A GENERALIZED LURIE'S PROBLEM

J. S. ANSARI

(*School of Automation, Indian Institute of Science, Bangalore 560012*)

Received on August 17, 1976 and in revised form on January 19, 1977

ABSTRACT

An indirect control problem of the type; $\dot{x} = A(t)\,x - b(t)\,\xi$ *with feedback* $\dot{\xi}(t) = \phi\,[t, c'(t)\,x(t) - f(t, \xi)]$, *is considered. This problem is more general than Lurie's problem, in that, A, b and c are allowed to be time varying, also f can be nonlinear and time varying. However, in one sense it is more restrictive, namely,* $\phi(t,a)/a < 0$, *as opposed to* $\phi(t,a)/a \geqslant 0$ *in Lurie's problem. A sufficient condition for absolute stability is derived.*

Keywords. Stability, Nonlinear systems, Control systems.

1. INTRODUCTION

An indirect control of the following type is considered;

Linear part:

$$\dot{x}(t) = A(t)\,x(t) - b(t)\,\xi(t) \tag{1 a}$$

Nonlinear feedback:

$$\dot{\xi}(t) = \phi\,[t, v'(t)\,x(t) - f\,(t, \xi(t))] \tag{1 b}$$

where, A is $n \times n$, b and c are $n \times l$ and f and ϕ are scalar functions such that

$$0 < a \leqslant \frac{\phi(t, a)}{a} \tag{2}$$

$$f(t, 0) = \phi(t, 0) = 0. \tag{3}$$

Linear part is input-output stable in the sense that if $|\xi(t)| < H$ for $t > 0$, then there exists an M depending on H and $x(0)$ such that $||x(t)|| < M$ for $t > 0$. M can be made as small as desired by choosing sufficiently small H and $||x(0)||$. This problem is more general than Lurie's indirect control in that, firstly, A, b and c have to be invariant in Lurie problem; secondly, in Lurie's problem f (t, ξ) can only be of the form $\rho\xi$ [1]; so that, under steady state, feedback in Lurie's problem is linear and time invariant.

However, in one respect the above system is more restrictive than Lurie's. This is so because ϕ $(t, a)/a$ is not allowed to be equal to zero.

It may be noted that because $A(t)$ can be time varying, frequency domain techniques, such as M. V. Popov method and circle criterion are not applicable. Other more general results are difficult to apply [2, 3]. The results obtained are similar to Lurie's sufficient condition for absolute stability which is of the form $\rho > \rho^*$ [1]. In the more general case under consideration here, $\rho\xi$ is replaced by a nonlinear time varying function $f(t, \xi)$, so that the condition obtained is of the type:

$$\frac{f(t, \xi)}{\xi} > \psi(t) \tag{4}$$

where $\psi(t)$ is known if the transition matrix of the linear part is known.

2. LEMMA

Let

$$\dot{y}(t) = F(t, y_t) \tag{5}$$

represent a scalar functional differential equation, where y_t in the argument of F shows dependence of F on $y(S)$, $0 \leqslant t - t_0(t) \leqslant S \leqslant t$. Note that equation (5) can contain integrals of functions of y within the limits zero to t. Let F be such that y and $\dot{y}$ are continuous for $t > 0$.

Suppose, there exist continuous, nonincreasing, continuously differenciable functions $H_1(t)$ and $H_2(t)$, such that for any t_a, $t_b > 0$,

(i) $F(t_a, y_{t_a}) < \dot{H}_2(t_a)$, if

$y(t_a) = H_2(t_a)$, and $-H_1(S) < y(S)$, $< H_2(S)$, for $S \epsilon I(t_a)$, where $I(t)$ represents the interval $(t - t_a, t)$

(ii) $F(t_b, y_{t_b}) > -\dot{H}_1(t_b)$, if

$y(t_b) = -H_1(t_b)$ and $-H_1(S) < y(S) < H_2(S)$, for $S \epsilon I(t_b)$.

Then

$$-H_1(t) < y(t) < H_2(t), \quad \text{for all } t > 0$$

if

$$-H_1(0) < y(S) < H_2(0), \quad \text{for } S \epsilon I(0), \text{ and } S = 0$$

Proof: Let t_1 be the smallest value of $t > 0$ such that $y(t)$ is equal to $H_2(t)$ or $-H_1(t)$. Let us assume that

$$y(t_1) = H_2(t_1) \tag{6}$$

and

$$-H_1(t') < y(t') < H_2(t'), \qquad t' \epsilon I(t_1) \tag{7}$$

Then, it is necessary that

$$\dot{y}(t_1) \geqslant \dot{H}_2(t_1). \tag{8}$$

If assumption (6) is true, it follows from condition (i) and equation (5), that

$$\dot{y}(t_1) < \dot{H}_2(t_1). \tag{9}$$

Since, inequalities (8) and (9) are contradictory, assumptions (6) and (7) cannot be true. Similarly, it can be shown that $y(t_1)$ cannot be equal to $-H_1(t)$. Hence, $y(t)$ remains strictly bounded by $-H_1$ and H_2.

3. Generalized Lurie's Indirect Control

Consider the feedback control problem described by equation (1) and constraints (2) and (3).

Theorem: If for every H_0, $0 < H_0 < \bar{H}$, there exists an $\epsilon > 0$ such that:

$$\text{(i)} \ (-1)^n \frac{f\left(t, (-1)^n H_0\right)}{H_0} - \int_0^t | k(t, S) | \, dS \geqslant \epsilon > 0, \quad n = 1, 2,.$$

where k represents the impulse response of the linear part defined as follows;

$$k(t, S) = c'(t) X(t, S) b(S)$$

$$\frac{\partial X(t, S)}{\partial t} = A(t) X(t, S); \ X(S, S) = I.$$

(ii) Given any $\delta > 0$, a set $u_0(\delta)$ can be chosen such that,

$x(0) \in u_0$ implies $g(t, u_0) < \delta$ for all $t > 0$, where, $g(t)$ is defined as

$$g(t, u_0) = \max_{X(0) \in u_0} \mid c'(t) X(t, 0) X(0) \mid$$

Then system (1) is stable in the sense that for any given H_0, $0 < H_0 < \bar{H}$, there exists a region U_0 in the $(n+1)$ dimension state space and positive number M such that $(x(0), \xi(0)) \in U_0$ implies $\mid \xi(t) \mid < H_0$ and $\| x(t) \| < M$ for $t > 0$.

Proof.—Elimination of x from equations (1 *a*) and (1 *b*) leads to

$$\dot{\xi}(t) = \phi\, [t, c'(t) X(t, 0) x(0) - f(t, \xi(t)) + \int_0^t k(t, S)\, \xi(S)\, dS] \tag{10}$$

Since equation (10) is of the form of equation (5), the above stated Lemma is applicable. Sufficient condition for $\mid \xi(t) \mid < H(t)$ obtained through the application of the Lemma are

$$\dot{\xi}(t) = \phi\, [t, c'X(t, 0)\, x(0) - f(t, H(t)) + \int_0^t k(t, S)\, p(S)\, dS] < \dot{H} \tag{11}$$

and

$$\dot{\xi}(t) = \phi\, [t, c'\, X(t, 0)\, x(0) - f(t, H(t)) - \int_0^t k(t, S)\, p(S)\, dS] > -\dot{H} \tag{12}$$

for any $p(S)$, such that $-H(S) < p(S) < H(S)$, for $0 \leqslant S < t$. Inequalities (11) and (12) are satisfied for any $x(0) \in u_0$, if

$$(-1)^n\, \phi\, [t, (-1)^n \{g(t, u_0) - (-1)^n f(t, (-1)^n H(t)) + \int_0^t \mid k(t, S) \mid H(S)\, dS\}] < \dot{H} \quad n = 1, 2. \tag{13}$$

It follows from constraint (2) that $\phi(t, a) > 0$, if $a > 0$. Hence, inequality (13) is satisfied for H equal to a constant, H_0, if

$$(-1)^n f\big(t, (-1)^n H_0\big) > g(t, u_0) + H_0 \int_0^t |k(t, S)|\, dS \qquad n = 1, 2 \tag{14}$$

If conditions (i) and (ii) of the theorem are satisfied, given an H_0, one can find an $\epsilon > 0$ and a set u_0 such that for any $x(0) \in u_0$.

$$(-1)^n f\big(t, (-1)^n H_0\big) - H_0 \int_0^t |k(t, S)|\, dS \geq \epsilon > g(t, u_0), \qquad n = 1, 2. \tag{15}$$

Since, inequalities (14) and (15) are equivalent, it follows that given any H_0 one can find a set u_0 such that $x(0) \in u_0$ and $|\xi(0)| < H_0$ implies $|\xi(t)| < H_0$ for $t > 0$. From the assumption of stability of the linear part, $|\xi(t)| < H_0$ implies existence of an M such that $\|x(t)\| < M$. Hence, the theorem is proved.

4. *Example 1:*

Consider a system described by relations (1 a), (1 b), (2) and (3), where $A(t)$ is $n \times n$ diagonal matrix, with diagonal elements $k_i(t) > 0$. In this case, impulse response $k(t, S)$ is given by

$$k(t, S) = \sum_1^n c_i(t)\, b_i(t) \exp\Big[-\int_{\infty}^{t} k_i(p)\, dp\Big]. \tag{16}$$

Choose ϵ_i, $i = 1, 2, \cdots, n$ and H_0 such that

$$|x_i(0)| < \epsilon_i H_0, \qquad i = 1, 2, \cdots, n \tag{17}$$

and

$$|\xi(0)| < H_0 \tag{18}$$

Define an upper bound g on the initial condition response as

$$g(t) = H_0 \sum_1^n \epsilon_i\, |c_i(t)| \exp\Big[-\int_0^t k_i(t')\, dt'\Big]. \tag{19}$$

Let us assume that the c_i and k_i are such that $g(t) \to 0$. This system is stable in $\bar{H}$ if for every $0 < H_0 < \bar{H}$, there exists an $\epsilon > 0$ such that

$$(-1)^n f\big(t, (-1)^n H_0\big) > H_0 \int_0^t | k(t, S) | \, dS + \epsilon, \; n = 1, 2. \quad (20)$$

Lurie's method and frequency domain methods are not applicable for this case.

In the special case, where k_i, b_i and c_i are constants, and $f(t, \xi)$ is linear and time invariant given by $\rho\xi$, the above condtion reduces to

$$\rho > \sum_1^n \frac{| c_i b_i |}{k_i} \quad (21)$$

If this condition is satisfied the system is absolutely stable for any ϕ such that $\phi(t, a)/a > 0$.

It is of interest to investigate the sufficient condition for stability obtained by Lurie's method for the special case of time invariant linear part with $\phi(a)/a < 0$. If one chooses a Liapunov function

$$V = \bar{x}' B \bar{x} + \int_0^\sigma \phi(a) \, da \quad (22)$$

where $\sigma = c' x - \rho\xi$ and B is such that

$$-C = A'B + BA = -I \quad (23)$$

where I is identity matrix, then the condition obtained is same as (21) [1]. By choosing a different C, one may get a better result.

Since, ϕ in Lurie's method belongs to a wider class as compared to the present problem, and the condition for stability in the special case by Lurie's method is not stronger, obviously, Lurie's method gives better results for the special case of the above example. However, the strength of the method described above primarily lies in that it can be applied to time varying linear part and nonlinear time varying f in the feedback, to which Lurie's method is not applicable.

Example 2: Let

$$\dot{x} = p(t)\, x + b(t)\, \xi \quad (24)$$

$$\dot{\xi} = x - \xi - \tfrac{1}{2} \xi^{1/3} \quad (25)$$

$$x(0) = 0 \tag{26}$$

$$\int_a^t p(t')\,dt' \leqslant (a - t) \text{ for } t \geqslant a \geqslant 0 \tag{27}$$

$$|b(t)| < 3. \tag{28}$$

According to the Theorem stated earlier, a sufficient condition for stability is that there exist an $\bar{H} > 0$ such that for every H_0, $0 < H_0 < \bar{H}$,

$$H_0 + \tfrac{1}{2} H_0^{1/3} > H_0 \, |b(t)| \int_0^t \exp\left(\int_{t^1}^t p(a)\,da\right) dt' \tag{29}$$

From conditions (27) and (28) it is seen that

$$H_0 \, |b(t)| \int_0^t \exp\left(\int_{t^1}^t p(a)\,da\right) at' < 3H_0. \tag{30}$$

From inequalities (29) and (30) it is seen that, inequality (29) is satisfied if

$$H_0 + \tfrac{1}{2} H_0^{1/3} > 3H_0. \tag{31}$$

Inequality (31) is satisfied for every $H_0 < 8$. Hence, $|\xi(t)|$ is bounded by H_0 for any pair of initial conditions such that $x(0) = 0$ and $|\xi(0)| \leqslant H_0 < 8$.

References

[1] Lefschetz, S. .. *Stability of Nonlinear Control Systems*. Academic Press, 1965, pp. 17–26.

[2] Damborg, M. J. .. Stability of the basic nonlinear operator Feedback systems. *Ph.D. Thesis*, University of Michigan, 1970.

[3] Sandberg, I. W. .. Some results on the theory of physical systems governed by nonlinear functional equation. *Bell System Tech. J.*, 1965, **44**.