Solution of certain simultaneous multiple integral equations involving inverse Mellin transforms

K. N. SRIVASTAVA, R. M. PALAIYA AND ARATI CHOUDHARY
M.A. College of Technology, Bhopal 462 007, India.

Received on January 9, 1978; Revised on June 30, 1978

Abstract

Closed form solutions are obtained for three sets of simultaneous multiple integral equations involving inverse Mellin transform. The method is illustrated by obtaining closed form solutions of a set simultaneous dual integral equations and a system of simultaneous triple integral equations involving inverse Mellin transform.

Key words : Simultaneous multiple integral equations, Inverse Mellin transform, Riemann boundary value problem, simultaneous dual integral equations, simultaneous triple integral equations.

1. Introduction

The Riemann boundary value technique used in ref. 1 is applied here for solving the following system of simultaneous multiple integral equations involving inverse Mellin transform:

$$\left.\begin{aligned} &M^{-1}[\beta\psi(s) + a\tan(\pi s/n)\varphi(s);\ r] = f_1(r),\ r \in L_1 \\ &M^{-1}[B\varphi(s) - a\tan(\pi s/n)\psi(s);\ r] = f_2(r),\ r \in L_1 \\ &\quad M^{-1}[\varphi(s);\ r] = 0,\ r \in L'_1 \\ &\quad M^{-1}[\psi(s);\ r] = 0,\ r \in L'_1 \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} &M^{-1}[\beta\psi(s) + a\cot(\pi s/n)\varphi(s);\ r] = f_1(r),\ r \in L_1 \\ &M^{-1}[\beta\varphi(s) - a\cot(\pi s/n)\psi(s);\ r] = f_2(r),\ r \in L_1 \\ &\quad M^{-1}[\varphi(s);\ r] = 0,\ r \in L'_1 \\ &\quad M^{-1}[\psi(s);\ r] = 0,\ r \in L'_1 \end{aligned}\right\} \tag{1.2}$$

$$\begin{aligned} &M^{-1}[\beta\psi(s) + a\cot(\pi s/n)\varphi(s);\ r] = f_1(r^{n/2}),\ r \in L_1 \\ &M^{-1}[\beta\varphi(s) + a\tan(\pi s/n)\psi(s);\ r] = f_2(r^{n/2}),\ r \in L_1 \\ &\quad M^{-1}[\varphi(s);\ r] = 0,\ r \in L'_1 \\ &\quad M^{-1}[\psi(s);\ r] = 0,\ r \in L'_1 \end{aligned}$$

where α and β are constants, n is a positive integer, $0 < s < n/2$; $L_1 = \cup_{j=1}^{m} (a_j, b_j)$, $L' = R' - L_1$, R' being the positive real line such that $0 \leqslant a_1 < b_1 < a_2 < b_2 \ldots a_m < b_m$ and $M^{-1} [f(s); r]$ denotes the inverse Mellin transform of the function $f(s)$.

This type of integral equations occur in problems of determining the distribution of stresses in composite wedges. When two wedges of dissimilar elastic properties are bonded together and the interface contains flaws in the form of cracks, the problem of determination of stress distribution reduces to that of solving the above system of integral equations.

2. Reduction of the system of multiple simultaneous equations to Riemann boundary value problem

In order to solve the system of equations (1.1), we assume that

$$M^{-1} [\varphi(s); r] = p(r); M^{-1} [\psi(s); r] = q(r), r \in L_1$$

so that

$$\varphi(s) = \int_{L_1} p(u) u^{n-1} du; \quad \text{and} \quad \psi(s) = \int_{L_1} q(u) u^{n-1} du$$

here $p(r)$ and $q(r)$ are unknown functions to be determined. By using the result (ref. 2, p. 342)

$$M^{-1} [\tan(\pi s/n); r/u] = -\frac{n (ru)^{n/2}}{\pi (u^n - r^n)}, \quad 0 < s < n/2$$

we obtain

$$M^{-1} [\tan(s/n) \psi(s); r] = \int_{L_1} u^{-1} q(u) M^{-1} [\tan(\pi s/n); r/u] du$$

$$= -\frac{n \cdot r^{n/2}}{\pi} \int_{L_1} \frac{q(u) u^{n/2-1}}{u^n - r^n} du. \tag{2.1}$$

Similarly

$$M^{-1} [\tan(\pi s/n) \varphi(s); r] = -\frac{n \cdot r^{n/2}}{\pi} \int_{L_1} \frac{p(u) u^{n/2-1}}{u^n - r^n} du. \tag{2.2}$$

So that the system of integral equations (1.1) reduces to the following simultaneous integral equations

$$\beta q(r) - \frac{\alpha n \cdot r^{n/2}}{\pi} \int_{L_1} \frac{p(u) u^{n/2-1}}{u^n - r^n} du = f_1(r), r \in L_1 \tag{2.3}$$

$$\beta p(r) + \frac{an \, . \, r^{n/2}}{\pi} \int_{L_1} \frac{q(u)\, u^{2/n-1}}{u^n - r^n}\, du = f_2(r), r \in L_1 \tag{2.4}$$

The substitution

$$p(u) + iq(u) = u^{n/2}\, \lambda(u) \tag{2.5}$$

$$f_2(r) + if_1(r) = r^{n/2} f(r) \tag{2.6}$$

helps us to reduce the equations (2.3) and (2.4) into a single integral equation

$$\beta\lambda(r) + \frac{a}{\pi i} \int_{L_1} \frac{\lambda(u)\, n\, u^{n-1}}{u^n - r^n}\, du = f(r), \; r \in L_1 \tag{2.7}$$

Next we consider the system of equations (1.2). We assume the same solution as for the system (1.1) and after using the result (ref. 2, p. 346)

$$M^{-1}[\cot(\pi s/n);\; r/u] = n \,.\, u^n/\pi\,(u^n - r^n), 0 < s < n/2,$$

and noting that

$$M^{-1}[\cot(\pi s/n)\, \varphi(s);\, r] = \frac{n}{\pi} \int_{L_1} \frac{u^{n-1}\, p(u)}{u^n - r^n}\, du \tag{2.8}$$

$$M^{-1}[\cot(\pi s/n)\, \psi(s);\, r] = \frac{n}{\pi} \int_{L_1} \frac{u^{n-1}\, q(u)}{u^n - r^n}\, du \tag{2.9}$$

we find that the system of equations (1.2) reduces to

$$\beta q(r) + \frac{\pi}{a} \int_{L_1} \frac{n \,.\, u^{n-1}\, p(u)}{u^n - r^n}\, du = f_1(r), \; r \in L_1 \tag{2.10}$$

$$\beta p(r) - \frac{a}{\pi} \int_{L_1} \frac{n \,.\, u^{n-1}\, q(u)}{u^n - r^n}\, du = f_2(r), \; r \in L_1 \tag{2.11}$$

The substitutions

$$q(r) + ip(r) = \lambda(r), \qquad f_1(r) + if_2(r) = f(r)$$

helps us to reduce the equations (2.10) and (2.11) into a single singular integral equation (2.7).

If we get

$$\Lambda(z) = \frac{n}{2\pi i} \int_{L_1} \frac{u^{n-1}\, \lambda(u)}{u^n - z^n}\, du, \; Re(z) = r \tag{2.12}$$

then after using the formula[4]

$$\Lambda^{+}(r) - \Lambda^{-}(r) = \lambda(r), \ \Lambda^{+}(r) + \Lambda^{-}(r) = \frac{n}{\pi i} \int_{L^1} \frac{u^{n-1}\lambda(u)}{u^n - r^n} du \tag{2.13}$$

we find that (2.7) is equivalent to Riemann boundary value problem

$$\Lambda^{+}(r) = \mu \Lambda^{-}(r) + (a+\beta)^{-1} f(r) \tag{2.14}$$

where $\mu = (a-\beta)/(a+\beta)$. The solution of (2.14) is [ref. 4, p. 514]

$$\Lambda(z) = \frac{n \,.\, X(z)}{2\pi(a+\beta)i} \int_{L_1} \frac{u^{n-1} f(u)}{X^{+}(u)(u^n - z^n)} du + X(z) P(z^n) \tag{2.15}$$

where $P(z^n)$ is an arbitrary polynomial in z^n and $X(z)$ is the solution of the homogeneous Riemann boundary value problem

$$X^{+}(t) = kX^{-}(t)$$

The solution of this problem as given in ref. 4 (p. 514) is

$$X(z) = \{\prod_{j=1}^{m} \{(z^n - b_j^n)(z^n + a_j^n)/(z^n - a_j^n)(z^n + b_j^n)\} \tag{2.16}$$

where $\gamma = (\log \mu)/2\pi i$. By using (2.13) and (2.15) the value of $\lambda(r)$ can be obtained. The solutions of the systems of equations (1.1) and (1.2) can be obtained once we have obtained the values of $p(r)$ and $q(r)$.

3. Finally, we consider the system of equations (1.3). In this case we assume the solution in the form

$$\varphi(s) = \int_{L_1} p_1(u^{n/2}) u^{n-1} du, \quad \psi(s) = \int_{L_1} q_1(u^{n/2}) u^{n-1} du.$$

Proceeding as in the last section we get the following equations

$$M^{-1}[\tan(\pi s/n)\psi(s); r] = -\frac{n}{2\pi} \int_{L_1} u^{\frac{n}{2}-1} q_1(u)^{n/2} \left[\frac{1}{u^{n/2} - r^{n/2}} - \frac{1}{u^{n/2} - r^{n/2}}\right] du$$

$$= -(1/\pi) \int_{S_1} q_1(t) [\{1/(t-x)\} - \{1/(t+x)\}] dt = -(1/\pi) \int_{S} \frac{q(t)}{t-x} dt \tag{3.1}$$

$$M^{-1}[\cot(\pi s/n)\varphi(s); r] = \frac{n}{2\pi} \int_{L_1} p_1(u^{n/2}) \left[\frac{1}{u^{n/2} - r^{n/2}} + \frac{1}{u^{n/2} + r^{n/2}}\right] du$$

$$= (1/\pi) \int_S \frac{p(t)}{t - x} dt \tag{3.2}$$

where $S = S_1 \cup S'_1$, $S_1 = \cup_{j=1}^{m} (A_j, B_j)$, $S'_1 = \cup_{j=1}^{m} (-B_j, -A_j)$, $A_j^n = a_j^{n/2}$, $B_j = b_j^{n/2}$ $x = r^{n/2}$ and $p(t)$ and $q(t)$ are odd and even extensions of $p_1(t)$ and $q_1(t)$ respectively to the region S'_1. Hence the system of equations (1.3) reduce to integral equations

$$\beta q(x) + \frac{a}{\pi} \int_S \frac{p(t)}{t - x} dt = f_1(x), \qquad x \in S \tag{3.3}$$

$$\beta p(x) - \frac{a}{\pi} \int_S \frac{q(t)}{t - x} dt = f_2(x), \qquad x \in S. \tag{3.4}$$

The substitution

$$\lambda(x) = q(x) + ip(x); \; f(x) = f_1(x) + if_2(x)$$

helps to combine the equations (3.3) and (3.4) into a single equation

$$\beta\lambda(x) + \frac{a}{\pi i} \int_S \frac{\lambda(t)}{t - x} dt = f(x), \quad x \in S. \tag{3.5}$$

If we define

$$\Lambda(z) = \frac{1}{2\pi i} \int_S \frac{\lambda(u)}{u - z} du, \; \text{Re}(z) = x$$

and use Pelmelj formula[4]

$$\Lambda^+(x) - \Lambda^-(x) = \lambda(x), \; \Lambda^+(x) + \Lambda^-(x) = \frac{1}{\pi i} \int_S \frac{\lambda(u)}{u - x} du \tag{3.6}$$

we find that (3.5) reduces to Riemann boundary value problem

$$\Lambda^+(x) = -k\Lambda^-(x) + (a + \beta)^{-1} f(x), \; x \in S \tag{3.7}$$

where $k = (a - \beta)/(a + \beta)$. The solution of this problem is (ref. 3, p. 451)

$$\Lambda(z) = \frac{X(z)}{2\pi(a + \beta)i} \int_S \frac{f(t)}{X^+(t)(t - z)} dt + P(z).X(z) \tag{3.8}$$

where $P(z) = C_1 z^{m-1} + C_2 z^{m-2} + \ldots + C_m$, $C_1, C_2, \ldots C_m$ are arbitrary complex constants and $X(z)$ is the solution of the homogeneous Riemann boundary value problem

$$X^+(z) = -kX^-(z). \tag{3.9}$$

The solution of this problem is (ref. 3)

$$X(z) = \prod_{j=1}^{m} [(z - A_j)(z + B_j)]^{-1/2+i\omega} [(z + A_j)(z - B_j)]^{-1/2-i\omega} \qquad A_1 \neq 0,$$

$$= (z + B_1)^{-1/2+i\omega} (z - B_1)^{-1/2-i\omega} \prod_{j=2}^{m} [(z - A_j)(z + B_j)]^{-1/2+i\omega}$$

$$[(z + A_j)(z - B_j)]^{-1/2-i\omega}, \; A_1 = 0,$$

where $\omega = (1/2\pi) \log k$. When $f(t)$ is a polynomial from ref. 3, (p. 457) we have

$$\int_S \frac{f(t)}{X^+(t)(t - z)} dt = \frac{2\pi i}{1 + k} [\{f(z)/X(z)\} - L(z)],$$

where $L(z)$ is the term independent of t in the Laurant series expansion of $[tf(t)/X(t)(t - z)]$. Hence from (3.8) we have

$$\Lambda(z) = (2a)^{-1} [f(z) - L(z) X(z) + P(z) X(z)].$$

4. Particular cases

We now consider some particular cases of the system of integral equations (1.3).

(i) Simultaneous dual integral equations

Let $m = 1$, $a_1 = 0$, $b_1 = 1$, $f_1(r) = f = \text{constant}$, $f_2(r) = 0$. Then $L_1 = (0 < r < 1)$, $L'_1 = (r > 1)$ $f(x) = f$ and

$$X(z) = (z + 1)^{-1/2+i\omega} (z - 1)^{-1/2-i\omega}, \; P(z) = C_1, \; L(z) = (z - 2i\omega) f,$$

$$\Lambda(z) = f(2a)^{-1} [1 - (z + C_1) X(z)]. \tag{4.1}$$

If we write $C_1 = C' + iC''$ where C' and C'' are real arbitrary constants, then we have

$$p(x) = f(a^2 - \beta^2)^{-1/2} (1 - x^2)^{-1/2} [(x + C') \cos \omega\theta - C'' \sin \omega\theta], \quad 0 < x < 1 \tag{4.2}$$

$$q(x) = -f(a^2 - \beta^2)^{-1/2} (1 - x^2)^{-1/2} [(x + C') \sin \omega\theta + C'' \cos \omega\theta], \quad 0 < x < 1 \tag{4.3}$$

where $\theta = \log (1 + x)/(1 - x)$. Since $p(x)$ and $q(x)$ are odd and even functions we must have $C' = 0$. The other constant C'' is determined from the physical conditions of the problem.

(ii) Simultaneous triple integral equations

Let $m = 1$, $a_1 = a$, $b_1 = b$, $f_1(r) = f$, $f_2(r) = 0$. Then $L_1 = (a < r < b)$, $L'_1 (= 0 < r < a)\ U\ (r > b)$

$$f(x) = f \text{ and } A = a^{n/2},\ B = b^{n/2},\ x = r^{n/2};$$

$$X(z) = [(z - A)(z + B)]^{-1/2 + i\omega} [(z + A)(z - B)]^{-1/2 - i\omega},\ P(z) = C_1 z + C_2$$

$$L(z) = f(z^2 + hz + g),\ \Lambda(z) = f(2a)^{-1}[1 - (z^2 + C_1 z + C_2) X(z)] \tag{4.4}$$

where g, h are known constants. From these equations we get

(i) for $A < x < B$

$$X^+(x) = -ik^{1/2}[(x^2 - A^2)(B^2 - x^2)]^{-1/2}[\cos \omega\theta + i \sin \omega\theta],$$

$$X^-(x) = ik^{-1/2}[(x^2 - A^2)(B^2 - x^2)]^{-1/2}[\cos \omega\theta + i \sin \omega\theta],$$

(ii) for $-B < x < -A$,

$$X^+(x) = ik^{1/2}[(x^2 - A^2)(B^2 - x^2]^{-1/2}[\cos \omega\theta + i \sin \omega\theta],$$

$$X^-(x) = -ik^{-1/2}[(x^2 - A^2)(B^2 - x^2)]^{-1/2}[\cos \omega\theta + i \sin \omega\theta],$$

If we set $C_1 = C'_1 + i C''_1$, $C_2 = C'_2 + iC''_2$, then we have for $A < x < B$

$$q_1(x) = -f(\alpha^2 - \beta^2)^{-1/2}[(x^2 - A^2)(B^2 - x^2)]^{-1/2}[(x^2 + C'_1 x + C''_2) \times \sin \omega\theta + (C''_1 x + C''_2) \cos \omega\theta]$$

$$p_1(x) = f(\alpha^2 - \beta^2)^{-1/2}[(x^2 - A^2)(B^2 - x^2)]^{-1/2}[(x^2 + C'_1 x + C'_2) \times \cos \omega\theta - (C''_1 x + C''_2) \sin \omega\theta]$$

and for $-B < x < -A$,

$$q(x) = f(\alpha^2 - \beta^2)^{-1/2}[(x^2 - A^2)(B^2 - x^2)]^{-1/2}[(x^2 + C'_1 x + C'_2) \times \sin \omega\theta + (C''_1 x + C''_2) \cos \omega\theta]$$

$$p(x) = -f(\alpha^2 - \beta^2)^{-1/2}[(x^2 - A^2)(B^2 - x^2)]^{-1/2}[(x^2 + C'_1 + C'_2) \times \cos \omega\theta - (C''_1 x + C''_2) \sin \omega\theta]$$

where $\theta = \log\{(x - A)(B + x)/(x + A)(B - x)\}$. Since $q(x)$ and $p(x)$ are odd and even extensions of $q_1(x)$ and $p_1(x)$ respectively on $-B < x < -A$, we must choose C'_1 and C''_2 such that they are zero. The functions $p(x)$ and $q(x)$ still involve two arbitrary constants C''_1 and C'_2. These constants are determined from the physical conditions of the problem.

References

1. SRIVASTAVA, K. N., PALAIYA, R. M. AND ARATI CHOUDHARY — System of Griffith cracks lying at the interface of two bonded dissimilar elastic half-planes (Under communication).
2. ERDELYI, A. *et al.* — *Tables of Integral Transforms*, Vol. 1, 1954, McGraw-Hill, New York.
3. MUSKHELSHVILLI, N. I. — *Some Basic Problem of the Mathematical Theory of Elasticity*, 1963, Noordhoff Publishing Company.
4. GAKHOV, F. D. — *Boundary Value Problems*, 1966, Pergamon Press.