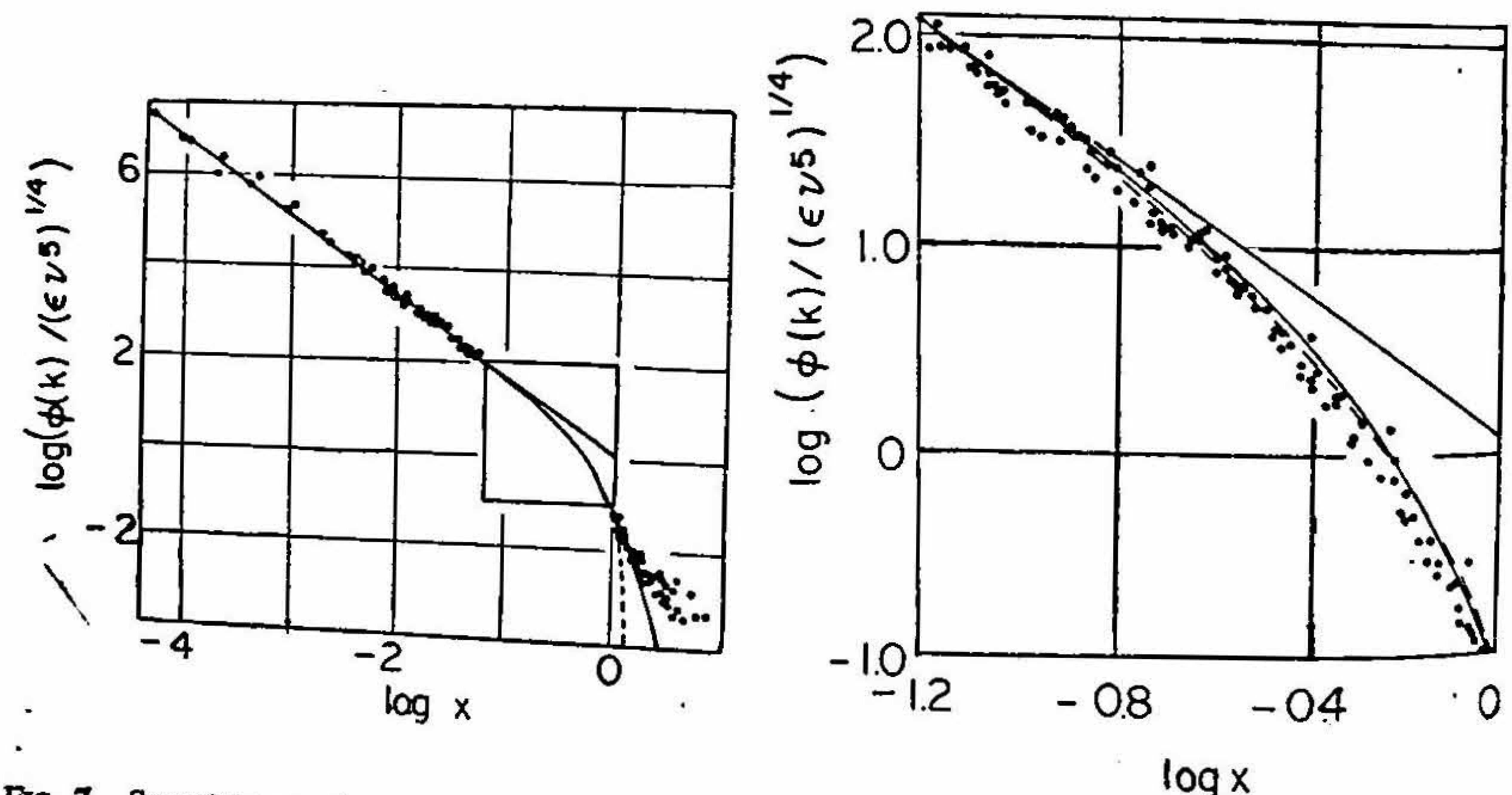


FIG. 7. Seventeen spectra, measured in a tidal channel, compared with the theories of Kolmogorov (straight line), Heisenberg (full curved line) and Kovasznay (dashed line). The data within the square are too crowded for display in the same figure, and so are shown separately. The measurements agree closely with the Kolmogorov $k^{-5/3}$ spectrum over more than three decades in the wave number[37].

A simple extension to shear flows* has also been made[44]. This shows that a plane shear flow subjected to rapid acceleration experiences a decrease in the streamwise intensity, an increase in the normal intensity, and practically no change in the Reynolds shear stress. (Although the theories for homogeneous flow involve no shear stresses, the trend for streamwise and normal intensities is the same, thus reinforcing the conclusion.) The strange situation often encountered in turbulent shear flows, that when the flow is rapidly strained the shear stresses keep frozen at their initial values and do *not* change in step with the mean flow (see *e.g.* Narasimha and Sreenivasan[46]), is therefore quite simply the expected effect from rapid distortion.

Townsend[31] has extensively utilized rapid distortion results in formulating some general ideas about equilibrium in shear flows, where there are competing effects.

Although, in general, rapid distortion theory rarely provides completely satisfactory quantitative results in any real flow**, it is a simple limit which offers much insight into the behaviour of turbulence, and is a good corrective to the free (and often unjustified) use of concepts like eddy viscosity.

4.2. *Similarity arguments*

Inability to achieve a rational closure to the turbulence problem (mentioned in Section 2) has encouraged the formulation of certain general principles in the hope of deducing the broad features of turbulent flows without appealing to detailed models of questionable validity. Of course, as these general principles have not yet been deduced from the basic laws of fluid motion, they also remain open to some doubt ; they therefore remain hypotheses, and their value must be judged by their usefulness.

Townsend[31] stated explicitly two principles that had earlier been used implicitly by many workers. The first of these, called the principle of high Reynolds number similarity, may be stated as follows :

> at sufficiently high Reynolds numbers, any turbulent flow away from solid walls does not depend directly on the viscosity of the fluid. (4.2)

Normally, geometrically similar flows (*e.g.*, plane jets) would be dynamically similar as well only if Reynolds numbers were the same. Principle (4.2) states that if

* It is again often thought (erroneously in my view) that rapid distortion of a shear flow is rarely possible. For example, Bradshaw[45] quotes the condition necessary as $\partial u/\partial y \ll e$, the rate of strain. But this is unnecessarily stringent, in view of (4.1). It must also be noted that, in the outer part of a turbulent boundary layer, $\partial u/\partial y \ll U/\delta$ as the total change in mean velocity across the outer layer is a fraction of the free stream velocity ; furthermore the streamwise macro-scales in the boundary layer are appreciably larger than δ (as the data of Favre *et al*[34] mentioned in Section 3 show). Rapid distortion is therefore far more frequent than is generally thought.

** For a method of applying viscous corrections, see Tucker and Reynolds[47].

the flows are turbulent and the Reynolds numbers are sufficiently high, geometrically similar flows are also dynamically similar *independent* of the Reynolds number. Thus, all incompressible turbulent plane jets are dynamically similar among themselves at all (sufficiently high) Reynolds numbers.

The implication of (4.2) in general is that viscosity plays a secondary role in many turbulent flows ; even in the presence of a wall, *e.g.*, in a turbulent boundary layer, the dependence on Reynolds number is weaker than in laminar flows. Thus, the skin friction coefficient c_f is proportional to $Re_\theta^{-1/2}$ in a laminar boundary layer, to something like $Re_\theta^{-1/4}$ (see the Ludwieg–Tillmann relation, Section 4.3) in a turbulent boundary layer at moderate Reynolds numbers, and to an even smaller power of Re in a turbulent wall jet (Bradshaw and Gee[48]).

The second principle is that

> if an equilibrium solution to the development of a turbulent shear flow exists, consistent with the prescribed boundary conditions and with the first principle (4.2), then the flow will eventually obey that equilibrium solution. (4.3)

The word 'equilibrium' has been used in many different senses in turbulent flow, but we adopt here (following Narasimha and Prabhu[49]) the operational definition that

> a turbulent shear flow is in equilibrium in a region if, at every streamwise station in the region, the distributions of mean velocity and of the turbulent stresses exhibit similarity, with essentially identical velocity and length scales. (4.4)

To illustrate, consider the two variables $w(x, y)$ and $\tau_0(x, y)$, respectively an appropriate mean velocity and stress in a plane turbulent flow. The quantities w and τ_0 exhibit what we may call internal similarity if we can write

$$w(x, y) = w_0(x) f\,[y/\delta(x)],$$

$$\tau(x, y) = \tau_0(x)\, g\,[y/\delta_\tau(x)],$$

where x and y are coordinates along and normal to the stream, and w_0, τ_0, δ and δ_τ are suitable local scales. In equilibrium these scales are inter-related in such a manner that

$$w_0(x)/\tau_0^{1/2}(x), \quad \delta(x)/\delta_\tau(x),$$

and similar ratios of all other relevant mean velocity and stress scales, are independent of x.

Townsend[31] calls such flows self-preserving. For a flow to be in equilibrium, the existence of self-preserving solutions to the equations is a necessary but not sufficient condition.

The word 'eventually' in (4.3) is important. It implies the existence of a characteristic time beyond which the details of the initial conditions are forgotten, only certain gross parameters retaining significance (perhaps this should be stated as a separate principle). For example, the turbulent near-wake behind a body depends very much on the body and the flow conditions, including the flow Reynolds number ; far downstream, however, the wake attains a state of equilibrium determined entirely by the drag of the body, or the momentum thickness of the wake. By principle (4.2) the Reynolds number is irrelevant ; therefore, the equilibrium wake can depend only on the free stream velocity U, and the momentum thickness θ. It follows that the wake thickness δ and the maximum velocity defect w_0 must obey relations of the type

$$\delta/\theta = f(x/\theta), \quad w_0/U = g(x/\theta) \tag{4.5}$$

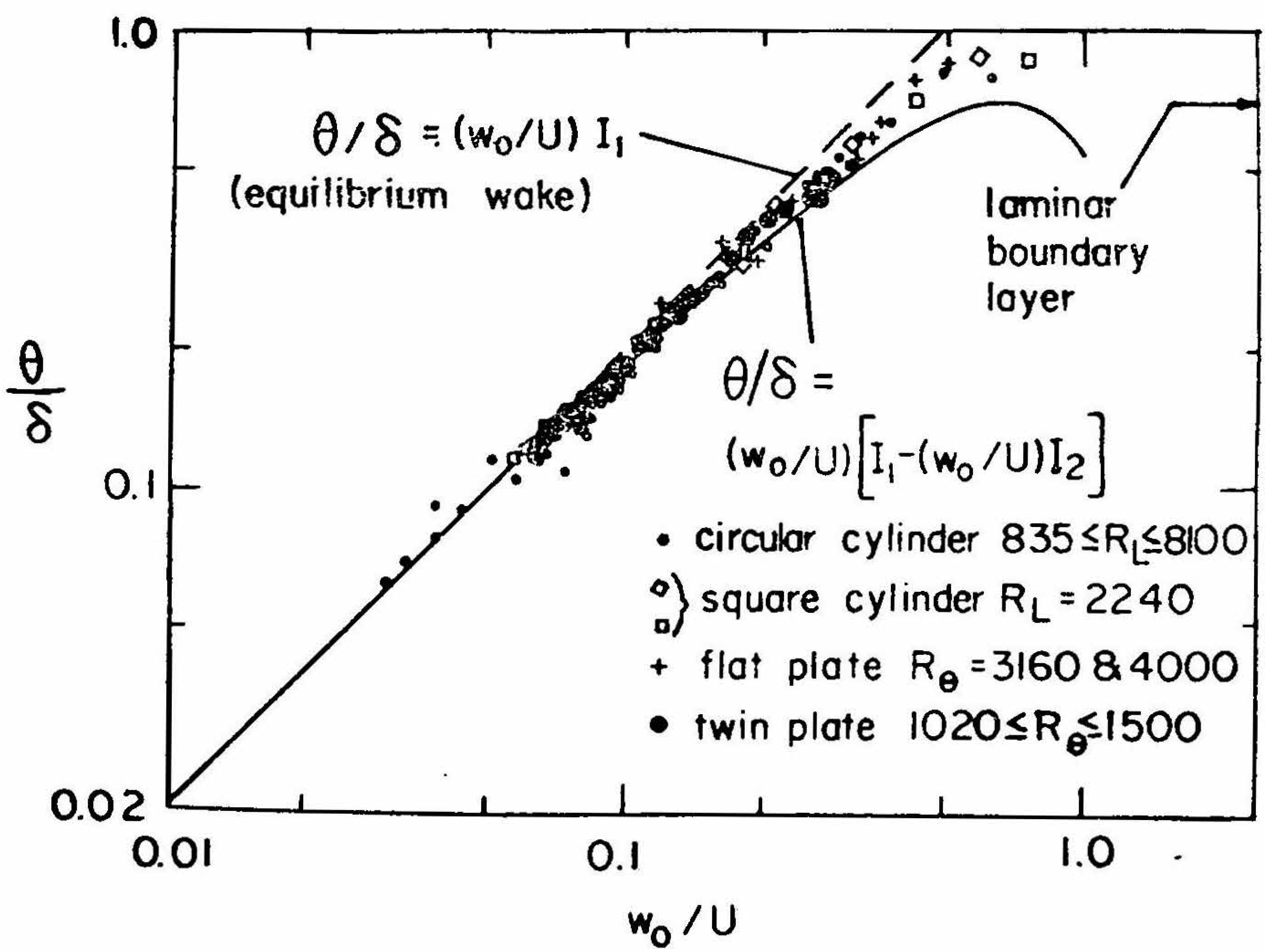


FIG. 8. Relation between thickness and maximum velocity defect in wakes behind different two-dimensional bodies. The straight line is the relation for an equilibrium wake, and seems to be a good approximation for $w_0/U \lesssim 0{\cdot}05$; the curve is a second order approximation. In the Reynolds numbers R_L, L = diameter of cylinder or side of square. It is interesting that the value of θ/δ near $w_0/U = 1$ is close to that for a laminar boundary layer[51].

etc., where the functions f, g are universal. If we substitute the forms (4.5) in the equations of motion (and carry out the self-preservation analysis : see *e.g.*, Townsend[31]) we get the further result that, in the far wake,

$$f(x/\theta) = (\ )\ (x/\theta)^{1/2}, \quad g(x/\theta) = (\ )\ (x/\theta)^{-1/2}. \tag{4.6}$$

This is essentially the 'self-preserving' solution of Townsend[31]. The constants of proportionality in (4.6) can be determined quite accurately[50], provided due care is taken about higher order terms.

As will be clear, these equilibrium solutions depict the streamwise development of the flow, but the conditions under which they obtain are quite restricted. For example, to obtain an equilibrium wake takes a distance of several hundred momentum thicknesses, the precise distance depending on the wake-generating body[49].

Fortunately, there is a principle whose application leads us to some general information even well before equilibrium is reached, or even before mean velocity profiles become similar. This principle is that

> under conditions far less severe than necessary for equilibrium, a turbulent flow attains a 'fully developed state' in which the flow parameters that characterise each streamwise station obey unique inter-relations among themselves. (4.7)

Many examples of successful application of this principle can be quoted. Thus the flow parameters δ (thickness) and w_0 (maximum velocity defect) in a turbulent wake are related to each other in a unique way (as shown in fig. 8), well before equilibrium is attained[51]. In a turbulent boundary layer, it has been well known for a long time (and was particularly emphasized by Coles[52]) that the inter-relation between the skin friction coefficient c_f and the local momentum thickness Reynolds number Re_θ is much better defined than, *e.g.*, the relation of c_f to a Reynolds number Re_x based on streamwise distance. (If the flow obeyed appropriate similarity laws, the relation with Re_θ can be converted, through the use of the momentum integral, to one with Re_x ; but the point of this principle is that there are useful relations in the state of full development even in the absence of strict similarity.) A particularly striking example of a successful (but empirical) relation of this type is that due to Ludwieg and Tillmann[53] for the skin friction coefficient, in terms of Re_θ and the shape factor H, to be discussed in Section 4.4.

The application of such principles to wall jets is discussed by Narasimhma *et al*[4] who find that the flow development far downstream depends only on the jet momentum flux and the viscosity very weakly, but unmistakably, on the latter.

The fourth principle concerns solutions in sub-domains where viscosity *is* important (*e.g.*, the mean velocity near the wall in a turbulent boundary layer, or the spectrum at the high wave numbers which cause direct dissipation of kinetic energy to heat). The hypothesis is that

> the viscous solution valid in some appropriate sub-region of the domain of interest in the turbulent flow must match the viscosity-independent solution away from the sub-domain. (4.8)

This has been called the Millikan-Kolmogorov Principle by Afzal and Narasimha[55], as its application was the basis of Millikan's argument for the log law in a turbulent boundary layer (Section 4.3), and of Kolmogorov's argument for the $k^{-5/3}$ spectrum (Section 3.2)[56]. At first sight the two applications seem quite different, but it is easy to see that the idea in both the cases is 'matchability'.

This matching here is not unlike that used with asymptotic expansions, as described *e.g.*, by Van Dyke[28]. There is however one important difference : because the equations for turbulent flow are not closed, the matching condition in general leads to a *functional* equation, and not merely to an evaluation of constants, as it often does in the problems described by Van Dyke[57]. The solution of this functional equation gives the functional form of the solution in the appropriate sub-region.

Because of its general success, and its avoidance of unnecessarily detailed models, the Millikan-Kolmogorov Hypothesis has been applied in a variety of different flows. In a zero-wall-stress separating boundary layer, the hypothesis suggests an inertial sub-layer governed by

$$u = \frac{2}{k_0}(\alpha y)^{1/2} + (\alpha \nu)^{1/3} C_0$$

where α is the stress gradient, and k_0 and C_0 are constants;[23,58] Kader & Yaglom[59], Yaglom[60] and Afzal & Narasimha[61] have studied the lowstress boundary layer in detail. A similar study in axisymmetric turbulent layers is Afzal and Narasimha[55].

In two-dimensional turbulence, the application of the hypothesis leads to an inertial sub-range with a k^{-3} spectrum[62] ; an interesting result here is a reverse cascade that transfers the energy from higher wave numbers to *lower* ones. There is some evidence from measurements in the atmosphere to support the k^{-3} spectrum[6].

4.3. *Validity of the principles*

What is the status of the principles that we have stated above ? There can be no doubt that they are extremely useful and have led to many results which are broadly realistic and have been confirmed by experiments. On the other hand, none of them can really be said to have been established beyond doubt. In each case the experimental evidence, when closely scrutinized, turns out to be slightly less than decisive.

Let us take a few examples. We have used earlier the idea that the dissipation of energy in turbulent flow of given large scales is independent of viscosity. However, the experimental evidence for this basic idea (see for example Batchelor[26] p. 106) is by no means overwhelming. A weak dependence on the Reynolds number, in the range covered in the experiments, cannot be ruled out. Similarly, the rate of growth of the mixing layer, say $d\delta/dx$ where δ is a measure of the thickness of the layer, may be expected on the basis of the principles which we have stated to be a universal number as the Reynolds number tends to infinity. Measurements quoted by Brown and Roshko[63] however show an appreciable scatter (see fig. 9). It is possible that the reason for this scatter is a dependence on initial conditions, such as for example the boundary layer at the tip of the splitter plate from which the mixing layer springs. Of course, we expect that sufficiently far downstream the state of the boundary layer at the tip is irrelevant to the characteristics of the mixing layer. What seems clear from the experiments is that in the range of downstream development covered in the experiments, a dependence on the Reynolds number or a memory of the initial conditions still persists. A dependence on initial conditions is of course a reflection of the long memory of a turbulent shear flow that we shall discuss in Section 5.

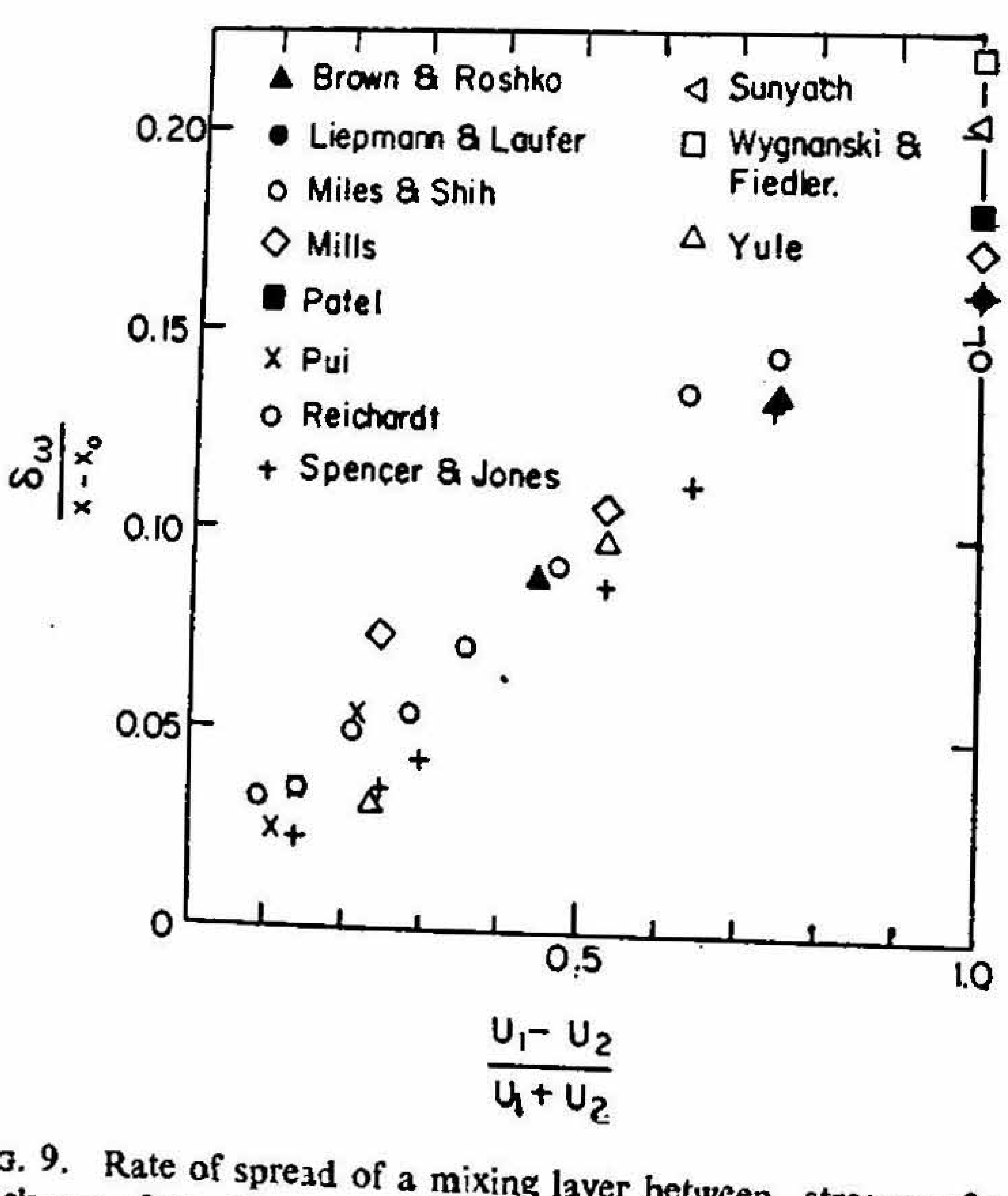


FIG. 9. Rate of spread of a mixing layer between streams of velocity U_1 and $U_2 \cdot \delta_\omega$ is a 'vorticity' thickness of the layer, as defined by Brown and Roshko[63], from whom this figure is taken.

The principle that a turbulent shear flow will eventually attain equilibrium and obey the self-preserving solution if the governing equations permit one, must again be qualified by various conditions. Certainly a time constant is involved: that is to say, even after the conditions required for a self-preserving solution are satisfied, the flow may take a finite time—indeed often a fairly long time— before it attains equilibrium. If flow conditions change before equilibrium has been attained, clearly the self-preserving solution is not of much use. A striking example is provided by the behaviour of a wake in a pressure gradient[49].

We finally come to the matching principle. Since the time of Prandtl's arguments leading to the log law in the inertial sub-layer of the turbulent boundary layer, it has been assumed that the log law rests on a secure foundation. Innumerable experiments have tended to confirm the existence of the log law, and Coles[64] has made an exhaustive analysis of all boundary layer data available and shown impressive agreement with the log law not only in constant pressure boundary layers but in those subjected to very strong pressure gradients (including nearly separating flows) as well. However, after a close reexamination of experimental data, Long and Chen[65] have recently sought to cast doubt on the validity of the log law and claim that the measurements show systematic (although slight) departures from the log law. If this is correct it would throw doubt on the argument leading to the log law, although it may not make the log law any less useful. Similarly (as we have already seen in Section 3) Kolmogorov's argument leading to the $k^{-5/3}$ spectrum in the inertial sub-region has also been doubted. Once again experimental evidence is largely in agreement with the $k^{-5/3}$ law as we have seen in Section 3. But recent theoretical and experimental work seem to suggest that there are small but systematic departures from the law.

The position of these so-called principles and laws is therefore slightly ambiguous. They are obviously very useful, being close to reality ; but they cannot still be elevated to the status of scientific laws, because the small departures noted from them cannot be dismissed as experimental error, and seem to indicate that the principles are strictly valid only under certain as-yet unstated conditions which would not always be easily obtained.

4.4. *The boundary layer*

Because of its importance in many applications, we consider the plane turbulent boundary layer in some detail.

If the pressure gradient is zero, the boundary layer must be determined in terms of three basic parameters: the free stream velocity U, the streamwise station x and the kinematic viscosity ν. Based on the hypothesis of local relations, we may expect that, when the flow is fully developed, all boundary layer parameters are determined in terms of U, ν and the boundary layer thickness δ. It is convenient to use the momentum thickness θ instead of δ, and look for relations of the type

$$c_f = c_f\,(Re_\theta),\ \ H = H\,(Re_\theta), \tag{4.9}$$

etc., for the skin friction coefficient c_f, the shape factor H, etc. ($H \equiv \delta^*/\theta$, where δ is the displacement thickness).

Now certain similarity arguments suggest the form of the functions (4.9). These arguments must begin by noting that, as the mean flow equations are not closed, we may admit a characteristic Reynolds stress – say the wall stress τ_w – as an additional variable. In laminar flow $c_f \, Re_\delta = 0(1)$; in turbulent flow c_f is higher, and we have

$$c_f Re_\delta \to \infty \text{ as } Re_\delta \to \infty, \; c_f \to 0. \tag{4.10}$$

Introducing the friction velocity $U_* = \tau_w^{1/2}$, we see that there are two vastly different length scales in the problem, namely ν/U_* and δ.

This suggests that the turbulent boundary layer needs an analysis of the singular perturbation type[64] ; indeed, in an early application of a kind of matching argument, Millikan[66] derived the log law in an inertial sub-layer of the boundary layer,

$$u_+ = u/U_* = \frac{1}{\kappa} \ln y_+ + B, \; y_+ = yU_*/\nu,$$

$$as \; y_+ \to \infty, \; y/\delta \to 0 \; ; \tag{4.11}$$

κ (known as Karman's constant) and B here are universal numbers (for the constant pressure boundary layer on a smooth surface). The argument has been cast in the language of matched asymptotic expansions by Yajnik[67], but to derive (4.11) he assumes specific asymptotic expansions for the stress. The application of the Millikan-Kolmogorov matchability hypothesis eliminates the need for such assumptions.

Coles[64], after a thorough analysis, shows that all the experimental data available on the velocity profile can be fitted very well to a composite 'standard profile'

$$\frac{u}{U_*} = f(y_+) + \frac{\Pi}{\kappa} w(y/\delta) \tag{4.12}$$

where $f(y_+)$ represents the law of the wall, and the second term the law of the wake, with

$$w(y/\delta) \doteq 2 \sin^2 \left(\frac{\pi}{2}\frac{y}{\delta}\right),$$

Π being a parameter that depends on the pressure gradient. There is no complete agreement on the precise values of the various parameters in these laws, but most of the data indicate[64] $\kappa \doteq 0.4$, $B \doteq 5 \cdot 0$, $\Pi = 0 \cdot 55$ (for a constant pressure boundary layer on a smooth surface).

As already mentioned, Long and Chen[65] have recently cast doubt on the validity of (4.3) (and hence also on the Millikan-Kolmogorov Hypothesis) ; but even the systematic departures from (4.3) noted by them appear to be so small that the log law will probably continue to be used as a standard.

If we accept the similarity arguments that lead to the composite profile (4.4), many other results follow. For example, an immediate consequence is the skin-friction law (obtained by putting $y = \delta$ in (4.4)),

$$\frac{U}{U_*} = f\left(\frac{U_*\delta}{\nu}\right) + 2\frac{\Pi}{\kappa}, = \left(\frac{2}{c_f}\right)^{1/2}. \tag{4.13}$$

This implies a unique relation between c_f, Re_δ and Π, or between c_f, Re_θ and H as in the Ludwieg-Tillman relation (see Section 5.1) which can therefore be looked upon as a fit to (4.13) (Re_δ and Π can be eliminated in favour of Re_θ and H using (4.12)). Expressions can similarly be derived for other boundary layer parameters[52,68].

5. Turbulence models and closure schemes

As will already be clear, a rational scheme for prediction of turbulent flow characteristics is not yet in sight. However, the need for making such predictions is very badly felt in technology, meteorology, oceanography, astrophysics, etc. Many calculation methods and numerical models have been devised to fill this need: some very simple, others quite elaborate, but all (necessarily) empirical to a greater or lesser extent. These models differ in the degree of generality they seek to achieve, and in the amount of information regarding the known structure of turbulence that they incorporate. In the author's view, however, none of the models achieves the kind of success that would suggest that the model may have hit the dynamical truth. Thus, although it is unlikely that these models will survive far into the future, their use at the present time seems inevitable for making the kind of parameter estimates that technology needs. Incidentally, it is worth reminding ourselves that these models have till now been used chiefly for 'post-diction', *i.e.* for comparing the results of calculations made with experimental data already available.

A brief and useful summary of the current status of turbulence modelling has recently been given by Ohji[69]. It has become convenient to distinguish between simple and complex turbulent flows: Table III lists some of the factors currently thought necessary to qualify a turbulent flow to be termed complex. In general, models for simple flows can now be considered satisfactory for engineering calculations, whereas for handling complex flows further development is necessary (Table IV), and is in fact vigorously taking place.

The schemes in use can be broadly divided into three classes: integral, differential, and spectral.

5.1. *Integral methods*

These methods deal only with integrals of the partial differential equations of motion. For example, let us consider plane incompressible boundary layer flow. The Reynolds momentum equation in this case is

Table III*

Factors characterising 'complex' turbulent flows

Wall properties	*Fluid properties*
Rough Curved Porous Flexible	Compressible Electro-conductive Reactive Multi-phase Non-Newtonian Baroclinic
Flow conditions	*Additional forces*
Externally turbulent Viscosity-dependent Recirculating Three-dimensional Unsteady Non-isothermal Polluted	Normal stress terms Buoyancy Centrifugal force Coriolis force Lorentz force

* Based on Ohji[89].

Table IV*

Comparative summary of current numerical models for turbulent flows

	Integral methods	*Differential methods*	*Numerical simulations*
Feature	Economy	Accuracy	Universality
Generation	Past	Present	Future
Equations	Ordinary differential equations	Partial differential equations	Unsteady differential equations
Assumptions	Empirical functions	Empirical constants	Universal constants
Application	Simple flows	Complex flows	Large-scale flows
Utilization	Routine design work	Advanced design work	Numerical forecast
Example**	Entrainment method	The K ϵ method	Large eddy simulations

* Based on Ohji[89].

** Only a typical method is cited here.

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau}{\partial y}, \tag{5.1}$$
$$\tau \equiv -\overline{u'v'};$$

the corresponding momentum integral equation, often known after Karman, is (see *e.g.* Schlichting[70])

$$\frac{d\theta}{dx} + (H+2)\,\theta\,\frac{d\ln U}{dx} = \tfrac{1}{2}\,c_f, \tag{5.2}$$

where θ is the momentum thickness, $H = \delta^*/\theta$ is the shape factor, $U = U(x)$ is the free stream velocity as a function of the streamwise coordinate x, and c_f is the skin friction coefficient.

In laminar flow, (5.2) is solved by assuming a velocity profile $u = Uf(y, \delta)$, where f is a function of the distance y normal to the surface and the boundary layer thickness δ. Such an assumption leads directly to expressions for H and c_f as well. This is not so in turbulent flow, where H and c_f appear as independent variables ; the reason is that it is not possible to use a profile which (without introducing additional parameters) is sufficiently good all across the boundary layer that it can yield useful estimates of H and c_f as well. The usual practice is to find additional relations for H and c_f.

A very large number of proposals have been made for such relations : fig. 10 shows some of them, from a survey made by Rotta[71]. Among the most successful of these proposals is the entrainment method of Head[72] which uses the equation

$$\frac{d}{dx}(UG\theta) = U\,\phi\,(G), \tag{5.3}$$

$$G = \frac{1}{U\theta}\int_0^\delta U\,dy = \psi\,(H).$$

Here G is clearly proportional to the mass flow in the boundary layer, and (5.3) is a statement about how this mass flow varies downstream, *i.e.* about entrainment. The equation involves two empirically determined functions $\phi\,(G)$ and $\psi\,(H)$.

The third relation is provided by the Ludwieg-Tillmann skin friction formula,

$$c_f = 0{\cdot}246 \times 10^{-0{\cdot}678H}\,(U\theta/\nu)^{-0{\cdot}268}, \tag{5.4}$$

which, although now more than thirty years old, has proved remarkably durable.

The above method suffers from the defect of taking insufficient account of the 'history' of the flow (such methods have been called 'pre-historic'). There is considerable evidence, however, that in turbulent flows the stress cannot be uniquely related to the local flow field. For example, Narasimha and Prabhu[49] showed that a characteristic memory length for a turbulent wake was of the order of $10^3\,\theta$, *i.e.*

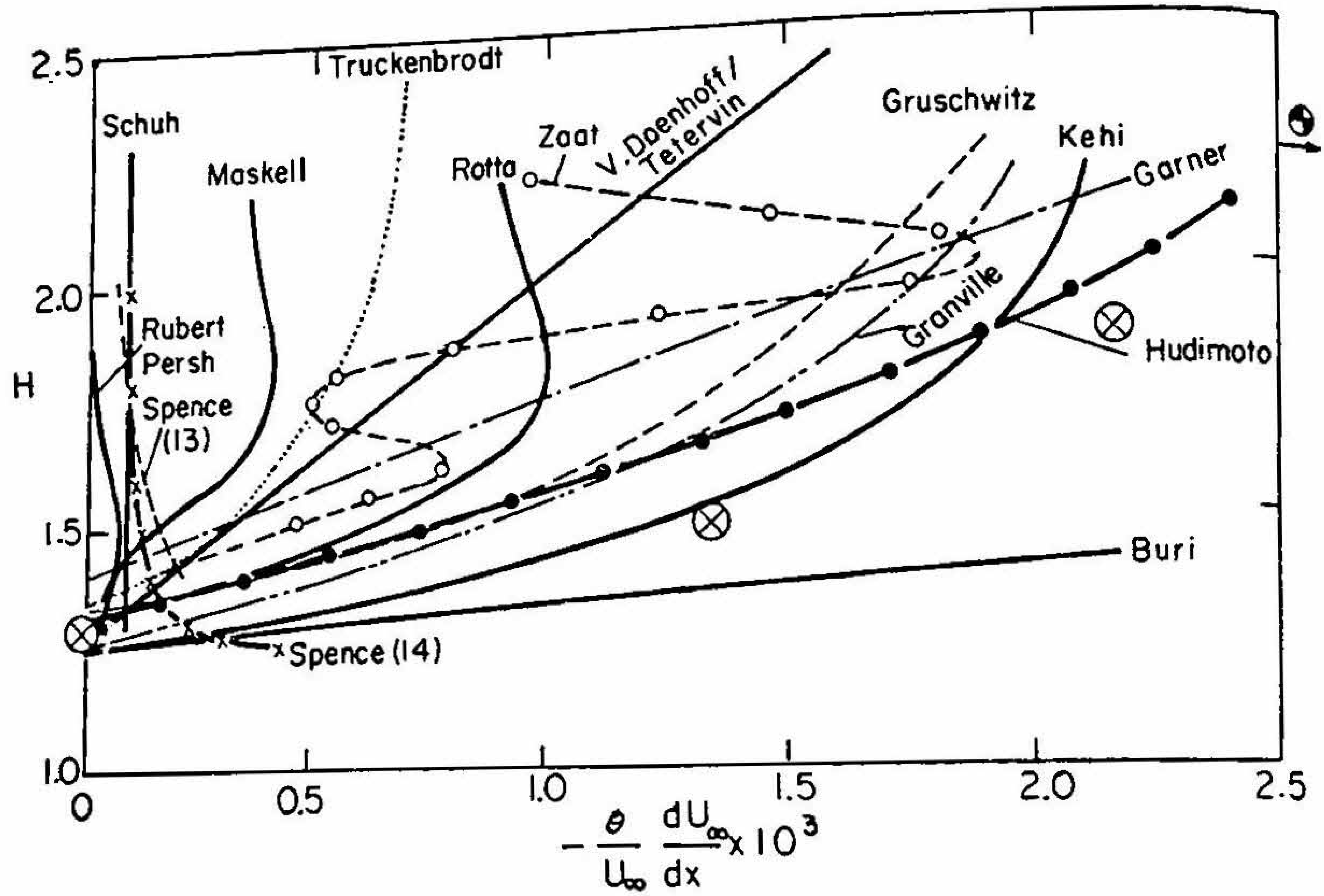


FIG. 10. All proposed equations for the shape parameter P, for use in integral methods, can be put in the form

$$L\theta \frac{dP}{dx} = -M\frac{\theta}{U}\frac{dU}{dx} - N,$$

where L is either 0 or 1, and on smooth surfaces M and N are functions of only P and Re_θ. Figure shows H as a function of the pressure gradient parameter $(\theta/U)\,dU/dx$ when $dp/dx = 0$, in the numerous proposals made : note how much of the plane is covered by one proposal or the other ! (From Rotta,[71], Figure 22·1).

the stresses at any point are influenced by the previous history of the fluid that has reached that point over a distance of that order (fig. 11).

This objection has been overcome to a considerable extent by a modification known as the 'lag-entrainment' method. Here, in addition to equations (5.2) and (5.3), a further first order differential equation for $\phi(G)$ is introduced, involving appropriately defined 'equilibrium' values of $\phi(G)$. Details can be found in Green *et al*[73]. Variants of the method for use in three-dimensional flow have also been formulated[74,75].

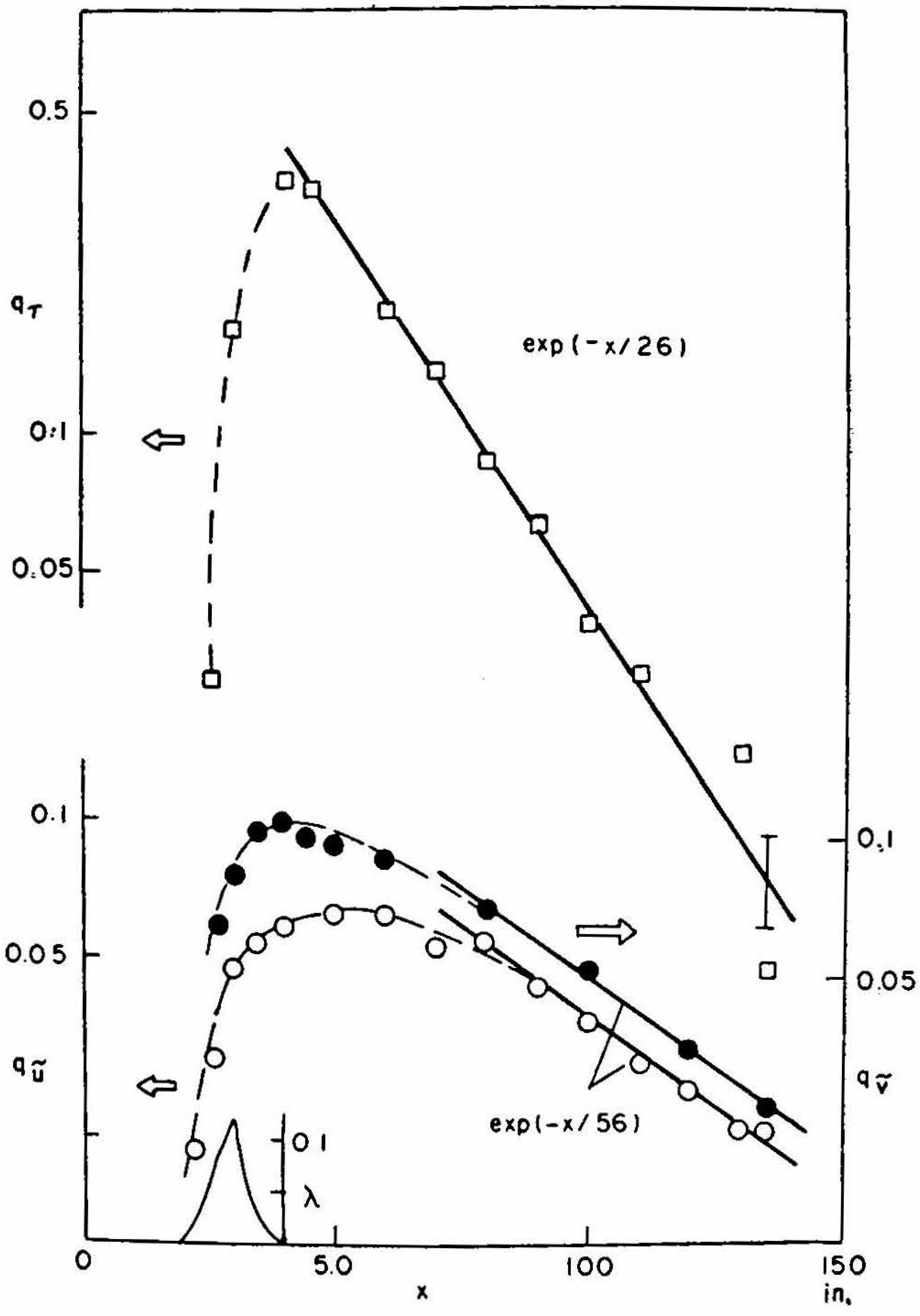


FIG. 11. The memory of a turbulent shear flow, as revealed by the approach of a perturbed wake to a new equilibrium state. The quantities q in the diagram represent non-dimensional measures of departure from an equilibrium state in a plane wake subjected to a pressure gradient, as indicated by the parameter λ: in the experiment, the free-stream velocity is increased over a short distance from one constant value upstream to a different value downstream. The subscript on q indicates its value as judged from measurements of (i) $\hat{u}$, the longitudinal r.m.s. velocity, (ii) v, the normal r.m.s. velocity, (iii) the Reynolds shear stress τ. Note that the approach to equilibrium is exponential and that the 'relaxation' distances involved are very large (56 in. = 1·42 m for $\hat{u}$, $\hat{v}$, and 26 in. = 0·66 m for τ). That is, there are measurable departures from equilibrium at distances of the order of 100 in. $\doteq$ 2·5 m downstream of a perturbing pressure gradient[76]!

5.2. *Differential methods*

There are an enormous number and variety of turbulence models used with 'differential' methods, which directly tackle the Reynolds equations (2.5). An '*n*-equation' model adds *n* partial differential equations to the Reynolds equations.

A well-known 'zero-equation' model introduces the eddy viscosity ν_T defined by

$$\tau = \nu_T \frac{\partial u}{\partial y} \tag{5.5}$$

(in two-dimensional flow). Note that ν_T (first proposed by Boussinesq in 1897) is a property of the class of flows considered and could be a function of position in that flow, but it is *not* a property of the fluid. Thus ν_T is often taken as the product of appropriate velocity and length scales, say q and l :

$$\nu_T = aql. \tag{5.6}$$

In Prandtl's mixing length theory

$$q = l \left| \frac{\partial u}{\partial y} \right| . \tag{5.7}$$

Typical variations of ν_T and l across the boundary layer, as assumed in turbulence models, are shown in fig. 12. As an example of such schemes, we may cite the work of Cebeci and Smith[76]. They assume that the eddy viscosity is given in the wall region by

$$\nu_T = (\kappa y)^2 \, [1 - \exp(-y/A)]^2 \, | \partial u/\partial y |,$$

where $\kappa = 0{\cdot}4$ is the Karman constant, and A is a damping length given by

$$A = 26(\nu/U_*) \, [1 - 11{\cdot}8P]^{1/2},$$

$$P_+ = \frac{\nu U}{U_*^3} \frac{dU}{dx} = \frac{-\nu}{\rho U_*^3} \frac{dp}{dx} .$$

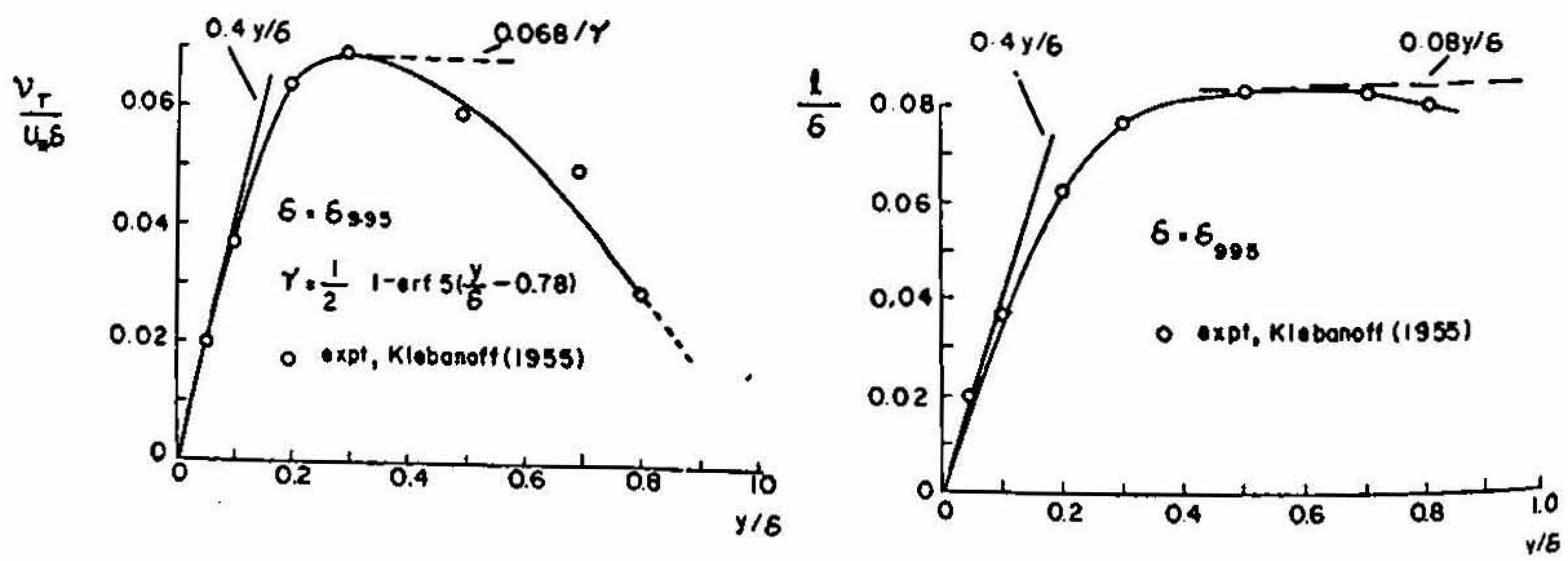


FIG. 12. Typical eddy viscosity and mixing length distributions across a turbulent boundary layer, as assumed in the calculation procedure of Cebeci and Smith[76].

In the outer layer the eddy viscosity is taken as

$$\nu_T = 0 \cdot 0168\, U\delta^* \gamma$$

where $$\gamma = \frac{1}{1 + 5 \cdot 5\,(y/\delta)^6}$$

is the intermittency; omission of γ alters the stress distribution slightly, but hardly affects the mean velocity profile.

Zero-equation models achieve closure at the mean velocity equation level of (2.5), without any additional differential equations. Such models are therefore also called 'mean velocity field closure' models.

The use of an eddy viscosity in certain flow situations has been justified by Townsend[31] through considerations of the large eddy structure; the idea has been used extensively in turbulent flow computations by Cebeci and Smith[76].

Both mixing length and eddy viscosity concepts have well-known limitations (see *e.g.* Corrsin[77]). For example, they imply that $\tau = 0$ when $\partial u/\partial y = 0$, but measurements in a wall jet, for example, show that at the velocity maximum, τ is not only not zero but quite appreciable (fig. 13). We have already referred to wake flow experiments[49] which show that *no* local theory would be satisfactory. Nevertheless, in a surprisingly wide class of flows the eddy viscosity model provides reasonable estimates of gross engineering parameters, even where the predictions of the stresses are in considerable error*.

These models are also 'pre-historic', and modifications to account for history effects have been made. A one-equation model which does not use eddy viscosity concepts but accounts for effects of history successfully (certainly in plane wakes) is a relaxation-diffusion equation[78,79],

$$\frac{d\tau}{dt} = A\,(\bar{\tau} - \tau) + \frac{\partial}{\partial y}\left(\nu_\tau \frac{\partial \tau}{\partial y}\right),$$

where A is a relaxation frequency, ν_τ a stress diffusion coefficient and $\bar{\tau}$ an equilibrium stress distribution.

Nee and Kovansznay[80] proposed an equation of the form

$$\frac{d\nu_T}{dt} = A\nu_T \left|\frac{\partial u}{\partial y}\right| - B\,\frac{\nu_T^2}{L^2} + \frac{\partial}{\partial y}\left(\nu_T \frac{\partial \nu_T}{\partial y}\right), \qquad (5.8)$$

but this can be cast in the form of an equation for the stress by substituting $\nu_T = \tau/(\partial u/\partial y)$, and so is properly considered an example of a 'one-equation' model. Relaxation equations for the mixing length have also been used.

* This must surely mean that the same rusults can be obtained without the use of an eddy viscosity; thus, a suitably designed integral method should in general be equally effective, and perhaps simpler and cleaner as well.

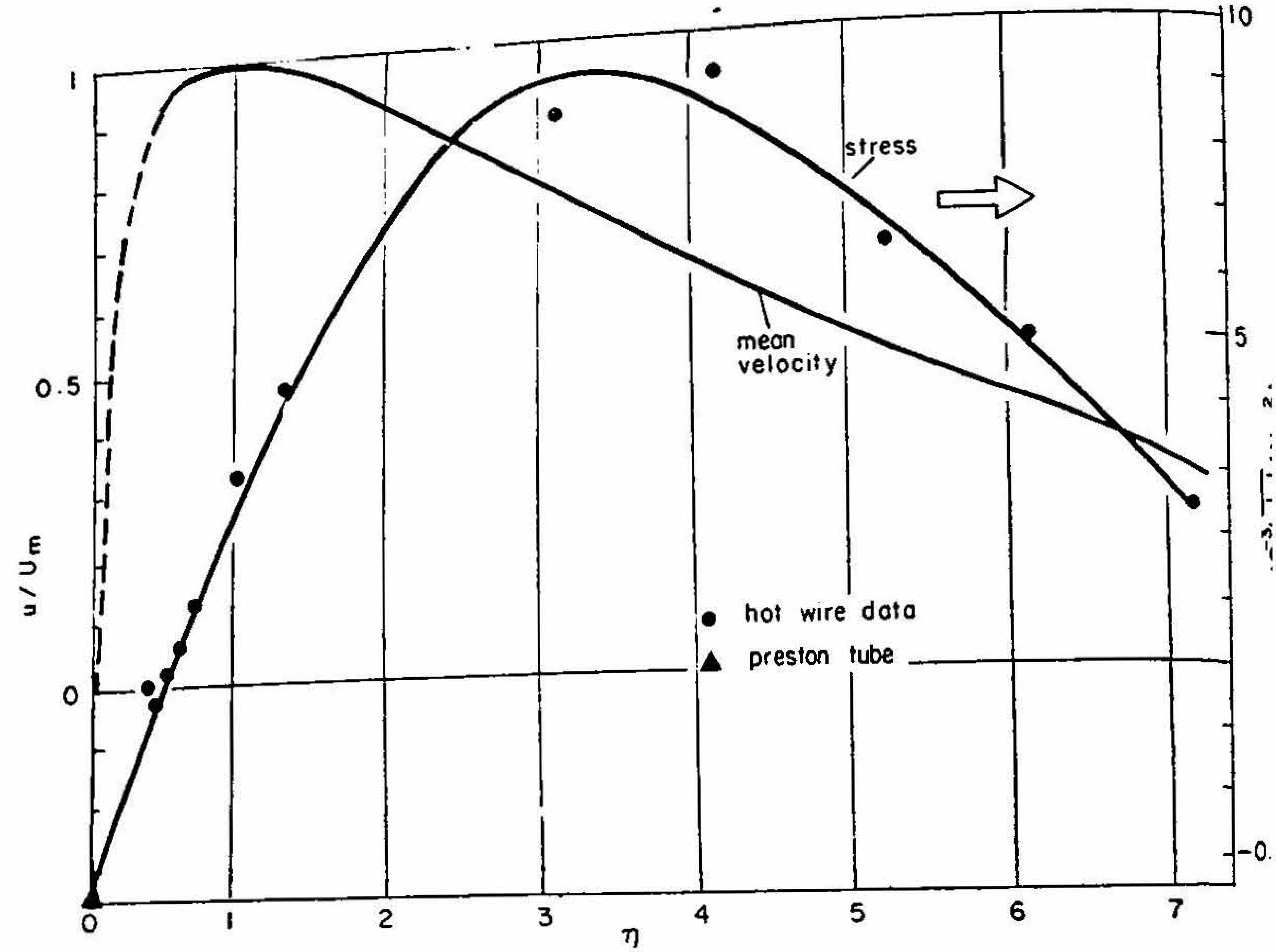


FIG. 13. Reynolds shear stress in a wall jet in still air[129]. Note how at the velocity maximum ($\eta = 1$, $\partial u/\partial y = 0$), the stress is comparable to the wall value, but opposite in sign; the stress vanishes at $\eta \doteq \frac{1}{2}$.

Among one-equation models perhaps the best-known is Bradshaw's, which models the equation for turbulent energy (obtained from (2.6) by putting $i = j$ and summing). It is further assumed that the stress is proportional to the turbulent energy :

$$\tau = 2a_1 K, \; K = \tfrac{1}{2} \, \overline{u_i' u_i'}. \tag{5.9}$$

Obviously this cannot be universally correct, as it implies that τ, like K, must always be positive (a counter-example is already provided in fig. 13). The model is

$$\frac{d\tau}{dt} = 2a_1 \left[\tau \frac{\partial u}{\partial y} - \tau_{max}^{1/2} \frac{\partial}{\partial y} (G\tau) - \frac{\tau^{3/2}}{L} \right] \tag{5.10}$$

where τ is in kinematic units, and τ_{max} is its maximum value ; a_1 is a constant ($= 0{\cdot}15$), and the non-dimensional quantities G and L/δ are taken to be (empirically selected) universal functions of y/δ (fig. 14). Equation (5·10) is hyperbolic, and Bradshaw *et al*[81] have used a method of characteristics to integrate it.

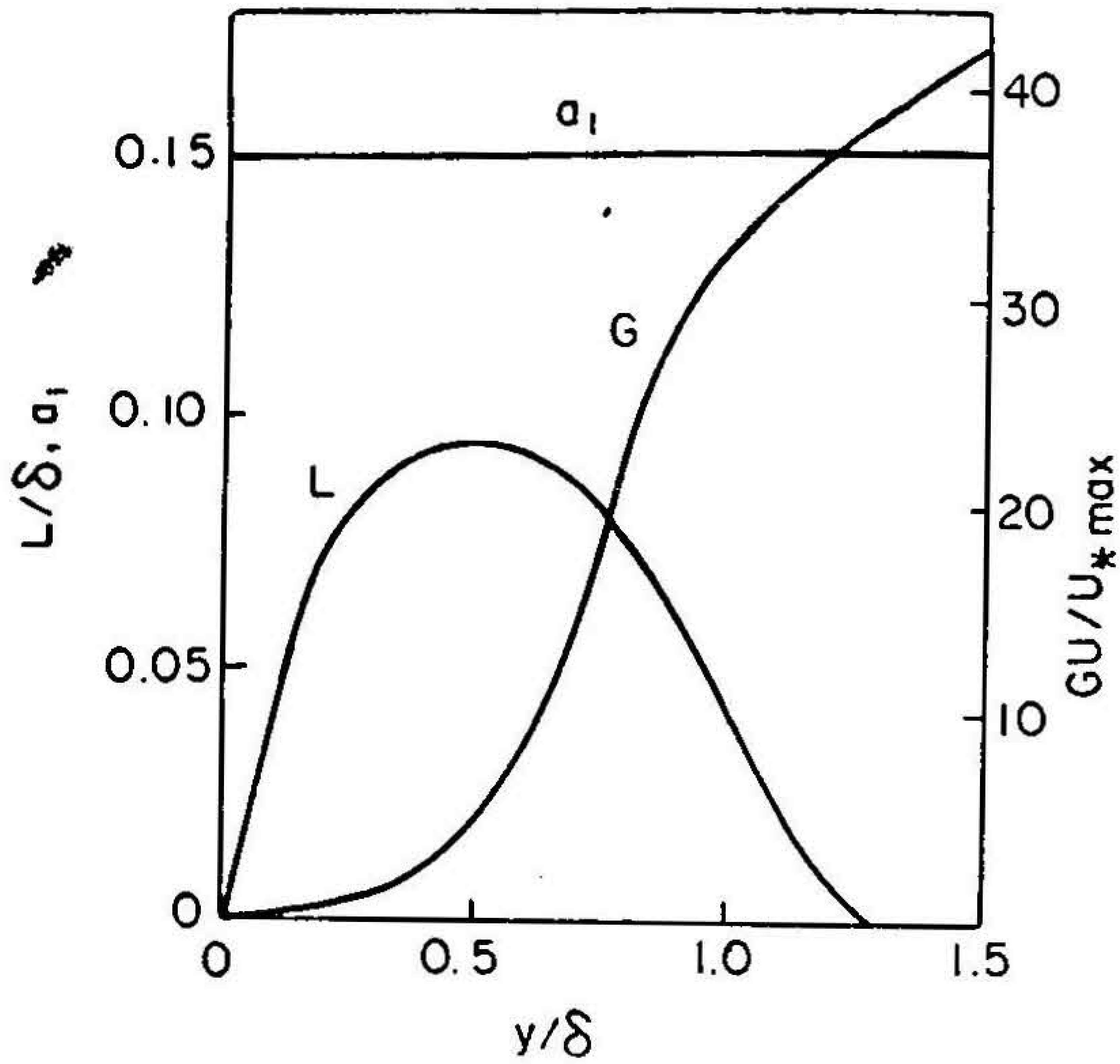


FIG. 14. Empirical function used in Bradshaw's model for turbulent boundary layers. δ here is the height above the surface at which the velocity is 99·5% of the free stream value. U_{*max} is the friction velocity based on τ_{max}.

Although (as already mentioned) the assumption (5.9) about stress in the Bradshaw model has obvious limitations, much information from experiments on the structure of turbulent means has gone into modelling the terms in the energy transport equation, which has therefore been widely used with much success.

Two-equation models have been studied extensively in recent years ; the basic argument behind such models of the ' Prandtl-Kolmogorov ' class is that, in addition to the energy equation (5.10) which in essence determines a velocity scale $K^{1/2}$, a separate equation for a length scale may also be necessary. Such an equation has for example been proposed by Ng and Spalding[82]. But the dynamics governing the length scale is obscure. Instead of an equation for a length scale, Saffman[83] proposes one for a pseudo-vorticity, which may be considered the reciprocal of a time scale.

Perhaps the most widely investigated two-equation system is the so-called ' $K\epsilon$ model ' Here an eddy viscosity is assumed in the form

$$\nu_T = C_\mu K^2/\epsilon,$$

where C_μ is a constant and

$$\epsilon = k^{3/2}/L$$

is a dissipation rate. In plane flow, K and ϵ are supposedly governed by the equations

$$\frac{dK}{dt} = \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_K}\frac{\partial K}{\partial y}\right) + \nu_T\left(\frac{\partial U}{\partial y}\right)^2 - \epsilon,$$

$$\frac{d\epsilon}{dt} = \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial y}\right) + C_{\epsilon 1}\frac{\epsilon}{K}\nu_T\left(\frac{\partial u}{\partial y}\right)^2 - C_{\epsilon 2}\frac{\epsilon^2}{K}.$$

In the '$K\epsilon 1$' model, the constants are taken as follows.

C_μ	σ_K	σ_ϵ	$C_{\epsilon 1}$	$C_{\epsilon 2}$
0·09	1·0	1·13	1·43	1·92

Development of the $K\epsilon$ models is described by Jones and Launder[84], Jones[85] and Launder *et al*[86].

An even more elaborate model would consist of a set of equations that govern the development of *each* component of the Reynolds stress tensor. In these so-called 'Reynolds stress field closures', the equations for the τ_{ij} take the form

$$\left(\frac{\partial}{\partial \tau} + U\frac{\partial}{\partial x_k}\right)\overline{u_i' u_j'} = P_{ij} - 2\epsilon_{ij} + D_{ij} + \pi_{ij}$$

where each of the terms on the right is modelled. Among models of this type is one due to Hanjalic and Launder[87].

We have had space to mention only one example of each kind of turbulence model; there are now a large number of these models, and the number keeps growing. A conference held in 1968 at Stanford[88, 89] took stock of the experimental data available at the time, and the relative performance of various prediction methods. A similar attempt was made during 1980–81, to consider in particular complex turbulent flows, and to assess the data acquired and the new methods developed since the 1968 conference. No single method emerged as clearly superior in either conference, but the general consensus appears to be this[131]. Some of the simplest integral methods work very well in the classes of flows for which they have been tuned and tested; the $K\epsilon$ models seem to have wider applicability than the others that have been tried. It appears likely therefore that we will see continued use of some simple integral methods as well as further refinement of partial differential equation methods. Whether a universal model for all classes of turbulent flows will emerge is still not clear; I consider it unlikely.

6. Higher level models

Each of the methods we have described till now postulates a mathematical model for the turbulent mean quantities to enable closure of the Reynolds equations: they are basically empirical, in the sense that (as already pointed out) there is no implied rational procedure for improving any of the models. There are however some other approaches to the turbulence problem which involve a much lower order of modelling (in the sense that less is modelled, more is left to genuine dynamics). One of these is called 'large eddy simulation'. This class of methods, originally introduced by meteorologists, is now being investigated for application in technology.

6.1. *Large eddy simulation*

The basic idea here is to divide turbulent eddies into two broad classes, following the general description of Section 3. The large eddies carry most of the energy, and are chiefly responsible for so-called eddy transport : they have a long memory, depend on how the flow was created, and are correspondingly hard to model. On the other hand the small eddies are more nearly universal, and tend to depend only on certain gross quantities characteristic of the turbulence, such as *e.g.* the dissipation. It might therefore be fruitful to model the small eddies but compute ('exactly') the large ones. If the computational grid is chosen to suit the large eddies, the small eddies are sub-grid scale motions ; so this kind of simulation is also often called sub-grid modelling.

The large eddy motion is defined by a filtered or coarse-grained variable.

$$q' = \langle q(x,t) \rangle' = \int G(x,y\,;\,\Delta)\, q(y,t)\, Dy,$$

where the dashed angular brackets indicate the filtering operation and G is a filter function, of effective width Δ in the sense that scales smaller than Δ are smeared out. For example, a Gaussian filter would be

$$G(x,y\,;\,\Delta) = (6/\pi)^{3/2}\,\Delta^{-3} \exp -\left(\frac{6\,(x-y)^2}{\Delta^2}\right).$$

Then we split q,

$$q = \langle q \rangle' + q'' = \bar{q} + q',$$

where q'' is the sub-grid variable ; note that in general

$$\langle q'' \rangle' = \langle q \rangle' - \langle \langle q \rangle' \rangle' \neq 0.$$

The filtered Navier-Stokes equations can be written as

$$\frac{\partial u_i'}{\partial x_i} = 0$$

$$\frac{\partial u_i'}{\partial t} + u_j' \frac{\partial u_i'}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial u_i'}{\partial x_j} + S_{ij} + A_{ij}\right),$$

where

$$S_{ij} = \langle u_i'' u_j'' \rangle - \langle u_i'' u_j \rangle - u_i u_j''$$

$$= \text{the 'sub-grid stress'},$$

$$A_{ij} = u_j' u_i' - \langle u_j' u_i' \rangle.$$

The idea is to model the sub-grid stress here as in mean velocity field closures, and compute the filtered velocity field.

Sub-grid modelling could use a corresponding sub-grid eddy viscosity, or could be more elaborate, involving equations analogous to Reynolds stress closures. Reviews of the present status of large eddy simulation are presented by Ferziger[90] and Herring[91]. Although there are still some unresolved issues concerning the modelling of sub-grid motions[91], it would appear that the final results are not too sensitive to the sub-grid model.

Indeed the solutions so obtained are quite close to observations, not only regarding mean parameters (fig. 15), but also on the turbulent flow structure[92].

However, the time taken to compute shear flows of practical interest is still very large ; *e.g.* channel flow at a Reynolds number of 5000 demands about 100 hr of the CDC-7600 for one solution ; to establish the validity of this solution it probably needs about 1000 hr. It would therefore appear that, as a method of making engineering calculations, large eddy simulation is still prohibitive in cost and time, but with advances in computing systems the situation may well change. Even otherwise, it is possible that 'exact' numerical solutions on the computer would provide information on quantities needed in lower-order models — quantities that are very hard to measure, *e.g.* the pressure-strain terms of (2.6).

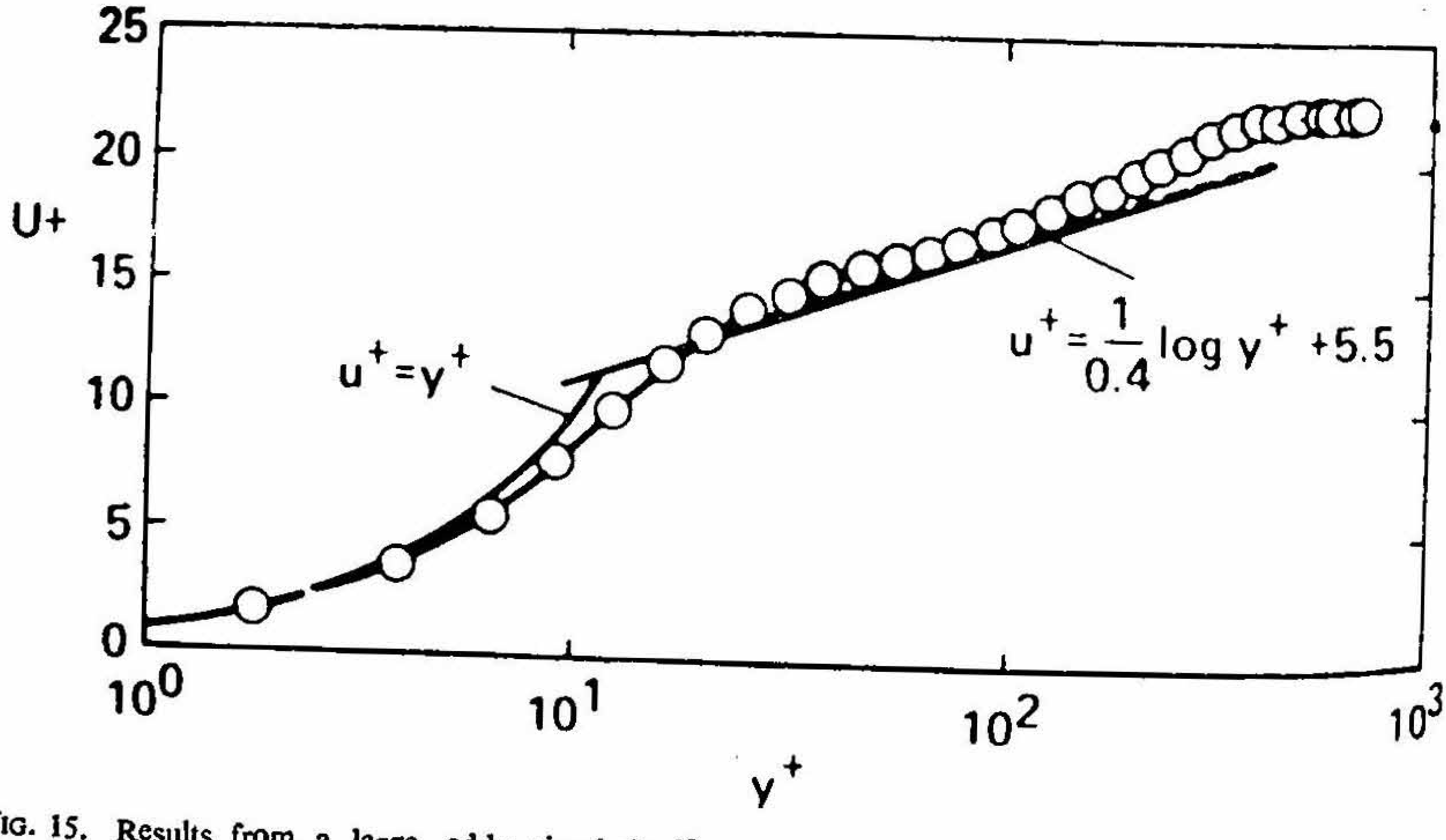


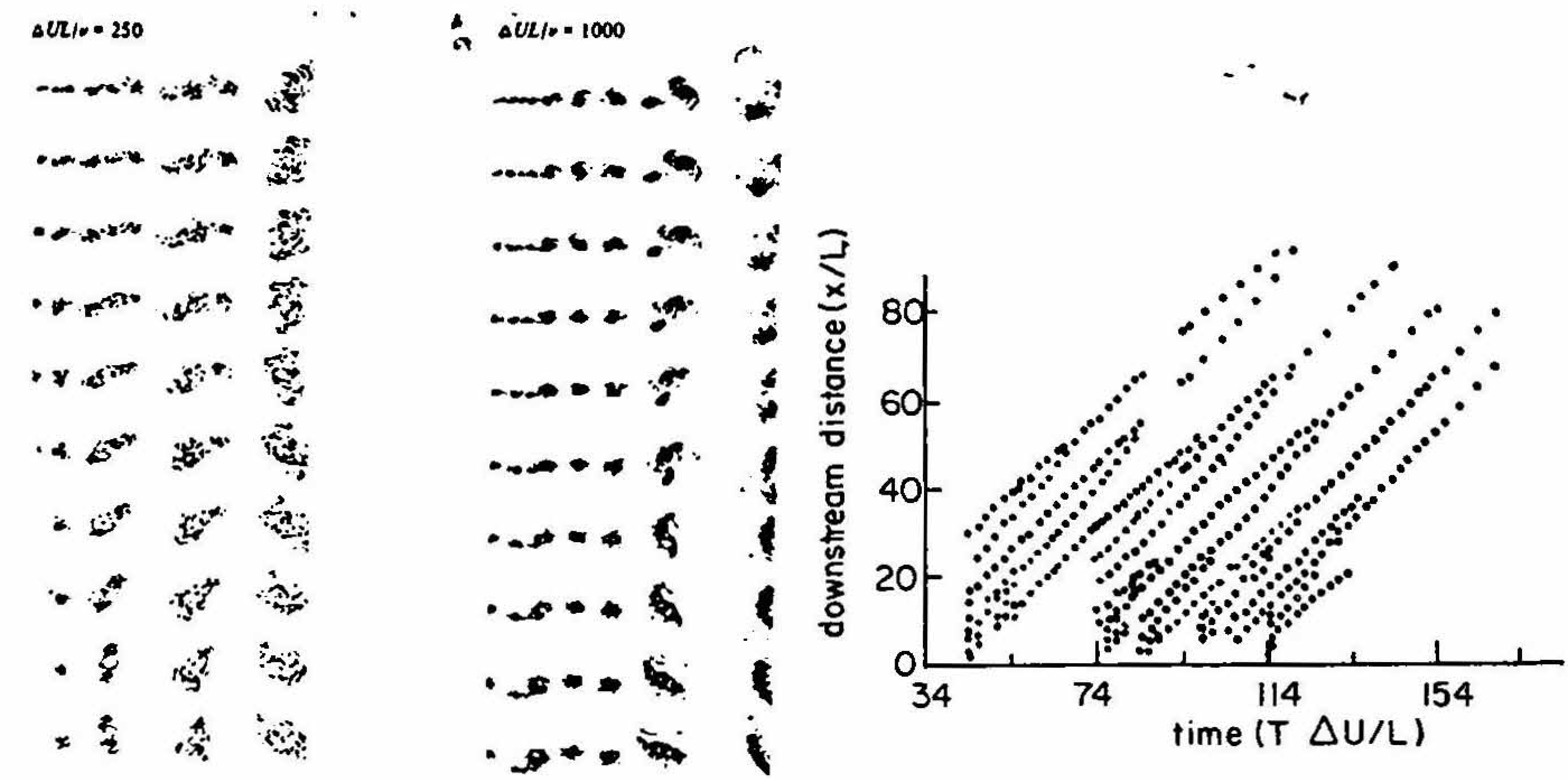
FIG. 15. Results from a large eddy simulation[92] of flow in a channel (points), compared with standard loglaw profile.

6.2. *Vortex dynamics*

We saw in Section 3 that the dynamics of much of turbulent flow, especially the large eddies, is largely inviscid. Recent experiments have suggested, as we shall see later, that in many turbulent flows there is a considerable degree of spatial order ; in particular, mixing layers have revealed recognizable vortices.

Several very interesting calculations have recently been made working out the dynamics of vortex filaments[93-95]. For example, a plane mixing layer is computed by studying the evolution in time of a layer of vortices (fig. 16) which interact through the Biot-Savart law. The calculations are particularly simple and effective in a mixing layer, where at Reynolds numbers that are not too high the vortex structures are largely two-dimensional; thus a collection of vortex filaments provides a reasonably faithful representation of the actual flow.

The effect of viscosity has been sought to be introduced in different ways, but none of them seems completely satisfactory. An obvious way is to include a viscous core, whose diameter will increase like the square-root of time by viscous



(i) (ii)

Fig. 16. (i) Streakline plots of each discrete vortex for a time ($L/\Delta U$) with respect to the average velocity. The field of view is equal to 40 L. The top plots have the origin at the left with each succeeding plot displaced two units downstream and two units later in time. Left column is Re = 250 flow and right column is Re = 1000.

(ii) Trajectories of vorticity clusters in Re = 250 flow. Flow field shown in (i) starts at time 124 (from Ashurst[95]).

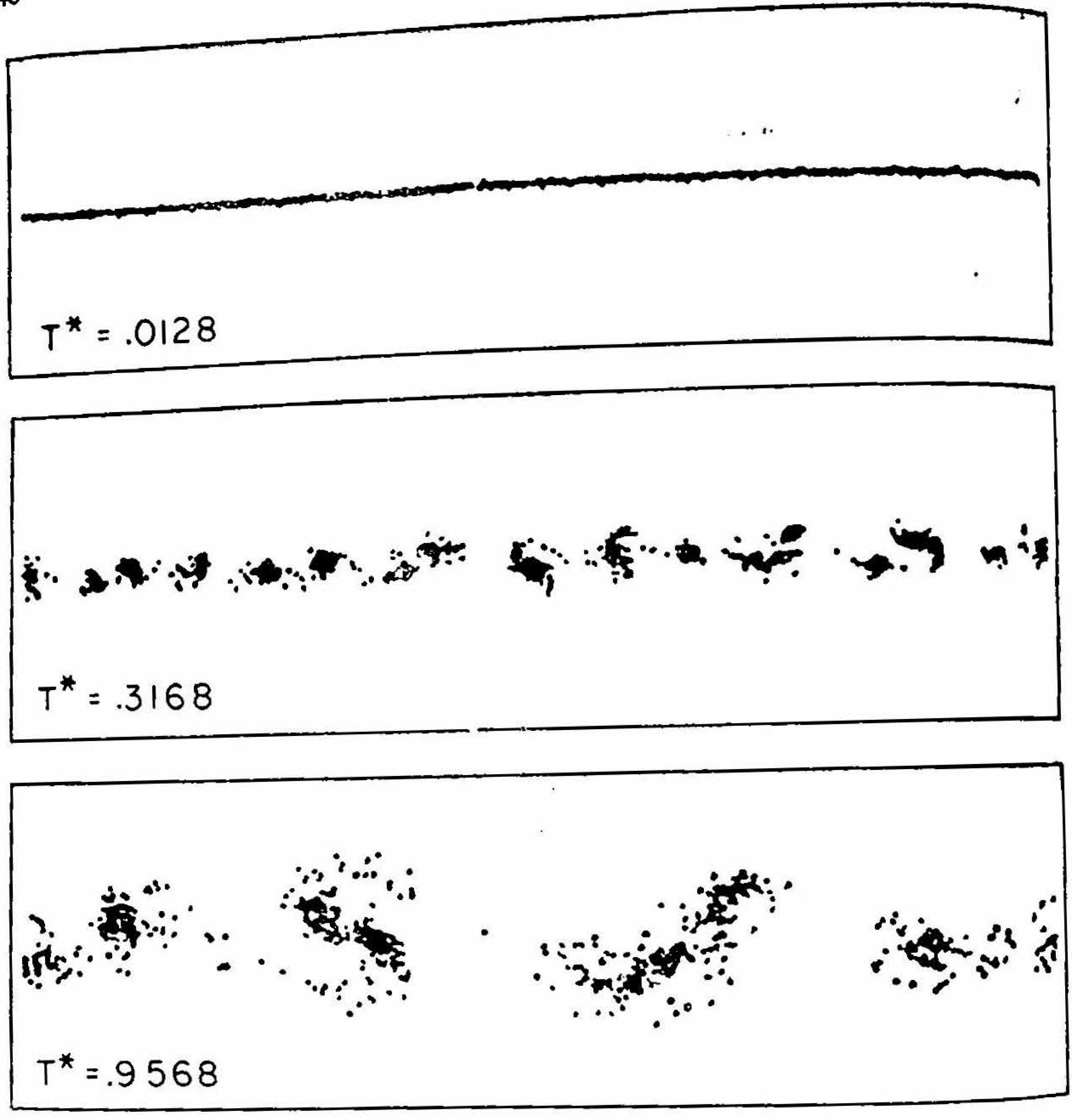


FIG. 16. (iii) Typical results from the unsteady one-dimensional vortex calculations of Delcourt[94].

diffusion. A more elegant way[96] is to impose a random walk on the filaments: it is well known that in the limit of small steps and times such a random walk is equivalent to diffusion. Delcourt[94] divides physical space into a large number of cells; the position of each vortex is calculated at the end of each time step, and the vortex is then placed at the middle of the cell it happens to reach before the next round of displacements is calculated. It may be shown that this procedure is equivalent to the use of an effective viscosity.

In contrast to the calculations of Acton and Delcourt, who used *one* space dimension + time, Ashurst[95] has computed the mixing layer using vortex trajectories in *two* space dimensions + time. Ashurst uses the method suggested by Chorin[96]; the inviscid flow is solved *via* discrete vortices (no finite difference mesh is required), the viscous effect being incorporated in the form of a superimposed random walk of the vortices. If the dispersion produced by this random walk is an r.m.s. distance σ_r, the effective kinematic viscosity is

$$\nu = \sigma_r^2/\Delta t$$

where Δt is the time step in the calculation.

Calculations using point vortices experience some difficulty, with the velocity divergence at the centre ; this is tackled in Chorin's method by using a short-range cut-off δ, so that for a radius $r < \delta$ the induced velocity has a constant value. The difference between the Euler solution and the numerical results can be shown to be $0(\delta^2)$.

7. Experimental work on coherent structures

For about three decades following G. I. Taylor's pioneering studies in the statistical theory of turbulence (Section 3), experimenters concentrated on measurements of spectra, correlations and turbulent energy balance in a variety of shear flows. These studies gave considerable insight into the dynamics of such flows, but perhaps led to an over-emphasis on an approach to turbulence that looked upon the phenomenon as a rather complicated kind of ' noise ' (in the electrical engineer's sense). Although measurements of correlations often pointed to the presence of ordered structures, and these were identified by Townsend[31] in several flows, the inferences drawn in these cases often seemed like inspired guesses at best and made no strong impact on research. However, in recent years experimental evidence has accumulated, indicating the presence of a far greater degree of spatial organization than had been conceded earlier. In particular, the experiments of Brown and Roshko[63] in a turbulent mixing layer provided spectacular visual evidence for the presence of organized motion or—to use terminology that is now getting to be widely accepted—' coherent structures '.

According to Coles[97], the adjective ' coherent ' means ' having an orderly and logical arrangement of parts such as to assist in comprehension or recognition. The meaning of the word ' structure ' is less transparent '. Hussain[98] defines a coherent structure as ' a turbulent fluid mass connected by a phase correlated vorticity '.

As already mentioned, the most convincing evidence of coherent motion appeared in the work of Brown and Roshko[63], one of whose shadowgraphs of a mixing layer between fluids of slightly different density is reproduced in fig. 17. It was at first difficult to believe that this picture represented *turbulent* flow, but the observed rate of

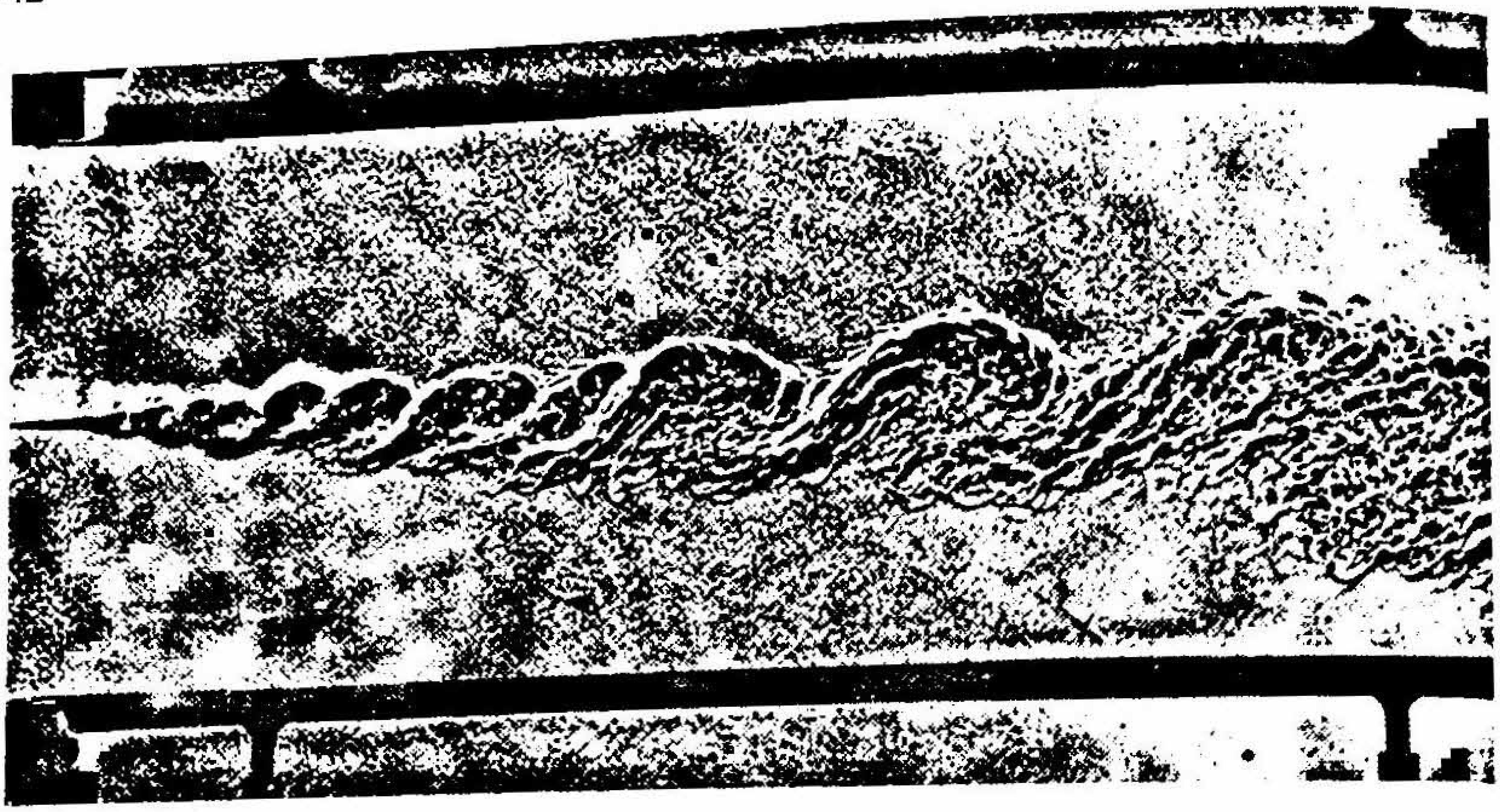


FIG. 17. Spark shadowgraph of turbulent mixing layer between nitrogen flowing at 8·8 m/s (upper stream) and helium flowing at 3·3 m/s (lower stream); pressure = 4 atm.[63]. Large organized vortical structures are clearly revealed.

growth and the chaotic nature of the fluctuations at any fixed point in the flow leave no doubt about the point. At relatively low Reynolds number the flow organization is surprisingly two-dimensional, but as Re increases transverse structures develop. There is still some controversy about the degree of organization present at very high Reynolds numbers; *e.g.*, Bradshaw[22,100] holds the view, based on his own mixing layer experiments, that what coherence may be present gets eventually lost. Coles's definition of coherence, quoted above, poses a pattern recognition problem; it is possible that an underlying pattern may be masked by a lot of 'hash', making recognition difficult in many situations. If however the hash is dynamically irrelevant (and is there only to confuse the observer, so to speak !) and much of the significant transport occurs through the coherent structures, it becomes important to understand and identify them. The real significance of Brown and Roshko's work is perhaps its demonstration that a high degree of spatial organization is not inconsistent with turbulent transport as we have always known it.

Is there a way of recognizing a coherent structure even when it is not visually apparent, as may happen either when visualization is difficult or when it is masked by 'hash'? How do we recognize and eliminate 'hash'? Most attempts at answering this question use the concept of 'conditional' averaging or sampling, the condition often being a phase. For example, we may decompose any quantity in a turbulent flow into three parts,

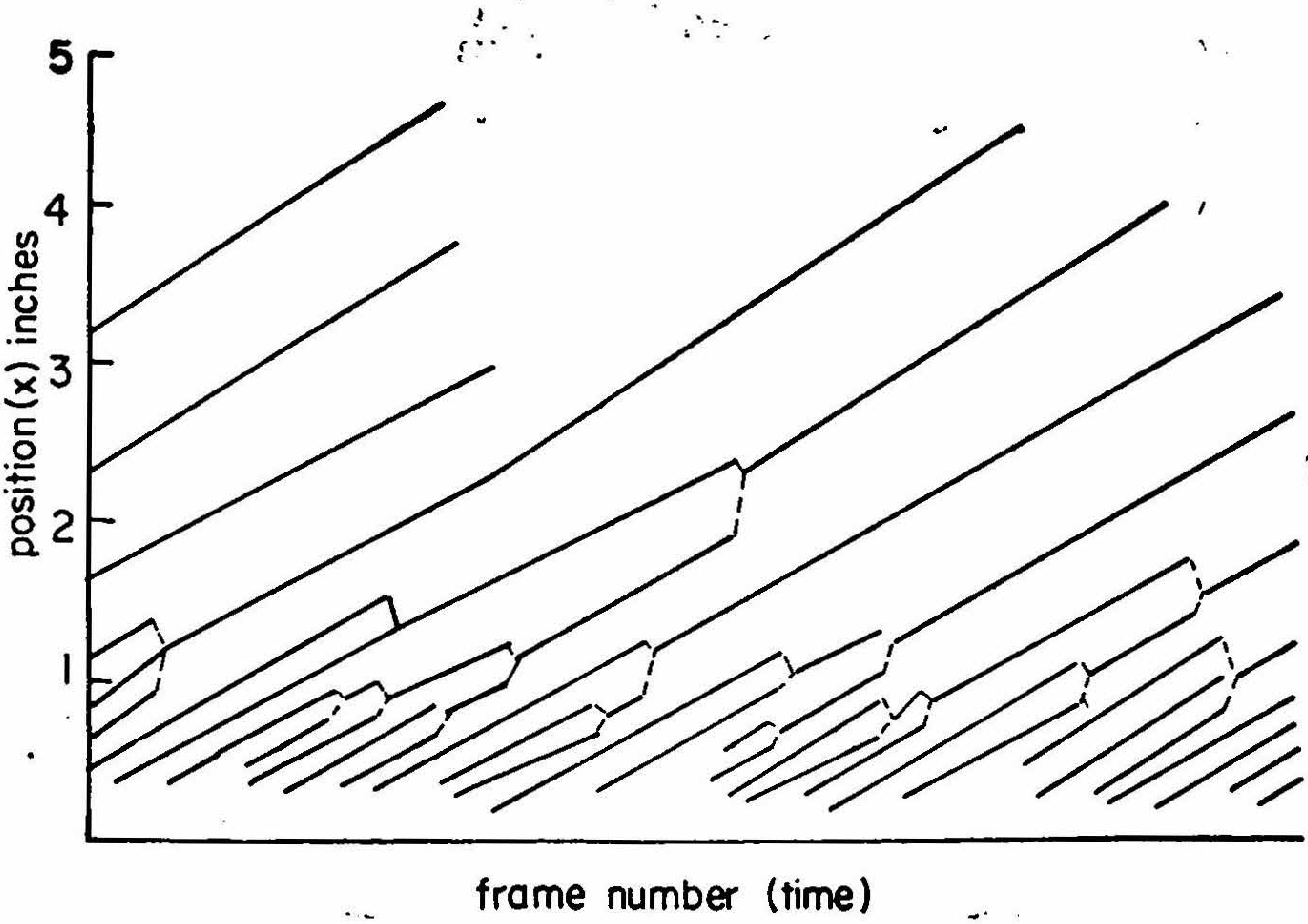


FIG. 18. Trajectories, in the *xt* plane, of vortices like those shown in (i). Note the vortex amalgamations taking place where two neighbouring trajectories meet and result in a single trajectory downstream; also the gradual increase in separation distance between vortices as one goes downstream, in general agreement with turbulence similarity arguments requiring the increase to be linear.

$q(x)$	$+\ \tilde{q}(x, t)$	$+\ \check{q}(x, t)$.
(mean)	(coherent part)	(incoherent or 'random' part)

To call $\check{q}$ random may be misleading, because it would suggest that $\tilde{q}$ is deterministic—which would be wrong; even the coherent part has a strong random element. For example, the vortices in the mixing layer of fig. 17 are not periodic; their separation, life time and other characteristics are best described in terms of probability distributions (fig. 18).

Now if a 'phase' can be defined in some meaningful way, the coherent part can be 'educed' by averaging at constant phase; the assumption is that the so-called phase-average

$$\langle q^{\vee} \rangle_{\phi} = 0, \quad but \langle q^{\vee} \rangle \neq 0,$$

where $\langle q^{\vee} \rangle_{\phi}$ denotes an average at fixed ϕ, *not* an average over all phases.

As an example of such eduction, we may examine the work of Cantwell[101] on the wake behind a circular cylinder at a Reynolds number (based on free-stream velocity and diameter) of about 140 000. He used a transducer on the surface of the cylinder upstream of the point of separation, and defined the phase from the nearly periodic signal from this transducer. (In actuality there is a jitter in the period of this signal, and this blurs the educed structure to some extent.) The velocity field in the wake was measured by a 'flying hot wire', *i.e.* one which was in motion *relative to the cylinder*, to ensure that there was an appreciable relative velocity *between the probe and the fluid* even in those regions where the fluid was nearly stagnant relative to the cylinder.

The educed flow field is shown in fig. 19; it is immediately apparent that there is a strong well-organized motion even at the high Re of the experiment. This coherent motion accounted for about half the total Reynolds shear stress in the flow.

Perry and Watmuff[102] have studied the wake behind an oblate ellipsoid and concluded that the coherent contribution is 'probably more' than 15–30%. They can recognize the structures up to a distance of 20 diameters downstream of the cylinder.

Because of the problems connected with phase jitter and eduction, these figures for the coherent contribution to momentum transport can only be a *lower* bound : *i.e.* the sum of the contributions from all the individual (jittering) structures is certain to be more, although how much more is not easy to establish.

The main conclusion from these experiments is that the educed field is rather like unsteady laminar flow. We are led to suspect that, if we could properly educe the flow from a chimney in cross-wind, we would find it very much like that from a cigarette.

A host of other flows have been studied in a similar way ; it is not possible to cite all of them here, but particular mention may be made of the many investigations of Hussain and his colleagues in axisymmetric jets and other free flows[98], and of the vastly more complex flow in the turbulent boundary layer by Kline *et al*[17], Corino and Brodkey[103] and many otders.

The boundary layer is worth some further attention, because of its importance. The flow near a surface is more complex and less evidently organized than free shear flows, but Kline *et al*[17] showed from visual observations that most of the turbulent production occurred in short periods of activity or 'bursts'. Heathershaw[104] reported measurements in the Irish Sea that showed that events outside three standard deviations, lasting only 3% of the time, accounted for 31% of the shear stress.

Corino and Brodkey[103] showed that each burst is part of a cycle of events now known as ejection, inrush and interaction respectively (fig. 20). The formation of

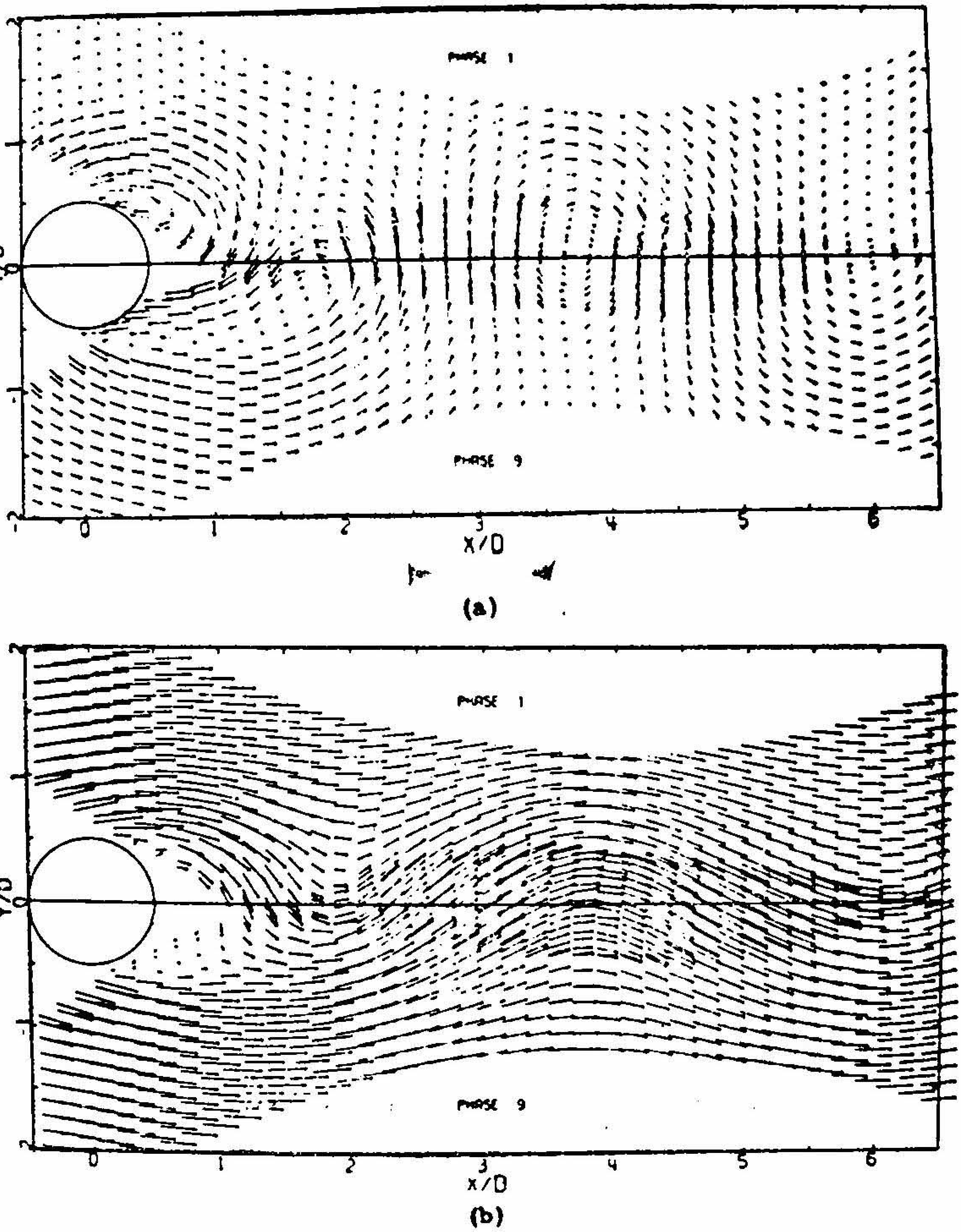


FIG. 19. The velocity field averaged at a selected fixed phase behind a circular cylinder at a Reynolds number of 140000 [101]. Sixteen phases were defined in each cycle of the signal from a pressure transducer on the surface of the cylinder; the signal showed a mean vortex-shedding frequency of 37 Hz, the actual frequency being within 10 Hz of the mean 90% of the time.

The pictures show the velocity field in a frame of reference (*a*) moving with the vortices, (*b*) moving with the cylinder.

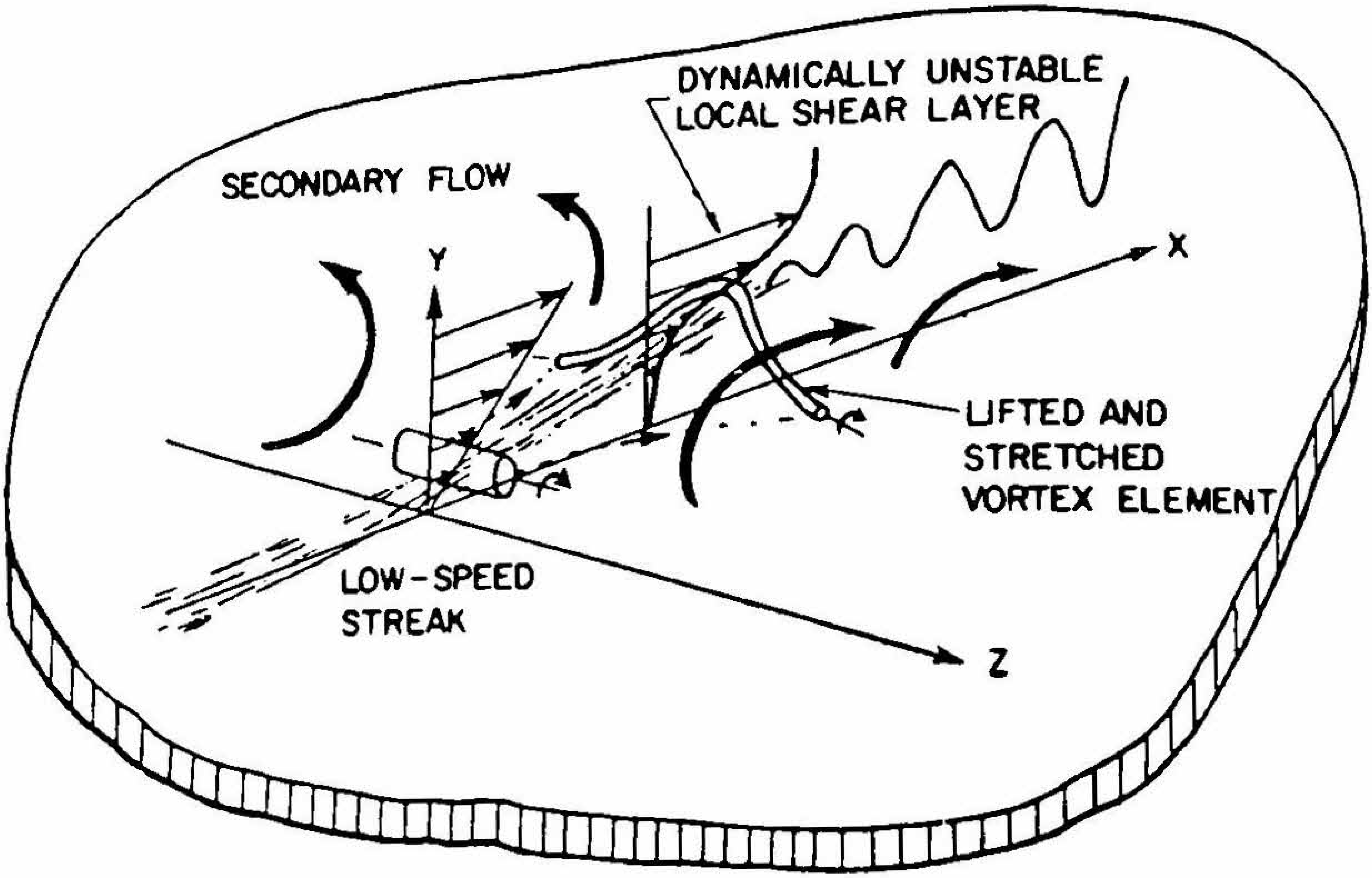


FIG. 20. Schematic sketch of the bursting phenomenon in a turbulent boundary layer[17]. The low speed streaks occur at distances of order $100\,\nu/U_*$ from each other along z; a vortex element near the wall is lifted and stretched by the secondary flow associated with the low speed streak. Th resulting unstable shear layer may be responsible for a rapid increase in high frequency turbulent energy. The bursts occur at a frequency of approximately $U/5\delta$[105].

intense shear layers during ejection and inrush, leading to vortices of the mixing layer type, appears to play a leading role in the generation of turbulent energy and its transport. These observations suggest that turbulence may in a sense be continuous transition. The frequency of occurrence of bursts scales on outer variables (say free-stream velocity U and boundary layer thickness δ) ; this finding[105] indicates that the flow in the wall layer is coupled with the outer flow, being on the one hand forced into catastrophic bursts every now and then by outer events, and, on the other, feeding the energy so created outwards during the rest of the cycle. The whole process by which turbulence sustains itself is in fact rather similar to that by which an internal combustion engine runs. The power is created during a short spark (=burst), which is made at a frequency determined by the rotation speed of the output shaft, which itself is determined by the rate of energy production. The closed loop here is very similar to that in turbulent flows; the only difference is that the cycles in turbulence are more irregular; it is as if the engine were sputtering and misfiring all the time !

Unfortunately, all this insight has not yet been translated into hard theory; there is no prediction method based on our knolwedge of coherent structures. The formulation of such theories must surely figure prominently on the agenda for future research,

8. Turbulence management

Methods of achieving turbulence control have been the subject of much research especially because of possible applications in technology. Perhaps 'management' is a more appropriate word than 'control', because both suppression and enhancement, of turbulence are of interest in technology, depending on the particular application. Suppression is useful if skin friction or fluctuating loads on a surface are to be reduced; enhancement is useful if *e.g.* greater heat transfer is desirable or if a boundary layer has to overcome an adverse pressure gradient. As we noted in Section 1, Reynolds had already listed factors promoting and suppressing turbulence: see Table I. A classic example of turbulence management was Wieselsberger's use of a trip wire on a sphere, whose drag was thereby reduced appreciably at what would otherwise have been sub-critical Reynolds numbers as the enhanced turbulence delayed separation of the boundary layer[106].

Most methods of turbulence management now in use may be termed 'direct', in the sense that they involve injection or exchange of appreciable amounts of energy. Of course, where the goal is say a reduction of drag, there must be a net gain in energy for the control to be worthwhile. For example, the use of vortex generators on an air-craft wing entails a certain additional drag, but at higher angles of attack the energiza-tion of the boundary layer by the vortices will reduce the pressure (and hence the total) drag of the wing by preventing or delaying the separation that would otherwise have occurred. The use of (steady) blowing for similar purposes, or of (steady) suction of the wall boundary layer to reduce turbulence intensities in a wind tunnel test section, or the use of screens in the contraction of a tunnel to damp turbulence, all belong to this category. Extensive studies of boundary layer control techniques of this 'direct' type have been made for a long time[107,108]. Although these studies have led to the use of certain boundary layer devices in practice, the spectacular savings that have always seemed in principle possible, specially by total laminarization of the flow, have not been realized in practice till today.

This brings us to a discussion of laminarization.

A detailed analysis and survey of such 'laminarizing flows' has recently been published[46], so it is sufficient here to recapitulate the conclusions of that survey. First of all, there is the remarkable fact that the number of agencies that can cause laminarization is so large: curvature and rotation (of the right sign*), heating, cooling, suction, favourable pressure gradients, magnetic fields (when the fluid is electrically conducting)—all these can, under the right conditions, render turbulent transport negligible or irrelevant.

* The right sign for suppression of turbulence is that the sense of the rotation, or the curved stream-lines, must be the same as that of the vorticity in the shear layer.

Secondly, one needs to be clear about what 'laminarization' means[109]. For some, laminarization can only mean the total disappearance of turbulence, or at least a tendency towards zero turbulence as an asymptotic state. This concept is clearly what is important to engineers who might be interested, for example, in estimating the loading due to random pressure fluctuations on a surface in a flying vehicle, or the disturbance levels likely to be encountered in a wind tunnel. If the intensities in these cases are not sufficiently low, it would be justifiable to hesitate in calling the flow laminar. On the other hand, there are many situations where intensities may not tend to zero but many flow parameters, including such important ones as skin friction and heat transfer coefficients, attain laminar values. Again, to engineers interested in estimating the drag of a surface in a flying vehicle or the heat transfer in a rocket nozzle, the possible presence of turbulent fluctuations in the flow is of secondary importance compared to the fact that the momentum and heat transport can be estimated as in laminar flows. It may be worthwhile to call the first type, in which turbulence eventually vanishes, 'hard' laminarization, and the second type, in which only turbulent transport is rendered negligible, 'soft' laminarization. We adopt here a rather pragmatic definition of laminarization according to which it will have occurred if the development of a flow (or a part of it) can be understood without recourse to any model for turbulent shear flow.

Although in any given laminarizing flow several factors may be operating simultaneously, it is useful to realize that there are three different classes of mechanism that may be responsible[110]. In the first, turbulent energy is dissipated through the action of a molecular transport property like the viscosity or conductivity, and the governing parameter is typified by the Reynolds number. In the second class turbulence energy is destroyed or absorbed by work done against an external agency, like buoyancy forces or flow curvature; the typical parameter is a Richardson number. In both types experimental evidence indicates that the suppression of turbulence goes beyond the mere decay of energy to an actual decorrelation of the velocity components contributing to the crucial Reynolds shear stress that governs the mean flow.

The third class of reverting flows is exemplified by a turbulent boundary layer subjected to severe acceleration. Here a two-layer model is indicated[111]. In the outer layer, turbulence is fairly rapidly distorted and the Reynolds shear stress is nearly frozen; the inner viscous layer exhibits random oscillations in response to the forcing provided by the residue of the original turbulence. Reversion here is not so much the result of dissipation or destruction of energy (although these mechanisms are also operating), but rather of the domination of pressure forces over slowly-responding Reynolds stresses in the outer region, accompanied by the generation of a new laminar sub-boundary layer stabilized by the acceleration.

It is instructive to compare these different types of laminarization. In both dissipative and absorptive types of reversion, there is a net decrease in turbulence energy: in the first instance, this energy is dissipated, essentially by the action of a molecular

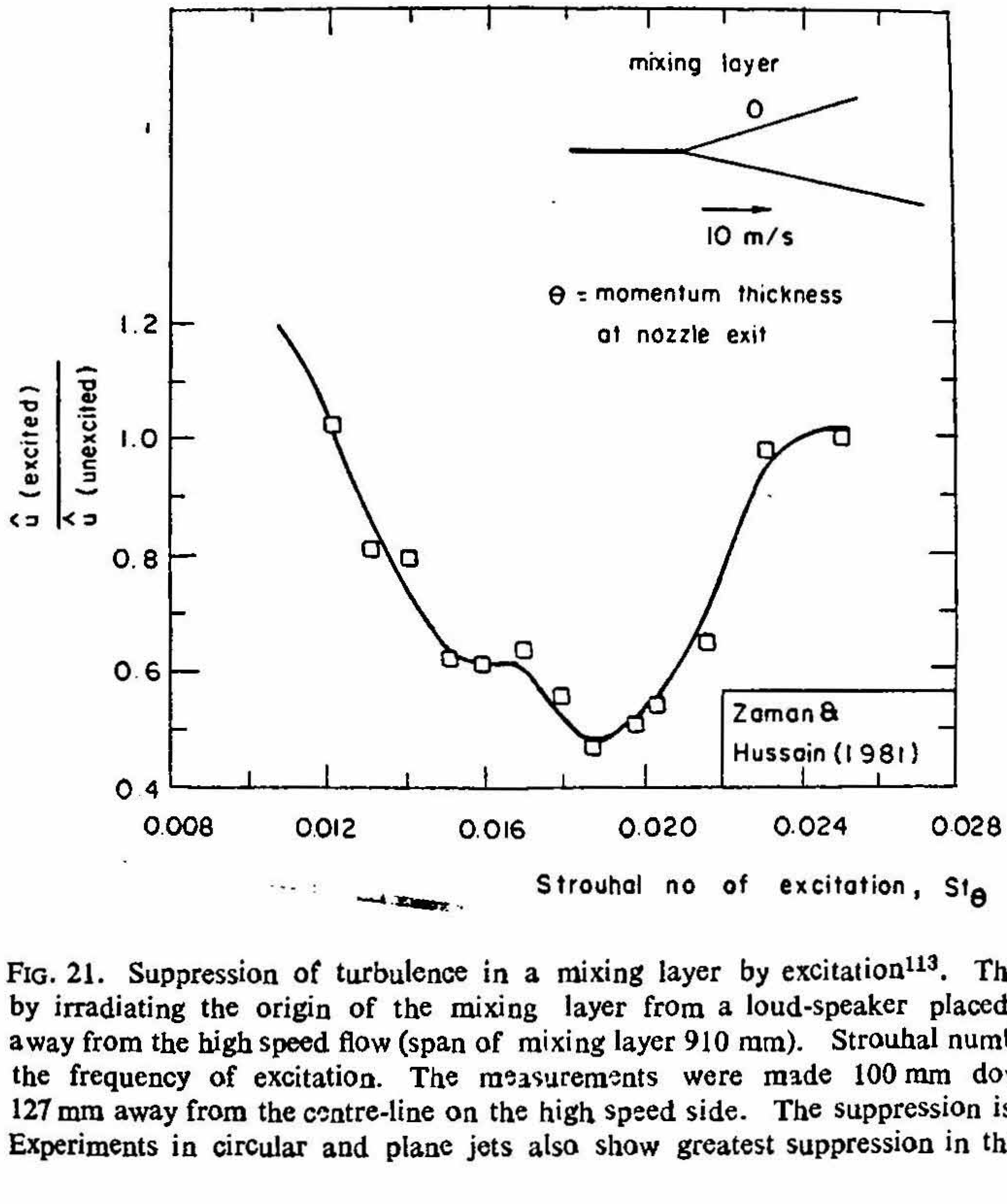


FIG. 21. Suppression of turbulence in a mixing layer by excitation[113]. The excitation was provided by irradiating the origin of the mixing layer from a loud-speaker placed 400 mm from midspan, away from the high speed flow (span of mixing layer 910 mm). Strouhal number was varied by altering the frequency of excitation. The measurements were made 100 mm downstream of the origin, 127 mm away from the centre-line on the high speed side. The suppression is largest at $St_\theta \doteq 0{\cdot}019$. Experiments in circular and plane jets also show greatest suppression in the St_θ range 0·016–0·019.

transport parameter like the viscosity or the electrical resistivity; in the second, it is destroyed by the work done against a body force like gravity. But dissipative reversion is very slow: if the Reynolds number in a pipe flow drops to a value as low as half of the critical value (following an enlargement in that pipe, say), the distance required to complete reversion is of the order of a hundred diameters. In contrast, in the absorptive type of reversion, the destruction of turbulence energy appears to proceed rapidly once the critical value of the parameter is exceeded: turbulence in a buoyant jet can be suppressed in a few jet widths. In both the cases, there is evidence that what happens goes beyond a mere decrease in turbulence energy; in fact some mechanism seems to be at work to decorrelate the velocity components that generate the crucial Reynolds shear stresses. In particular, the effects of even mild curvature on a turbulent shear flow seem astonishingly strong. Clearly one is not merely wearing the machinery of turbulence down in these cases – it is more as if one were throwing a spanner in the works ! In other words, these external influences imposed on the flow must be interfering with its organization - with the coherent structure and the bursting cycle that sustain turbulence.

This brings us to the possibilities of turbulence management by interference with the coherent structure in a turbulent flow. If turbulence energy production is patchy (and/or ordered) in space and intermittent in time, it would seem logical that novel methods of indirect control should be possible. Much attention is being devoted now to this subject; such 'structural' control need not necessarily be time-dependent. It is possible that Roshko's[112] experiments on the effect of a splitter plate on the wake behind a circular cylinder belong to this class. He found that a plate extending 5 diameters along the centre line behind the circular cylinder drastically altered the pressure field, reduced the drag coefficient of the cylinder from 1·1 to 0·7 and increased the base pressure coefficient from —0·5 to — 1·0.

There is now definite evidence that many free shear flows are sensitive to harmonic excitation. Zaman and Hussain[113] studied the turbulence intensity in the mixing layer at the exit of a nozzle when the flow was excited through a loudspeaker in the settling chamber; some results are shown in fig. 21. It is seen that at a Strouhal number of about 0·018 (based on momentum thickness at the exit, θ_e, and core velocity) the longitudinal intensity is down to a third of the value in the unexcited state. The suppression was detected as far downstream as 6000 θ_e; the excitation level corresponded to an r.m.s. value $\hat{u} \doteq 0\cdot3$ to $0\cdot5\%$ of U. Wygnanski[114] has reported how mixing can be enhanced or suppressed at different locations along the mixing layer, depending on the excitation.

Efforts to control boundary layer flow by similar intermittent operation or excitation have, on the other hand, given only ambiguous results. Spangler *et al*[115] used pulsed blowing though a flush transverse slot, at frequencies up to 340 Hz (estimated burst frequency in his boundary layer, using the data of Rao *et al*[105], was about 14 Hz). Wall stress reductions of 16% were achieved at a distance of 2·8 boundary layer thicknesses downstream of the slot. Spangler attributed the effects to the introduction of low turbulence air into the boundary layer.

Yajnik and Acharya[116] showed that wall stress decreased if a grid was inserted in the boundary layer, but of course there is a momentum loss at the grid that has also to be taken into account. Nagib *et al*[117] have studied the pressure loss in a channel whose walls have special grooves cut in them (presumably interfering with the formation of bursts). Bushnell[130] has recently reviewed work on a variety of turbulence control devices.

Ideally, one may visualize a system of boundary layer control in which the controlling agent operates at just those places and times where, say, a turbulent burst is about to occur; this would need suitable detectors distributed over the surface on which the

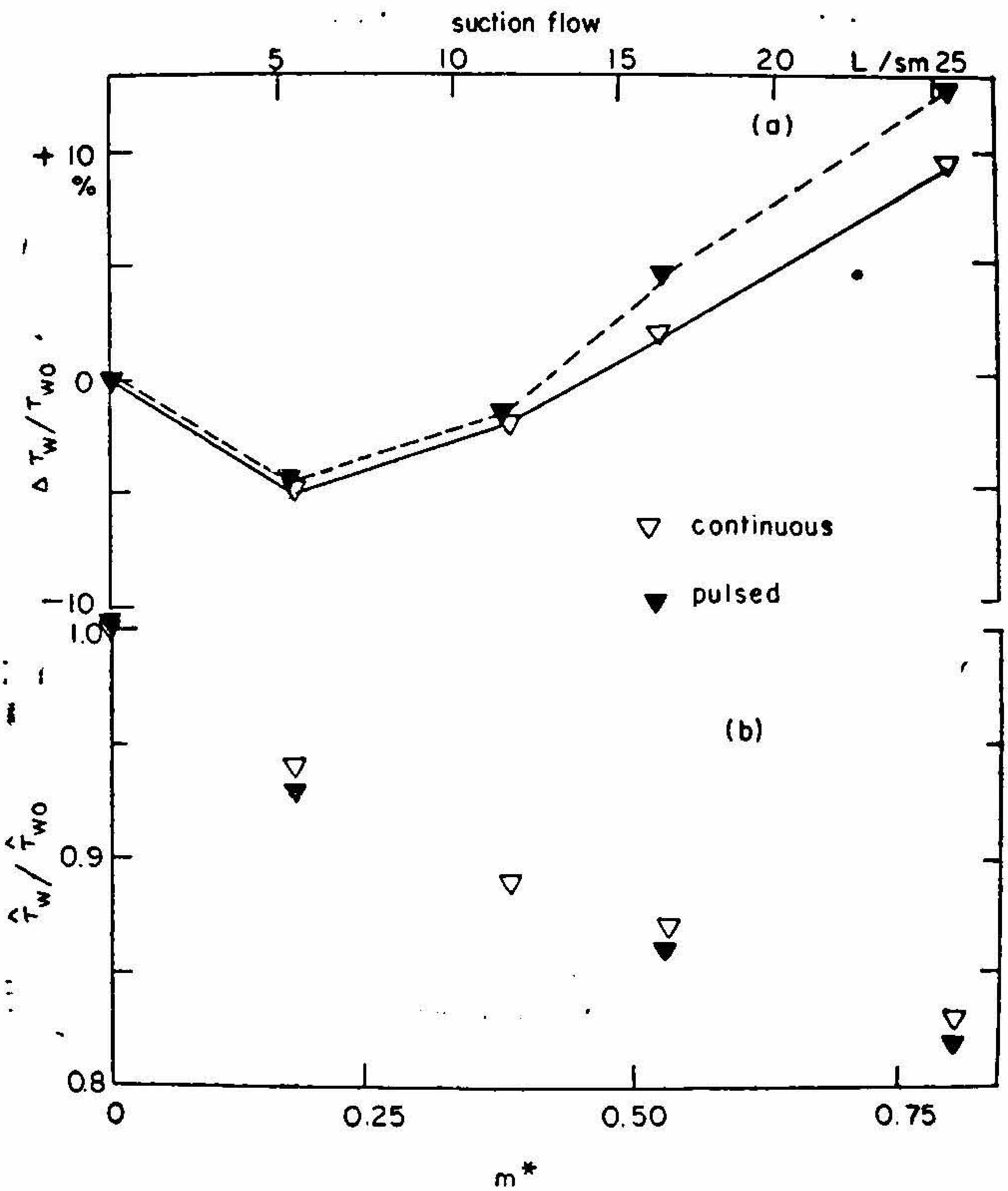


FIG. 22. The effect of continuous and pulsed suction through a flush two-dimensional slot on the wall stress in a turbulent boundary layer[119]. Measurements made at $2\frac{1}{2}$ boundary layer thicknesses downstream of the slot. Abscissa shows both actual suction flow in litres/second-metre, and as non-dimensionalized (m^*) by free stream velocity and displacement thickness of the unsucked boundary layer at the slot. Note that the reduction in wall stress ($\Delta \tau_w$) is a maximum of about 5% at $m^* \doteq 0{\cdot}18$, but that the r.m.s. value of the stress fluctuations, $\hat{\tau}_w$, decreases monotonically with increasing suction. The data displayed in the diagram show that pulsing frequencies up to burst rates in the boundary layer produce no significant differences compared to continuous suction. More recent work at frequencies up to five times the burst rate has revealed no change in trend. It is thought that the reduction in wall stress at moderate suction is a genuine effect on the turbulent boundary layer, the increase at higher suction being the consequence of the generation of a new laminar sub-boundary layer at and downstream of the suction slot.

flow is to be controlled. This has not yet been done; but the response to various degrees of pulsation in the control, at fixed points, is being attempted. If control is applied at a point, its effect will be felt over a certain characteristic area. Precise information over the mangitude of this area is not available, but it is clear that the chances of obtaining a beneficial effect due to pulsing are higher if the control is applied over a line rather than an area, because of the smaller smearing effect. Recently Parikh *et al*[118] studied the effect of controlled oscillations of the free stream on a certain *area* on a flat plate. They found no significant effect of pulsation up to the bursting rates. A similar negative conclusion was reached in an experiment with pulsed slot suction[119], whose results are shown in fig. 22 ; while optimum suction does result in lower wall stresses, and high suction in higher wall stresses (of use in preventing separation), it was found that *pulsing* the control, up to burst frequencies, produced no significant effects at given suction mass flow.

It thus appears that while pulsed excitation has strong effects on free shear flows, boundary layers present a harder problem for control; either selection of favourable control points, or use of frequencies much higher than bursting rates, may be necessary before significant gains can be made.

An interesting discussion of various turbulence 'manipulators' is given by Morkovin[120].

Another method of achieving significant modification of a turbulent flow is by the addition of polymers to liquids. Since Toms's discovery [121] of the effect in 1948 (see Lumley[122] and Virk[123] for reviews), many studies have been made; surface drag reductions of 50–80% may be achieved by the addition of small amounts (of the order of ten to hundred parts per million by weight) of such polymers as 'polyox' (= polyethylene oxide) in water. The additives do not alter the density or shear viscosity of the solvent appreciably; and the effect reaches a saturation level as concentration increases, independent of the solvent-molecule pair, the molecular weight of the polymer,[124] etc. In fact, the law of the wall appears to be modified, in the asymptotic saturation state, to a semilogarithmic form with a Karman constant of about 0·085 (instead of the classical value 0·4). Experiments show that the effects are stronger in wall flows than in free flows, and when the polymer is injected at the wall rather than in the outer flow. It therefore appears [125,126] that the polymer interferes in some way with the bursting cycle in the turbulent boundary layer (discussed in Section 7) . It has been suggested that the interference arises as a result of the coiling and uncoiling of the long polymeric molecular chain, possibly resulting in a sharp increase in the viscosity for *extensional* strain (Lumely [122] estimates that the increase may be by a factor of 10^4 !).

Somewhat similar drag reduction effects may arise from fibre suspensions (*e.g.* asbestos) as well[127].

9. Conclusion

This survey has touched on various aspects of the turbulence problem. A remarkable feature of the current scene is the proliferation of numerical models that have little relation to what is being learnt about the structure of turbulent flows by new experiments. These models are generally highly empirical ; although many of them are inspired by the dynamical equations (often for stress transport), the spirit in which the models are formulated is usually not very different from that in which empirical loss formulate are to be (and are still) constructed in, say, hydraulics. Karman is supposed to have called early hydraulics the science of 'variable constants'. The tools available for carrying out computations have now become very powerful, and so, instead of constructing algebraic formulae as in the last century, we can now construct (with no greater effort) partial differential equations instead. If some relatively simple equation of this type had proved spectacularly successful, we could have expected scientific advances. However, no such model is in sight ; instead, the models in use are becoming more complicated. Of course, such models are perhaps necessary in engineering work, but their near-total isolation from the work on the structure of turbulence makes one doubt whether they are of any lasting value. It is indeed to be hoped that they will be rapidly replaced by something more satisfactory, although what this will be is not yet clear ! Large eddy simulations still take too much computing time, and do involve some modelling of the small eddies ; it is not clear at the present time whether they will become practical in the not-too-distant future. My own view is that the most promising line of attack is *via* vortex dynamics : a great deal more study here would be justified, and could be very rewarding. One reason for suggesting this is again that contact here with observations seems more immediate.

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Much of the material in this survey is the result of several years' collaboration and discussion with many colleagues (on campus and elsewhere) and students, to all of whom I am greatly indebted.

Special Notation

$\equiv$	identically equal to
$f((x))$	the functional form $f(x)$ (if $f((x)) = x^2$, $f((y)) = y^2$)
Dk	element of volume in k-space
$\propto$	proportionality sign

$\langle x \rangle$	mean of x (over a coordinate in which x is homogeneous, or over an ensemble)
p	vector x
τ	tensor

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