On a self-adjoint extension of a type of a matrix differential operator

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Abstract

In Coddington and Levinson[1] we get the requisite boundary conditions to be associated with a $2n$-th order symmetric differential expression which defines a self-adjoint operator. Naimark[2] obtains the corresponding set of boundary conditions to be associated with a $2n \times 2n$ matrix whose elements involve first order derivatives.

Here we discuss self-adjoint extension of certain type of matrix differential operator with a set of non-separated boundary conditions at the end points a, b.

A similar problem associated with an $r \times r$ matrix differential operator with elements depending upon differential coefficients of orders up to $2n$ has also been discussed. Finally, we deal with the corresponding singular problem where the interval $[a, b]$ is replaced by $[0, \infty)$.

Key words: Self-adjoint extension, Quasi-derivatives, Domain of definition, Lagrange's identity, Deficiency indices, Square-integrable solutions.

1. Introduction

The form of the boundary conditions to be imposed at the end points of a finite interval so that a $2n$-th order symmetric differential equation together with boundary conditions should lead to a self-adjoint differential operator is known[1]. The problem when the $2n$-th order differential equation is replaced by a set of $2n$ first order equations with the corresponding set of boundary conditions has also been studied[2].

2. Construction of the matrix differential operator L

We define a 2×2 matrix differential expression Q by

$$Q(y) = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.1}$$

where

$$y = (y_1, y_2)^T = \{y_1, y_2\}$$

and

$$l_{ij}(Z) = (-1)^2 (p_{ij} Z^{(2)})^{(2)} + (-1)(q_{ij} Z^{(1)})^{(1)} + r_{ij} Z; \tag{2.2}$$

$(i, j = 1, 2)$. Then (2.1) is equivalent to

$$Q(y) = (py^{(2)})^{(2)} - (qy^{(1)})^{(1)} + ry \tag{2.3}$$

where $p = (p_{ij})$, $\det |p_{ij}| \neq 0$, $q = (q_{ij})$ and $r = (r_{ij})$ are 2×2 matrices such that $p(x)$, $q(x)$, $r(x)$ are Lebesgue measurable in the interval (a, b) and are Lebesgue-integrable in any closed subinterval $[\alpha, \beta]$ of (a, b).

In order to remove the restriction regarding the differentiability of the coefficients p, q, r up to the prescribed order, we introduce the " quasi-derivatives " of the vector function $y(x)$, defined as follows.

$$\left.\begin{aligned} y^{[0]}(x) &= y(x) \\ y^{[1]}(x) &= \frac{d}{dx} y(x) \\ y^{[2]}(x) &= p(x) \frac{d^2}{dx^2} y(x) \\ y^{[3]}(x) &= q(x) \frac{d}{dx} y(x) - \frac{d}{dx} y^{[2]}(x) \\ y^{[4]}(x) &= r(x) y(x) - \frac{d}{dx} y^{[3]}(x) \end{aligned}\right\} \tag{2.4}$$

Then (2.3) can be rewritten as

$$Q(y) = y^{[4]}(x). \tag{2.5}$$

Using Green's formula it follows that the adjoint of Q is

$$Q_1 = \begin{pmatrix} l_{11} & l_{21} \\ l_{12} & l_{22} \end{pmatrix}.$$

Hence for symmetry, we should have $l_{12} = l_{21}$.

Let the domain of the operator L be defined as

$$\mathfrak{D}(L) : f \in \mathcal{H} = \mathcal{L}^2(a, b),$$

the space of square integrable vector functions over the interval (a, b),

where (i) the components of $f = \{f_1, f_2\}$ have absolutely continuous quasi-derivatives up to the order three ;

(ii) $f^{[4]}(x) \in \mathcal{H}$

$$Lf = Q(f), \quad \forall f \in \mathfrak{D}(L).$$

Let $\mathfrak{D}_0$ represent the set of all vector functions $y(x)$ in $\mathfrak{D}$ which satisfy the conditions

$$y^{[k]}(a) = y^{[k]}(b) = [0], \qquad k = 0, 1, 2, 3. \tag{2.6}$$

Finally, let L_0 be defined by

$$L_0 y = Ly, \qquad \forall y \in \mathfrak{D}_0$$

where $\mathfrak{D}_0$ is the domain of definition of the operator L_0.

3. Existence of solutions

We begin by establishing the following theorem on existence and uniqueness of the solution of the vector-matrix differential equation $Q(y) = f$.

Theorem 3.1: If

(i) $f(x) = \{f_1(x), f_2(x)\}$ be $\mathcal{L}$-measurable in (a, b),

(ii) $f(x) \in \mathcal{L}[\alpha, \beta]$ for every $[\alpha, \beta] \subset (a, b)$,

(iii) $C = \{c_{10}, c_{20}, c_{11}, c_{21}, c_{12}, c_{22}, c_{13}, c_{23}\}$,
$c_{ij} \in \mathbb{C}$ for $i = 1, 2$; $j = 0, 1, 2, 3$,

(iv) $a < x_0 < b$

then there exist a unique vector function $y(x) = \{y_1(x), y_2(x)\}$ such that

$$Q(y) = f \tag{3.1}$$

and

$$y_i^{[k]}(x_0) = c_{ik},\ i = 1, 2;\ k = 0, 1, 2, 3 \tag{3.2}$$

where

$$y^{[k]}(x_0) = \{y_1^{[k]}(x_0), y_2^{[k]}(x_0)\}$$

are defined as in (2.4).

The theorem remains true if $x_0 = a$ or $x_0 = b$ (x_0, regular) of if the equation (3.1) is replaced by $Q(y) - \lambda y = f$, where λ is an arbitrary complex parameter.

Proof: From (2.4) and (2.5) it follows that the equation $Q(y) = f$ is equivalent to the following system of first-order equations.

$$\left.\begin{aligned} \frac{d}{dx} y(x) &= y^{[1]}(x) \\ \frac{d}{dx} y^{[1]}(x) &= p^{-1}(x)\, y^{[2]}(x) \\ \frac{d}{dx} y^{[2]}(x) &= q(x)\, y^{[1]}(x) - y^{[3]}(x) \\ \frac{d}{dx} y^{[3]}(x) &= r(x)\, y(x) - f(x). \end{aligned}\right\} \tag{3.3}$$

From these we obtain

$$\frac{d}{dx} Y(x) = A(x) Y(x) - F(x) \tag{3.4}$$

where

$$Y(x) = \{y_1, y_2, y_1^{[1]}, y_2^{[1]}, y_1^{[2]}, y_2^{[2]}, y_1^{[3]}, y_2^{[3]}\}$$
$$F(x) = \{0, 0, 0, 0, 0, 0, f_1, f_2\}$$

and

$$A(x) = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & q & 0 & -I \\ r & 0 & 0 & 0 \end{pmatrix}$$

0, I being respectively 2×2 null and identity matrices, also $P = p^{-1}(x)$.

The elements of the matrix $A(x)$ and the vector $F(x)$ are measurable in the interval (a, b) and $|A(x)|$, $|F(x)|$ are summable in every finite sub-interval $[\alpha, \beta]$ of the interval (a, b). Hence the theorem follows, since the equations (3.4) have one and only one solution satisfying the initial conditions

$$Y(x_0) = C$$

i.e., $y_i^{[k]}(x_0) = c_{ik}, i = 1, 2; k = 0, 1, 2, 3$ in (a, b).

[See Naimark[2], Th. 1, § 16.1].

Lemma 3.1. If $f(x) = \{f_1(x), f_2(x)\} \in \mathcal{L}^2(a, b)$, then the equation $Q(y) = f$ has a solution $y(x) = \{y_1(x), y_2(x)\}$ satisfying the conditions

$$y^{[k]}(a) = y^{[k]}(b) = [0]; \; k = 0, 1, 2, 3 \tag{3.5}$$

if and only if the vector function $f(x)$ is orthogonal to all solutions of the homogeneous system $Q(y) = [0]$.

Proof : Consider a particular solution $y(x)$ of the equation $Q(y) = f$ such that

$$y^{[k]}(a) = [0]; \; k = 0, 1, 2, 3 \tag{3.6}$$

and it follows from Theorem 3.1, that there exists precisely one such solution.

Let $Z_s = \{Z_{1s}, Z_{2s}\}$; $s = 1, 2, \ldots, 8$ form a fundamental system of solutions of the homogeneous system $Q(Z) = [0]$ satisfying the initial conditions

$$Z_{ij}^{[k]}(b) = \delta_{i+2k\, j} \tag{3.7}$$

for $i = 1, 2; k = 0, 1, 2, 3; j = 1, 2, \ldots, 8$.

By Lagrange's identity

$$(f, Z_s) - (Q(y), Z_s) = [y, Z_s]_a^b + (y, Q(Z_s)). \tag{3.8}$$

But $Q(Z_s) = [0]$ and by the conditions (3.5), $[y, Z_s]_{x=a} = 0$. Hence (3.8) takes the form

$$(f, Z_s) = [y, Z_s]_{x=b} = \sum_{k=1}^{2} \{\bar{y}^{[k-1]} Z_s^{[4-k]} - \bar{y}^{[4-k]} Z_s^{[k-1]}\}_{x=b}$$

where $\bar{y}$ is the transpose of y.

Finally making use of the relation (3.7), we get

$$\{(f, Z_{2s-1}), (f, Z_{2s})\} = \begin{cases} -y^{[4-s]}, & s = 1, 2 \\ y^{[4-s]}, & s = 3, 4 \end{cases} \tag{3.9}$$

From (3.9) it follows that the equations $y^{[k]}(b) = [0]$, $k = 0, 1, 2, 3$ are satisfied if and only if $(f, Z_s) = 0$ for $s = 1, 2, \ldots, 8$ which shows that $f(x)$ is orthogonal to all solutions of the homogeneous system.

Lemma 3.2. Given arbitrary real numbers

$$a_{ik}, \beta_{ik} \quad i = 1, 2; \; k = 0, 1, 2, 3$$

such that $a_k = \{a_{1k}, a_{2k}\}$, $\beta_k = \{\beta_{1k}, \beta_{2k}\}$. Then there exists a vector function $y(x) = \{y_1(x), y_2(x)\} \in \mathfrak{D}$, which satisfies the conditions $y^{[k]}(a) = a_k$ and $y^{[k]}(b) = \beta_k$, $k = 0, 1, 2, 3$.

Proof : Let $f(x) = \{f_1, f_2\}$ be an arbitrary element in $\mathcal{H}$ satisfying the conditions

$$\{(f, Z_{2s-1}), (f, Z_{2s})\} = \begin{cases} -\beta_{4-s}, & s = 1, 2 \\ \beta_{4-s}, & s = 3, 4 \end{cases} \tag{3.10}$$

where Z_s, $s = 1, 2, \ldots, 8$ are the same as in Lemma 3.1. The vector function $f(x)$ exists and $f(x)$ can be chosen to be an element of M, the set of all solutions of the equation $Q(Z) = [0]$. Since all these solutions are continuous functions in the interval $[a, b]$, they all belong to $\mathcal{L}^2(a, b)$. Hence $M \subset \mathcal{H}$. For let

$$f = \sum_{k=1}^{8} C_k Z_k, \; C_k = \text{constant}$$

then from (3.10) we obtain a system of equations in the constants C_k whose determinant is the same as the Gram-determinant for the linearly independent vector functions $Z_1, Z_2, \ldots, Z_8$ and therefore does not vanish.

Now let $v(x) = \{v_1(x), v_2(x)\}$ be a solution of (3.1) and let $v(x)$ satisfy the initial conditions

$$v^{[k]}(a) = [0],\ k = 0, 1, 2, 3 \tag{3.11}$$

then by applying Lagrange's identity to $v(x)$ and $Z_s(x)$ we get

$$(f, Z_s) = (Q(v), Z_s) = [v, Z_s]_a^b + (v, Q(Z_s)).$$

But $Q(Z_s) = [0]$, also from the conditions (3.11) we have $[v, Z_s]_{x=a} = 0$.

Further, by virture of the conditions (3.7)

$$\{(f, Z_{2s-1}), (f, Z_{2s})\} = \begin{cases} -v^{[4-s]}, & s = 1, 2 \\ v^{[4-s]}, & s = 3, 4 \end{cases} \tag{3.12}$$

Now comparing (3.10) and (3.12)

$$v^{[k]}(b) = \beta_k,\ k = 0, 1, 2, 3. \tag{3.13}$$

Thus there exists a solution, $v(x) \in \mathfrak{D}$, of (3.1) such that (3.11) and (3.13) hold. Similarly there exists a second solution $w(x) = \{w_1(x), w_2(x)\}$ of (3.1) satisfying the initial conditions

$$\left.\begin{aligned} w^{[k]}(a) &= \alpha_k \\ w^{[k]}(b) &= [0] \end{aligned}\right\} \quad k = 0, 1, 2, 3\,.$$

The lemma therefore follows by taking $y(x) = v(x) + w(x)$. Since $y(x)$ satisfies the stated conditions, belongs to $\mathfrak{D}$ and is the solution of (3.1).

4. Deficiency indices of the operator L_0

Let m be the number of linearly independent square-integrable solutions of $L_0 Z = \lambda Z$, λ a complex number and n, the same of $L_0 Z = \bar{\lambda} Z$, $\bar{\lambda}$ complex conjugate of λ. Then (m, n) is called the deficiency indices of the differential operator L_0. If the coefficients of the differential expression $Q(y)$, by means of which the differential operator L_0 was defined, are real, then $m = n$. Following § 2.2 and Naimark[2], § 17.3 we find the deficiency indices of the operator L_0 to be (8, 8).

5. Self-adjoint extension

Let L_s be the self-adjoint extension of the matrix differential operator L_0, such that its domain of definition $\mathfrak{D}_s$ satisfies $\mathfrak{D}_0 \subset \mathfrak{D}_s \subset \mathfrak{D}$. The following theorem characterises the domain of definition of L_s by means of the boundary conditions.

Theorem 5.1: The domain of definition $\mathfrak{D}_s$ of an arbitrary self-adjoint extension L_s of the operator L_0 with deficiency indices (h, h) consists of the set of all vector functions $y(x) \in \mathfrak{D}$, which satisfy the conditions

$$[y, \phi_k]_b - [y, \phi_k]_a = 0,\ k = 1, 2, \ldots, h \tag{5.1}$$

where $\phi_1, \phi_2, \ldots, \phi_h$ are certain vector functions belonging to $\mathfrak{D}$ and determined by L_s, which are linearly independent modulo $\mathfrak{D}_0$ and for which the relations

$$[\phi_j, \phi_k]_b - [\phi_j, \phi_k]_a = 0 \text{ hold for } k, j = 1, 2, \ldots, h. \tag{5.2}$$

Conversely, for arbitrary vector functions $\phi_1, \phi_2, \ldots, \phi_h$ belonging to $\mathfrak{D}$ which are linearly independent modulo $\mathfrak{D}_0$ and which satisfy the relations (5.2), the set of all vector functions $y(x) \in \mathfrak{D}$ which satisfy the conditions (5.1) is the domain of definition of a self-adjoint extension of the operator L_0.

$y(x)$ and $\phi_k(x)$ are 2-dimensional column vectors. The proof follows exactly in the same way as Naimark[2], Th. 4, § 18.1.

We are now in a position to prove the following theorem.

Theorem 5.2: Every self-adjoint extension L_s of the operator L_0 is determined by linearly independent boundary conditions of the form

$$\sum_{k=1}^{4} M_k y^{[k-1]}(a) + \sum_{k=1}^{4} N_k y^{[k-1]}(b) = [0] \tag{5.3}$$

with

$$\sum_{s=1}^{2} M_s M_{5-s}^T - \sum_{s=1}^{2} M_{5-s} M_s^T = \sum_{s=1}^{2} N_s N_{5-s}^T - \sum_{s=1}^{2} N_{5-s} N_s^T \tag{5.4}$$

where $M_k = [m_{ts}^k]$, $N_k = [n_{ts}^k]$, $s = 1, 2$; $k = 1, 2, 3, 4$; $t = 1, 2, \ldots, 8$.

Conversely, every system of linearly independent boundary conditions of the form (5.3) defines a certain self-adjoint extension L_s of the operator L_0 provided that the relations (5.4) are satisfied.

Proof: We apply theorem 5.1 to the operator L_0. In this case $h = 8$. Let the domain of definition $\mathfrak{D}_s$ of a self-adjoint extension L_s of the operator L_0 be given in the sense of theorem 5.1 by the elements $\phi_1, \phi_2, \ldots, \phi_8$; where $\phi_k = \{\phi_{1k}, \phi_{2k}\}, k = 1, 2, \ldots, 8$.

We put

$$\left.\begin{aligned} \{m_{t1}^k, \ m_{t2}^k\} &= -\phi_t^{[4-k]}(a) \\ \{m_{t1}^{2+k}, \ m_{t2}^{2+k}\} &= \phi_t^{[2-k]}(a) \\ \{n_{t1}^k, \ n_{t2}^k\} &= \phi_t^{[4-k]}(b) \\ \{n_{t1}^{2+k}, \ n_{t2}^{2+k}\} &= -\phi_t^{[2-k]}(b) \end{aligned}\right\} \tag{5.5}$$

$k = 1, 2$; $t = 1, 2, \ldots, 8$.

Now the conditions (5.1) and the relation (5.2) can be put in the form (5.3) and (5.4) respectively. Hence the linear independence of the vector functions $\phi_1, \phi_2 \ldots, \phi_8$ confirm the linear independence of the conditions (5.3) (*cf.* Theorem 5.1).

Conversely, let the linearly independent boundary conditions (5.3) satisfying the conditions (5.4) be prescribed. By lemma 3.2, there are elements $\phi_1, \phi_2, \ldots, \phi_8$ in $\mathfrak{D}$ satisfying the conditions (5.5), but then the conditions (5.3) and (5.4) can be written in the form (5.1) and (5.2) respectively. Hence the conditions (5.5) separate out the domain of definition of a certain self-adjoint extension of the operator L_0. Hence the theorem follows.

The present theorem can be generalized as follows.

Every self-adjoint extension L_s of the operator L_0 generated by an $r \times r$ matrix differential expression with elements depending upon differential coefficients of order $2n$, can be determined by

(i) linearly independent non-separated boundary conditions of the form

$$\sum_{k=1}^{2n} E_k\, y^{[k-1]}(a) + \sum_{k=1}^{2n} F_k\, y^{[k-1]}(b) = [0]$$

with

$$\sum_{s=1}^{n} E_s E_{2n-s+1}^{T} - \sum_{s=1}^{n} E_{2n-s+1} E_s^T = \sum_{s=1}^{n} F_s F_{2n-s+1}^{T} - \sum_{s=1}^{n} F_{2n-s+1} F_s^T$$

and conversely.

(ii) linearly independent separated boundary conditions of the form

$$\sum_{k=1}^{2n} G_k\, y^{[k-1]}(a) = [0] \quad \text{and} \quad \sum_{k=1}^{2n} H_k\, y^{[k-1]}(b) = [0]$$

with

$$\sum_{s=1}^{n} H_s H_{2n-s+1}^{T} - \sum_{s=1}^{n} H_{2n-s+1} H_s^T = [0]$$

$$\sum_{s=1}^{n} G_s G_{2n-s+1}^{T} - \sum_{s=1}^{n} G_{2n-s+1} G_s^T = [0]$$

and conversely; where $y(x)$ is r-dimensional column vector; E_k, F_k are $2nr \times r$ matrices and G_k, H_k are $nr \times r$ matrices.

The proof follows by dimensional generalisation of Theorem 5.2.

6. The operator L_0 in the singular case

We consider the operator L_0 as discussed in § 4. Let us suppose that the end-point a is regular and the end-point b is singular.

Theorem 6.1: The domain of definition $\mathfrak{D}_0$ of the operator L_0 consists precisely of those vector functions $y(x) = \{y_1(x), y_2(x)\}$ belonging to $\mathfrak{D}$ which satisfy the following conditions

(1) $y^{[k]}(a) = [0]$. $k = 0, 1, 2, \ldots, (2n-1)$

(2) $[yz](b) = 0$, for vector functions $z(x) = \{z_1(x), z_2(x)\}$.

The proof follows by Naimark[2], § 17.5.

Theorem 6.2: If the operator L_0 has the deficiency indices (4, 4) then for any arbitrary vector elements $y, z \in \mathfrak{D}$

$$[yz](b) = 0. \tag{6.1}$$

Proof: Let $\triangle = [a, \beta]$ be a fixed finite interval lying entirely within (a, b). Then the matrix differential expression $Q(y)$ of order 4 is regular in $\triangle$. In $\mathfrak{D}_\triangle$ we choose the vector functions $z_s = \{z_{1s}, z_{2s}\}$; $s = 1, 2, \ldots, 8$ such that

$$\left.\begin{array}{l} z_{ij}^k(a) = \delta_{j+2k\ i} \\ z_{ij}^k(\beta) = 0 \end{array}\right\} \qquad \begin{array}{l} i = 1, 2; k = 0, 1, 2, 3 \\ j = 1, 2, \ldots, 8. \end{array}$$

Such vector functions do exist by Lemma 3.2. Beyond the limits of the interval $[a, \beta]$ these vector functions are equal to zero, *i.e.*,

$$z_{ij}^{[k]}(b) = [0].\ i = 1, 2;\ j = 1, 2, 3, \ldots, 8;\ k = 0, 1, 2, 3.$$

Now by Naimark[2], lemma VI of § 17.5, p. 71 and Theorem 6.1 the proof follows.

The theorems 5.1 and 5.2 can be restated as

Theorem 5.1: The domain of definition $\mathfrak{D}_s$ of an arbitrary self-adjoint extension L_s of the operator L_0 with deficiency indices (4, 4) consists of the set of all vector functions $y(x) \in \mathfrak{D}$ which satisfy the conditions,

$$[y, \phi_k](a) = 0, \quad k = 1, 2, 3, 4 \tag{6.2}$$

where $\phi_1, \phi_2, \phi_3, \phi_4$ are certain vector functions belonging to $\mathfrak{D}$ which are linearly independent modulo $\mathfrak{D}_0$ and for which the relations

$$[\phi_j \phi_k](a) = 0 \tag{6.3}$$

hold.

Conversely, for arbitrary vector functions $\phi_1, \phi_2, \phi_3, \phi_4$ belonging to $\mathfrak{D}$ which are linearly independent modulo $\mathfrak{D}_0$ and which satisfy the relations (6.3), the set of all vector functions $y(x) \in \mathfrak{D}$ which satisfy the conditions (6.2) is the domain of definition of a self-adjoint extension of the operator L_0.

Theorem 5.2: Every self-adjoint extension L_s of the operator L_0 is determined by linearly independent boundary conditions of the form

$$\sum_{k=1}^{2n} E_k\, y^{[k-1]}(a) = [0] \tag{6.4}$$

with

$$\sum_{s=1}^{n} E_s E^T_{2n-s+1} - \sum_{s=1}^{n} E_{2n-s+1} E^T_s = [0] \tag{6.5}$$

and conversely.

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