TRANSFER FUNCTION APPROACH TO POLE-PLACEMENT

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Received on August 18, 1976 and in revised form on December 30, 1976

ABSTRACT

A method for the design of a linear time invariant system for realizing arbitrarily specified closed loop poles, via *the transfer function approach is presented. The design technique is compared with that which utilizes the state variable approach.*

Keywords: Control system design, pole-placement, feedback compensation, pole assignment.

1. INTRODUCTION

The modern approach to control system design is to use the state variable techniques. In order to assign arbitrarily specified poles to the closed loop system through state variable feedback, we require the system to be completely controllable. In order to generate the state variables, we require the system to be completely observable. Hence, design by state variable method utilizes only the completely controllable and completely observable subset of the overall state variable equations. Since the transfer function corresponds to the completely controllable and completely observable subset of the state variable description, it stands to reason that the transfer function approach should also enable us to design the system for achieving arbitrarily specified closed loop poles. Such a design procedure enables a nonspecialist (who is unable to comprehend the abstract state variable techniques) to appreciate the technique which works with physical variables of the process.

It is interesting to note that when designing an observer, we specify the eigenvalues of the observer arbitrarily. Further, it can be recalled that the introduction of the dynamic observer in the feedback path does not affect the stability of the closed loop system. More specifically, the closed loop poles of the overall system containing the observer in the feedback path (*see* Fig. 1) are the required closed loop poles and the poles of the observer [1]. Conventional transfer function approach tells us that

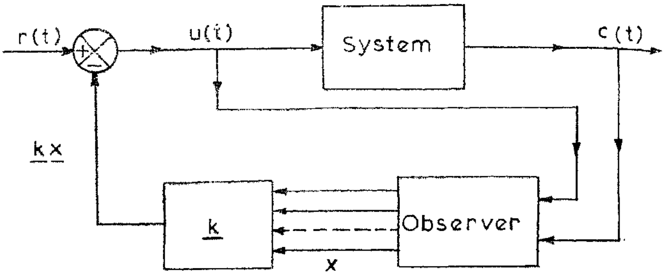

FIG. 1. State variable feedback.

unless there is a pole zero cancellation, a feedback loop pole cannot be a pole of the closed loop. Hence we conclude that all the poles of the observer get cancelled.

The above observations motivate us to develop a design procedure to realize arbitrarily specified closed loop poles through the transfer function approach. Such a design procedure is presented in the present paper with special reference to a single input-single output system. The technique can be extended to multiple input-multiple output systems [2].

2. DEVELOPMENT OF THE DESIGN PROCEDURE

Let the transfer function of the system whose design is contemplated be

$$G(s) = \frac{g(s)}{d(s)}. \tag{1}$$

Let the degree of $d(s)$ be (n) and that of $g(s)$ be $(n-1)$, so that the above transfer function corresponds to that encountered generally in practice. Let the feedback design be achieved according to Fig. 2.

Let

$$H(s) = \frac{h(s)}{p(s)} \tag{2}$$

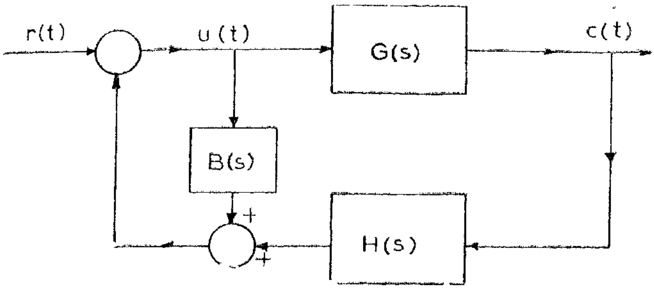

FIG. 2. Feedback design.

and

$$B(s) = \frac{b(s)}{p(s)} \tag{3}$$

The degrees and coefficients of the polynomials $p(s)$, $h(s)$ and $b(s)$ are yet to be specified. There is no loss of generality in assuming the coefficients of the highest powers of (s) in $p(s)$ and $d(s)$ to be unity. The design objective is to select $H(s)$ and $B(s)$ so that the closed loop of Fig. 2, is assigned arbitrarily specified eigenvalues. Simple calculations show that the closed loop transfer function is given by

$$\frac{C(s)}{U(s)} = \frac{G(s)}{1 + H(s)\,G(s) + B(s)} \tag{4}$$

Now

$$H(s)\,G(s) + B(s) = \frac{h(s)\,g(s)}{p(s)\,d(s)} + \frac{b(s)}{p(s)} = \frac{h(s)\,g(s) + b(s)\,d(s)}{p(s)\,d(s)} \tag{5}$$

If it is possible to select $h(s)$ and $b(s)$ such that

$$h(s)\,g(s) + b(s)\,d(s) = p(s)\,k(s) \tag{6}$$

where $k(s)$ is an arbitrarily specified polynomial, then equation (5) becomes

$$H(s)\,G(s) + B(s) = \frac{k(s)}{d(s)} \tag{7}$$

and the closed loop transfer function becomes

$$\frac{C(s)}{U(s)} = \frac{g(s)/d(s)}{1 + \frac{k(s)}{d(s)}} = \frac{g(s)}{d(s) + k(s)} = \frac{g(s)}{r(s)} \tag{8}$$

We recall that the coefficient of s^n in $d(s)$ is unity. Hence it is sufficient to require $k(s)$ to be a polynomial of degree $(n-1)$ with arbitrary coefficients so that $d(s)+k(s)$ becomes an arbitrarily specified polynomial. Using this information in equation (6), we can conclude that the degrees of $h(s)$, $b(s)$ and $p(s)$ are $(n-1)$, $(n-2)$ and $(n-1)$ respectively, when minimum number of parameters are to be used in $H(s)$ and $B(s)$. With the above preliminary observations we let

$$p(s) = s^{n-1} + p_{n-2}\, s^{n-2} + \ldots + p_1 s + p_0 \tag{9}$$

$$h(s) = h_{n-1}\, s^{n-1} + h_{n-2}\, s^{n-2} + \ldots + h_1 s + h_0 \tag{10}$$

$$b(s) = b_{n-2}\, s^{n-2} + b_{n-3}\, s^{n-3} + \ldots + b_1 s + b_0 \tag{11}$$

$$r(s) = s^n + r_{n-1}\, s^{n-1} + \ldots + r_1 s + r_0 \tag{12}$$

where the coefficients p_i, h_i, b_i and r_i are yet to be specified. Let

$$g(s) = g_{n-1}\, s^{n-1} + \ldots + g_1 s + g_0 \tag{13}$$

$$d(s) = s^n + d_{n-1}\, s^{n-1} + \ldots + d_0 s + d_0 \tag{14}$$

where g_i and d_i are known coefficients.

From the perfoımance specifications, we can arrive at the desired closed loop characteristic equation, and hence r_i are determined from this.

We note that

$$r(s) = d(s) + k(s). \tag{15}$$

Hence

$$\begin{aligned} k(s) &= r(s) - d(s) \\ &= k_{n-1}\, s^{n-1} + \ldots + k_1 s + k_0 \end{aligned} \tag{16}$$

where

$$k_i = r_i - d_i. \tag{17}$$

Let us choose the polynomial $p(s)$ such that all its eigenvalues are in the left half of complex plane. This requirement is imposed on the polynomial $p(s)$, so that any imperfect cancellation of the poles of $H(s)$ and $B(s)$ will not significantly affect the transient response of the resulting system.

Design of the feedback network becomes complete with the computation of the coeffirients b_i and h_i of the polynomial $B(s)$ and $H(s)$.

Let

$$\begin{aligned} p(s)\, k(s) &= q(s) \\ &= s^{2n-2} + q_{2n-3}\, s^{2n-3} + \ldots + q_1 s + q_0. \end{aligned} \tag{18}$$

We note that since $p(s)$ and $q(s)$ are known polynomials, $q(s)$ can be evaluated. Using equation (18) in equation (6) we get

$$h(s)\,g(s) + b(s)\,d(s) = q(s). \tag{19}$$

Substituting for the various polynomials in equation (19) from equations (10), (11), (12), (13) and (18) and equating the coefficients of like powers of s on both sides, we get the following linear algebraic equation for the unknown coefficients b_i and h_i.

$$\left[\begin{array}{cccc|cccc} d_0 & 0 & \dots 0 & 0 & g_0 & 0 & \dots 0 & 0 \\ d_1 & d_0 & \dots 0 & 0 & g_1 & g_0 & \dots 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{n-3} & d_{n-4} & \dots d_0 & 0 & g_{n-3} & g_{n-4} & \dots 0 & 0 \\ d_{n-2} & d_{n-3} & \dots d_1 & d_0 & g_{n-2} & g_{n-3} & \dots g_0 & 0 \\ \hline d_{n-1} & d_{n-2} & \dots d_2 & d_1 & g_{n-1} & g_{n-2} & \dots g_1 & g_0 \\ 1 & d_{n-1} & \dots d_3 & d_2 & 0 & g_{n-1} & \dots g_2 & g_1 \\ 0 & 1 & \dots d_4 & d_3 & 0 & 0 & \dots g_3 & g_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots 1 & d_{n-1} & 0 & 0 & \dots g_{n-1} & g_{n-2} \\ 0 & 0 & \dots 0 & 1 & 0 & 0 & \dots 0 & g_{n-1} \end{array}\right] \left[\begin{array}{c} b_0 \\ b_1 \\ \vdots \\ b_{n-3} \\ b_{n-2} \\ \hline h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_{n-2} \\ h_{n-1} \end{array}\right] = \left[\begin{array}{c} q_0 \\ q_1 \\ \vdots \\ q_{n-3} \\ q_{n-2} \\ \hline q_{n-1} \\ q_n \\ q_{n+1} \\ \vdots \\ q_{2n-3} \\ q_{2n-2} \end{array}\right] \tag{20}$$

The determinant of the coefficient matrix is the resultant of the polynomials $d(s)$ and $g(s)$ and is non-zero, if the polynomials $d(s)$ and $g(s)$ are relatively prime 3. Therefore equation (20) gives a unique solution to the parameters b_i and h_i. This completes the design of the closed loop system for arbitrary assignment of closed loop poles.

3. A Numerical Example

Given:

$$G(s) = \frac{s^2 + s + 2}{s^3 + 4s^2 + s - 6}$$

Required closed loop characteristic polynomial $= (s^3 + 6s^2 + 11s + 6)$.

Solution

Assume $p(s) = s^2 + 4s + 20$ (arbitrary). Equation (20) for the above becomes

$$\begin{bmatrix} -6 & 0 & 2 & 0 & 0 \\ 1 & -6 & 1 & 2 & 0 \\ 4 & 1 & 1 & 1 & 1 \\ 1 & 4 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ h_0 \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 240 \\ 248 \\ 92 \\ 18 \\ 2 \end{bmatrix}$$

from which

$$\begin{bmatrix} b_0 \\ b_1 \\ h_0 \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} -15 \\ -10\cdot5 \\ 75 \\ 62\cdot5 \\ 12\cdot5 \end{bmatrix}$$

Hence

$$H(s) = \frac{12\cdot5\, s^2 + 62\cdot5 s + 75}{s^2 + 4s + 20}$$

$$B(s) = \frac{-(10\cdot5 s + 15)}{s^2 + 4s + 20}$$

It would be interesting to compare this solution with that through state variable feedback. The observer canonical form for the given transfer function is written below

$$\dot{x} = \begin{bmatrix} -4 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 6 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} u(t)$$

$$c(t) = [1 \quad 0 \quad 0]\, x.$$

Let us seek an observer (2nd order) such that $z(t) = [\alpha\ I]\ x$ where $z(t)$ is given by $\dot{z} = Dz + fc(t) + \omega u(t)$.

Assume the eigenvalues of D to be given by $s^2 + 4s + 20 = 0$ and let

$$D = \begin{bmatrix} -4 & 1 \\ -20 & 0 \end{bmatrix}$$

It can be shown that we get the following solution for the observer.

$$a = \begin{bmatrix} -4 \\ -20 \end{bmatrix}, \quad f = \begin{bmatrix} 19 \\ 6 \end{bmatrix}, \quad \omega = \begin{bmatrix} -3 \\ -18 \end{bmatrix}$$

Further it can be shown that the required state variable feedback which assigns the required closed loop poles is given by

$$u(t) = r(t) - v(t)$$

where

$$v(t) = -\underline{[0 \cdot 5 \quad 0 \cdot 5 \quad 0 \cdot 5]}\, x$$

which in terms of the state variable of the observer and the output of the system is given by

$$u(t) = -\underline{[12 \cdot 5 \quad 0 \cdot 5 \quad 0 \cdot 5]} \begin{bmatrix} c(t) \\ z(t) \end{bmatrix} + r(t)$$

The two feedback loop transfer function can now be calculated and are given by

$$\left.\frac{V(s)}{U(s)}\right|_{c(t)=0} = \underline{[0 \cdot 5 \quad 0 \cdot 5]}\, [sI - D]^{-1} \begin{bmatrix} -3 \\ -18 \end{bmatrix}$$

$$= \frac{-(10 \cdot 5s + 15)}{s^2 + 4s + 20}$$

and

$$\left.\frac{V(s)}{C(s)}\right|_{u(t)=0} = 12 \cdot 5 + \underline{[0 \cdot 5 \quad 0 \cdot 5]}\, [sI - D]^{-1} \begin{bmatrix} 19 \\ 6 \end{bmatrix}$$

$$= 12 \cdot 5 + \frac{12 \cdot 5s - 175}{s^2 + 4s + 20} = \frac{12 \cdot 5s^2 + 62 \cdot 5s + 75}{s^2 + 4s + 20}$$

4. Summary and Conclusions

In this paper it has been shown that the transfer function approach can give exactly the same results as given by the state variable approach for the design of a single input-single output system in order to realize arbitrarily specified closed loop poles.

Pearson [4] presents a method of solving the same problem. Here the feedback compensator is excited only by the output of the system. Further the method requires the specification of $(2n - 1)$ closed loop poles.

The present paper also requires, though indirectly, the specifications of $(2n - 1)$ poles, if we take into consideration the arbitrarily selected $(n - 1)$ poles of the feedback compensator excited by the both input and output of the given systems.

It would be worthwhile to examine which of these methods would cause less instrumentation problems (by demanding lesser gain terms/coefficients of the compensator transfer function) before choosing one in preference to the other.

REFERENCES

[1] Luenberger, D. G. .. Observers for multivariable systems. *IEEE Trans.*, 1966, *AC* **11** (2), 190–197.

[2] Wolovich, W. A. .. *Linear Multivariable Systems*. (Springer Verlag), 1974.

[3] Bocher, M. .. *Introduction to Higher Algebra*. (MacMillan), 1931.

[4] Pearson, J. B. .. Compensator design for dynamic optimization. *Int. J. Control*, 1969, **9** (4), 473–482.