PROPAGATION OF E_0 WAVE IN MODULATED ARTIFICIAL DIELECTRIC MEDIA

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Received on December 24, 1976

ABSTRACT

The problem of propagation of E_0 wave guided by a circular cylindrical non-uniformly corrugated metallic structure which is considered as an artificial dielectric medium having co-sinusoidal dielectric constant profile in the direction of propagation has been formulated in the form of Hill's equation which has been solved for the phase constant as a function of the parameters of the corrugated structure.

Key Words : Artificial dielectric; co-sinusoidal profile: Hill's equation.

1. INTRODUCTION

The characteristics of guided electromagnetic waves in a stratified dielectric medium can be determined by formulating the problem in terms of a pair of linear second-order differential equations with variable co-efficients, with one equation for each of two orthogonal polarisations. A rigorous solution of the problem is possible, if for a given polarisation the dielectric constant profile is such that the governing differential equation can be written in a form for which the solution can be expressed in terms of known functions. Tyras [1] Wait [2] Burman and Gould [3, 4], Gould and Burman [5] have suggested some dielectric constant profiles for which the governing differential equation is amenable to solution in terms of known functions. The differential equation can also be solved by W.K.B. and phase integral methods of approximation if the spatial variation of dielectric constant is very slow that is small over distances comparable to the wavelength of the guided wave.

Rigorous solution for the propagation characteristics and field distribution of waves guided by a sinusoidally modulated plane reactance surface has been obtained by Oliner and Hessel [6] who expressed the explicit field amplitudes and the determinantal equation for the propagation wave number in the form of a continued fraction which is rapidly covergent for all values of modulation. The problem of electromagnetic wave propagation in sinusoidally stratified dielectric media for the case of *H*-modes has been formulated in the form of a Mathieu differential equation and the dispersion properties and the fields of electromagnetic waves are analysed in terms of a "stability" chart by Tamir *et al.* [7]. Propagation characteristics of *E*-waves in a sinusoidally stratified plane dielectric medium have been studied by Yeh *et al.* [8] by formulating the problem in the form of Hill's equation. Numerical computations of the dispersion characteristics show that the stability diagrams for Hill's equation and those for Mathieu's equation are quite different. Casey [9] has introduced a novel method for determining electromagnetic fields in a plane-stratified medium by solving the governing differential equations for each polarisation in terms of Hill's function. This method is useful for a wide range of dielectric constant profile.

The reflection coefficient of electromagnetic waves at a stratified medium has been obtained by Ronchi [10] by using variational technique. The problem of reflection and transmission of electromagnetic waves directed at a dielectric slab, the relative permittivity of which varies symmetrically, either linearly or exponentially from a maximum value at the plane of symmetry has been solved by Haddendorst [11]. The reflection and transmission coefficients have also been computed for electromagnetic waves propagating through a dielectric slab, the permittivity of which decreases symmetrically according to an inverse square law profile from the middle plane to the slab walls bounded by air by Phillippe [12]. Numerical methods have been used to study scattering of plane electromagnetic waves from nonplanar periodic structures by Neurenther and Zaki [13]. Scattering of plane electromagnetic waves from a perfectly conducting surface with a sinusoidal height profile has also been analysed by Zaki and Neurenther [14] by formulating the problem in the form of integral equation and solving it by using the method [15] of moments. An exact mathematical procedure for the design of a modulated corrugated surface to support a specified group of surface waves has been described by Bolljahn [16]. The analysis is limited in application to two-dimensional corrugated surface radiators. The analytical technique developed is however not valid for the cylindrical geometry nor for the dielectric slab form of trapped-wave surface. The analysis does not apply to the case of dielectric structure as it postulates that there is no energy flow across the interface between the guiding structure and the free-space region above this structure. Bolljahn's technique of designing the surface of a surface wave modulated structure may be said to be a basic contribution so far as two-dimensional model of a corrugated surface is concerned. Tamir and Wang [17], Tamir [18] and Tamir and Wang [19] have also contributed significantly in the field of wave propagation over modulated surface.

It appears from the above survey of existing available literature on propagation of electromagnetic waves in a stratified dielectric medium that all work so far have been concerned with modulated natural planar dielectric surface except that of Bolljahn. It is also evident that no attempt has been made so far to study propagation of electromagnetic E_0-waves in a circular cylindrical non-uniformly corrugated structure which is simulated as an artificial dielectric co-sinusoidally modulated in the direction of propagation, though the propagation characteristics of E_0 wave in uniformly corrugated metal and dielectric structures have been the subject of intense study [20–27].

The object of the present paper is to present a report on the derivation of an expression for the phase constant of E_0-wave launched in an artificially simulated co-sinusoidally modulated dielectric medium in the form of non-uniformly corrugated circular cylindrical metallic structures. The study is a contribution of the work recently reported by the authors [28] on the simulation of co-sinusoidally modulated dielectric profile and the work [20–27] related to uniformly corrugated structures.

2. Wave Equation in a Modulated Dielectric Medium

Maxwell's electromagnetic field equations in a source-free, charge-free, non-conducting medium with varying electric permittivity $\kappa(z) = \epsilon^0 \epsilon(z)$ in the direction of propagation are

$$\nabla \times \vec{E} = -\mu \mu_0 \frac{\partial \vec{H}}{\partial t} \tag{1}$$

$$\nabla \times \vec{H} = \epsilon_0 \, \epsilon(z) \frac{\partial \vec{E}}{\partial t} \tag{2}$$

where the field intensity vectors $\vec{E}$ and $\vec{H}$ are assumed to have harmonic time dependence according to

$$\vec{E} = \vec{E} \exp(-i\omega t)$$
$$\vec{H} = \vec{H} \exp(-i\omega t) \tag{3}$$

and

$$\epsilon_0 = 8 \cdot 864 \times 10^{-12} \text{ farad/meter}$$
$$\mu_0 = 4\pi \times 10^{-7} \text{ henry/meter}$$

performing the Curl operation on (1) and substituting from (2), we obtain the following electric vector wave equation in a medium having spatially varying dielectric constant $\epsilon\ (z)$ in the z-direction and assuming the medium having constant permeability and is non-conducting

$$\nabla \times \nabla \times \vec{E} - k_0^2\, \epsilon(z) \vec{E} = 0. \tag{4}$$

In a similar way we arrive at the following magnetic vector wave equation

$$\nabla \times \nabla \times \vec{H} - k_0^2\, \epsilon(z) \vec{H} - \frac{\nabla \epsilon(z)}{\epsilon(z)} \times \nabla \times \vec{H} = 0 \tag{5}$$

3. WAVE EQUATION FOR E WAVE IN A SPATIALLY MODULATED DIELECTRIC MEDIUM

The electric $\vec{E}$ and magnetic $\vec{H}$ field intensity vectors can be expressed in terms of a scalar quantity $\Psi(\rho, \phi, z)$ in cylindrical coordinates (ρ, ϕ, z) by the following Curl relations

$$\vec{E} = \frac{i}{\omega \kappa(z)} \nabla \times \nabla \times [\Psi(\rho, \phi, z) \vec{i}_z] \tag{6}$$

$$\vec{H} = \nabla \times [\Psi(\rho, \phi, z) \vec{i}_z] \tag{7}$$

where $\vec{i}_z$ denotes unit vector in the z-direction.

We will deal with the magnetic vector wave equation (5) which after substituting for $\vec{H}$ from eq. (7) becomes

$$\nabla \times \nabla \times \nabla \times [\Psi(\rho, \phi, z) \vec{i}_z] - k_0^2\, \epsilon(z) \nabla \times [\Psi(\rho, \phi, z) \vec{i}_z]$$
$$- \frac{\nabla \epsilon(z)}{\epsilon(z)} \times \nabla \times \nabla \times [\Psi(\rho, \vec{\phi}, z) i_z] = 0 \tag{8}$$

which in cylindrical coordinates reduces to

$$\frac{\partial^2 \Psi_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \Psi_z}{\partial \rho} + \frac{\partial^2 \Psi_z}{\partial z^2} - \frac{1}{\epsilon(z)} \frac{\partial \epsilon(z)}{\partial z} \frac{\partial \Psi_z}{\partial z} + k_0^2\, \epsilon(z) \Psi_z = 0 \tag{9}$$

which represents the scalar wave equation for E-waves in a non-conducting, non-magnetic medium having spatially varying dielectric constant.

4. Solution of the Wave Equation

Using the method of separation of variables, the solution of equation (9) is obtained as

$$\Psi(\rho, \phi, z) = [A_1 J_0(k\rho) + B_1 Y_0(k\rho)]\, \Phi^{1,2}(z) = 0 \tag{10}$$

Where the function $\Phi^{1,2}(z)$ satisfies the following second-order differential equation

$$\left[\frac{d^2}{dz^2} - \frac{d\epsilon(z)}{dz}\frac{1}{\epsilon(z)}\frac{d}{dz} + \{k_0^2\, \epsilon(z) - k^2\}\right] \Phi^{1,2}(z) = 0 \tag{11}$$

where $k_0^2 = \omega^2 \mu_0 \epsilon_0$ and the radial propagation constant k is independent of the z-coordinate and is related to the axial phase constant $\beta(z)$ by the following separation constant equation

$$k^2 + \beta^2 = k_0^2\, \epsilon(z). \tag{12}$$

5. Solution of the Wave Equation for a Co-sinusoidally Modulated Dielectric Medium

We will consider E-wave propagation in a medium which has a co-sinusoidal dielectric constant profile of the form

$$\epsilon(z) = \epsilon^\circ \left(1 - \delta \cos \frac{2\pi z}{L}\right) \tag{13}$$

where L denotes the period of modulation in the z-direction and the modulation index δ satisfies the inequality condition

$$0 \leqslant \delta < 1. \tag{14}$$

The unmodulated dielectric constant ϵ° is a function of the radius b, spacing s between discs and radius a of the central supporting conductor of an uniformly corrugated circular cylindrical metallic structure and is given by [28]

$$\epsilon^\circ = \frac{20s \pm (400\, s^2 - sc)^{1/2}}{k_0^2\, s^3} \tag{13 a}$$

where

$$c = -768\left[\frac{s}{2} + \frac{l\gamma_0}{2k_0}\frac{K_0(\gamma_0 b)}{K_1(\gamma_0 b)}\frac{F_1(k_0 b)}{F_0(k_0 b)}\right]$$

$$F_0(k_0 b) = J_0(k_0 a)\, Y_0(k_0 b) - Y_0(k_0 a)\, J_0(k_0 b)$$

$F_1(k_0b) = J_0(k_0a)\, Y_1(k_0b) - Y_0(k_0a)\, J_1(k_0b).$

$l = s + t$

t = thickness of discs

γ_0 = radial propagation constant of uniformly corrugated structure.

For a co-sinusoidally stratified dielectric medium the differential equation (11) after substituting (13) becomes

$$\left[\frac{d^2}{dz^2} - \frac{d\left\{\epsilon_0\, \epsilon^\circ\left(1 - \delta\cos\frac{2\pi z}{L}\right)\right\}}{dz} \times \frac{1}{\epsilon_0\, \epsilon^\circ\left(1 - \delta\cos\frac{2\pi z}{L}\right)}\frac{d}{dz} + \left\{k_0^2\, \epsilon^\circ\left(1 - \delta\cos\frac{2\pi z}{L}\right) - k^2\right\}\right]\Phi^{1,2}(z) = 0. \tag{15}$$

Using a new variable $\xi\left(= \frac{\pi z}{L}\right)$ instead of the variable z, the above equation (15) can be transformed to

$$\left[\frac{\pi^2}{L^2}\frac{d^2}{d\xi^2} - \frac{1}{\epsilon^\circ(1 - \delta\cos 2\xi)}\frac{\pi^2}{L^2}\frac{d}{d\xi}\{\epsilon^\circ(1 - \delta\cos 2\xi)\}\frac{d}{d\xi} + \{k_0^2\, \epsilon^\circ(1 - \delta\cos 2\xi) - k^2\}\right](1 - \delta\cos 2\xi)^{1/2}\, W^{1,2}(\xi) = 0 \tag{16}$$

where

$$\Phi^{(1,2)} = (1 - \delta\cos 2\xi)^{1/2}\, W^{(1,2)}(\xi) \tag{16a}$$

The differential equation (16) can be written in the form

$$\left[\frac{d^2}{d\xi^2} + \lambda(\xi)\right] W^{(1,2)}(\xi) = 0 \tag{17}$$

where

$$\lambda(\xi) = \left[\frac{2\delta\cos 2\xi}{(1 - \delta\cos 2\xi)} - \frac{3\delta^2\sin^2 2\xi}{(1 - \delta\cos 2\xi)^2} + \left(\frac{k_0L}{\pi}\right)^2\left\{\epsilon^\circ - \epsilon^\circ\,\delta\cos 2\xi - \left(\frac{k}{k_0}\right)^2\right\}\right] \tag{17a}$$

The function $\lambda(\xi)$ being an even function can be written in the form

$$\lambda(\xi) = \theta_0 + 2\sum_{n=1}^{\infty}\theta_n\cos 2n\xi \tag{18}$$

substituting equation (18) into equation (17) we obtain the following governing differential equation for propagation of *E*-waves in a spatially stratified artificial dielectric medium having periodicity *L*

$$\left[\frac{d^2}{d\xi^2} + \theta_0 + 2 \sum_{n=1}^{\infty} \theta_n \cos 2n\xi\right] W^{(1,2)}(\xi) = 0 \tag{19}$$

which is the general form of well known Hill's differential equation. Since the artificial dielectric medium has been assumed to be periodically modulated the solution of equation (19) for $W^{(1,2)}(\xi)$ will include not only the fundamental but also higher order spatial harmonics. Hence using Floquet's theorem, the solution for $W(\xi)$ can be expressed in terms of forward (n is $+$ *ve*) and backward (n is $-$ ve) spatial harmonics by

$$W^{(1,2)}(\xi) = \exp(\pm i\beta\xi) \sum_{n=-\infty}^{n=\infty} C_n(B) \exp(\pm 2in\xi) \tag{20}$$

where $n = 0$ refers to the fundamental.

6. Determination of θ_0 and θ_n ($n \neq 0$)

$\lambda(\xi)$ in equation (17 *a*) can be written as

$$\lambda(\xi) = \left(\frac{k_0 L}{\pi}\right)^2 \left\{\epsilon^\circ - \epsilon^\circ \delta \cos 2\xi - \left(\frac{k}{k_0}\right)^2\right\} - \left(\frac{3}{2}\delta^2 + \frac{3}{8}\delta^4\right)$$
$$+ \left(2\delta + \delta^3 - \frac{3}{2}\delta^5\right)\cos 2\xi + \left(\frac{5\delta^2}{2} + \delta^4\right)\cos 4\xi$$
$$+ \left(2\delta^3 + \frac{3}{2}\delta^5\right)\cos 6\xi + \left(\frac{11}{8}\delta^4 + \frac{3\delta^5}{2}\right)\cos 8\xi \tag{17 b}$$

Expanding $\lambda(\xi)$ in equation (18) into Fourier Cosine series and comparing with $\lambda(\xi)$ given by equation (17 *b*), the following values for θ_0 and θ_n are obtained in terms of the unmodulated ($\delta = 0$) dielectric constant ϵ° and the modulation index δ which can be defined by

$$\delta = \frac{\epsilon_{max} - \epsilon_{min}}{\epsilon_{max} + \epsilon_{min}} \tag{21}$$

$$\theta_0 = \left(\frac{k_0 L}{\pi}\right)^2 \left[\epsilon^\circ - \left(\frac{k}{k_0}\right)^2\right] - \frac{3}{2}\delta^2\left(1 + \frac{\delta^2}{4}\right)$$

$$\theta_1 = -\frac{\epsilon^\circ \delta}{2}\left(\frac{k_0 L}{\pi}\right)^2 + \left(\delta + \frac{\delta^3}{2} - \frac{3}{4}\delta^5\right)$$

$$\theta_2 = \frac{5\delta^4}{4} + \frac{\delta^4}{2}$$

$$\theta_3 = \delta^3 + \frac{3}{4}\delta^5$$

$$\theta_4 = \frac{11}{16}\delta^4 + \frac{3}{5}\delta^5. \tag{22}$$

Since $\lambda(\xi)$ and hence θ's are related to δ and ϵ° which are functions of a, b and s, therefore, the values of respective θ's depend on the physical parameters of the structure, *i.e.*, $\theta = f(s\ b, a)$. Hence it may be said that the following differential equation

$$\left[\frac{d^2}{d\xi^2} + \theta_0 + 2\sum_{n=1}^{\infty} \theta_n \cos 2n\,\xi\right] \exp(\pm i\beta\xi)$$

$$\times \sum_{n=-\infty}^{\infty} C_n(B) \exp(\pm 2in\xi) = 0 \tag{23}$$

govern the propagation characteristics of E-waves in a non-uniformly corrugated circular cylindrical metallic structure which can be simulated [28] as a spatially modulated artificial dielectric medium.

7. FIELD COMPONENTS OF E_0 WAVE

Substituting equations (16 a) and (20) into equation (10) the solution for the scalar function $\Psi(\rho, \phi, z)$ becomes

$$\Psi(\rho, \phi, z) = \left(1 - \delta \cos\frac{2\pi z}{L}\right)^{1/2} [A\ J_0(k\rho) + B\ Y_0(k\rho)]$$

$$\cdot \exp\left(\pm i\beta\frac{\pi z}{L}\right) \sum_{n=-\infty}^{\infty} C_n\ (\beta) \exp\left(\pm 2in\frac{\pi z}{L}\right) \tag{24}$$

The field components for E_0 wave are H_ϕ, E_ρ and E_z which can be determined from equations (7) and (6) respectively. Equation (7) yields

$$H_\phi = -\frac{\partial \Psi_z}{\partial \rho}. \tag{25}$$

Hence using equation (24) in equation (25), we obtain

$$H_\phi = \left(1 - \delta \cos\frac{2\pi z}{L}\right)^{1/2} [A\ kJ_1(k\rho) + BkY_1(k\rho)]$$

$$\cdot \exp\left(\pm i\beta\frac{\pi z}{L}\right) \sum_{n=-\infty}^{\infty} C_n(\beta) \exp\left(\pm 2in\frac{\pi z}{L}\right) \tag{26}$$

Since from equation (6)

$$E_z = \frac{1}{\omega \kappa (z)} \left[\frac{\partial H_\phi}{\partial \rho} + \frac{1}{\rho} H_\phi \right] \tag{27}$$

therefore the axial component of the electric field is given by

$$E_z = \frac{k^2}{\omega \epsilon_0 \epsilon (z)} \left(1 - \delta \cos \frac{2\pi z}{L}\right)^{1/2} \exp\left(\pm i\beta \frac{\pi z}{L}\right)$$
$$. [A J_0 (k\rho) + B Y_0 (k\rho)] \sum_{n=-\infty}^{\infty} C_n (\beta) \exp\left(\pm 2in \frac{\pi z}{L}\right) \tag{28}$$

where the following recurrence relations have been used

$$J_1' (k\rho) = J_0 (k\rho) - \frac{J_1 (k\rho)}{k\rho}$$
$$Y_1' (k\rho) = Y_0 (k\rho) - \frac{Y_1 (k\rho)}{k\rho} \tag{28 a}$$

Since from equation (6)

$$E_\rho = \frac{i}{\omega \kappa (z)} \frac{\partial H_\phi}{\partial z} \tag{29}$$

the radial component of the electric field is given by .

$$E_\rho = \frac{i}{\omega \epsilon_0 \epsilon (z)} [A\, kJ_1 (k\rho) + B\, kY_1 (k\rho)] \,.\, \exp\left(\pm \frac{i\beta\pi z}{L}\right)$$
$$\times \left[\frac{\pi\delta}{L} \left(1 - \delta \cos \frac{2\pi z}{L}\right)^{-1/2} \sum_{n=-\infty}^{\infty} C_n (\beta) \exp (\pm 2in\pi z/L)\right.$$
$$\pm \left(1 - \delta \cos \frac{2\pi z}{L}\right)^{1/2} \left(\frac{i\pi z}{L} \frac{d\beta}{dz} \pm \frac{i\beta\pi}{L}\right)$$
$$\times \sum_{n=-\infty}^{\infty} C_n (\beta) \exp (\pm 2in\pi z/L) + \left(1 - \delta \cos \frac{2\pi z}{L}\right)^{1/2}$$
$$\times \sum_{n=-\infty}^{\infty} \left\{\frac{d\, C_n (\beta)}{dz} \exp\left(\pm \frac{2in\pi z}{L}\right) \pm \frac{2in\pi}{L} C_n(\beta)\right.$$
$$\left.\left.\times \exp\left(\pm \frac{2in\pi z}{L}\right)\right\}\right] \tag{30}$$

The field distributions in the stratified artificial dielectric medium given by the components, E_ρ, E_z, and H_ϕ can be evaluated by determining β from the solution of Hill's equation.

8. CHARACTERISTIC EQUATION FOR β

Since

$$\frac{d^2}{d\xi^2}\left[\exp(\pm i\beta\xi)\sum_{n=-\infty}^{\infty} C_n(\beta)\exp(\pm 2in\xi)\right]$$

$$= -\sum_{n=-\infty}^{\infty}(\beta+2n)^2 C_n(\beta)\exp(\pm i\beta\xi)\exp(\pm 2in\xi)$$

equation (23) reduces to

$$-\sum_{n=-\infty}^{\infty}(\beta+2n)^2 C_n(\beta)\exp(\pm i\beta\xi)\exp(\pm 2in\xi)$$

$$+\sum_{m=-\infty}^{\infty}\theta_m\cos 2m\xi\sum_{-\infty}^{\infty} C_n(\beta)\exp(\pm i\beta\xi)\exp(\pm 2in\xi)=0$$

which reduces to

$$\left[-(\beta+2n)^2 C_n(\beta)+\sum_{m=-\infty}^{\infty}\theta_m C_{n-m}(\beta)\right]\exp(\pm 2i\xi)^n=0 \qquad (31)$$

since

$$\exp(\pm i\beta\xi)\neq 0$$

Hence the governing differential equation (23) for the modulated corrugated structure reduces to an infinite number of linear homogeneous algebraic equation in C_n as follows

$$-(\beta+2n)^2 C_n(\beta)+\sum_{m=-\infty}^{\infty}\theta_m C_{n-m}(\beta)=0 \qquad (32)$$

with $\theta_{-m}=\theta_m$ and $n=\ldots-2,-1,0,+1,+2\ldots.$

The set of equations (32) is written in the form of an infinite determinant. The possibilities of infinite determinant were first brought into notice by Hill [26] in his researches on Lunar theory. In order to obtain a non-trivial solution the determinant must vanish to zero. It is also necessary that the determinant be convergent. The investigation of convergence of the infinite determinant is due to Poincare' [30] and by Koch [31].

9. CONVERGENCE OF INFINITE DETERMINANT

In order to secure the convergence of the infinite determinant in [32] the following discussion is considered to be worthwhile. Let the determinant be represented by

$$D_m=[A_{ik}]_{i,k=-m,\ldots,+m}$$

If $m \to \infty$, the determinant D_m is said to be convergent if it tends to a determinate limit D. In the determinant D, the elements A_{ii} form the principal diagonal. The subscripts i and k form the rows and column respectively. Any element A_{ik} is called a diagonal if $i = k$ or a non-diagonal if $i \gtrless k$. The element $A_{o,\,o}$ is the origin of the determinant.

According to Koch's condition of convergence, an infinite determinant converges, provided the product of the diagonal elements converges absolutely, and the sum of the non-diagonal elements converges absolutely, *i.e.*, if Π diag $(D_m]$ converges absolutely and also Σ non-diag $[D_m]$ converges. If the diagonal elements of an infinite determinant is denoted by $1 + a_{ii}$ and non-diagonal elements are denoted by a_{ik}, $(i \neq k)$, the determinant is written as

$$\begin{vmatrix} \cdots & \cdots & \cdots \\ \ldots 1 + a_{-1-1} & a_{-10} & a_{-11} \ldots \\ a_{0-1} & 1 + a_{00} & a_{01} \ldots \\ a_{1-1} & a_{10} & 1 + a_{11} \ldots \\ \cdots & \cdots & \cdots \end{vmatrix}$$

Since $\sum\limits_{i,k=-\infty}^{\infty} |a_{ik}|$ is convergent, the infinite product

$$\bar{P} = \prod_{i=-\infty}^{m} \left(1 + \sum_{k=-\infty}^{\infty} |a_{ik}|\right)$$

is also convergent. If we form the products

$$P_m = \prod_{i=-m}^{m} \left(1 + \sum_{k=-m}^{m} a_{ik}\right), \qquad \bar{P}_m = \prod_{i=-m}^{m} \left(1 + \sum_{k=-m}^{m} |a_{ik}|\right)$$

then it can be shown (32) that

$$|D_{m+p} - D_m| \leqslant \bar{P}_{m+p} - \bar{P}_m \,.$$

Therefore since P_m tends to a limit as $m \to \infty$, so also D_m tends to a limit D and we then say that the infinite determinant

$$[A_{ik}]_{i,\,k} = -\infty \ldots + \infty$$

is convergent and has the value D.

In order to secure convergence of eq. (32) divide it by $\theta_0 - 4n^2$ which yields

$$\frac{-(\beta + 2n)^2\, C_n(\beta) + \sum\limits_{m=-\infty}^{\infty} \theta_m\, C_{n-m}(\beta)}{\theta_0 - 4n^2} = 0$$

Which can be written as

$$\frac{(\beta+2n)^2 C_n(\beta) - \sum\limits_{m=-\infty}^{\infty} \theta_{n-m} C_n(\beta)}{4n^2-\theta_0} = 0 \tag{33}$$

10. Determinant of the Characteristic Equation

In order that the coefficients C_n in (33) may not vanish, the determinant of its coefficients must vanish to zero. Hence the definite determinantal equation is

$$\begin{array}{c|ccccc|c}
n \rightarrow & -2 & -1 & 0 & 1 & 2 & \\
i \downarrow & & & & & & \\
\hline
-2 & \frac{(\beta-4)^2-\theta_0}{4^2-\theta_0} & \frac{-\theta_1}{4^2-\theta_0} & \frac{-\theta_2}{4^2-\theta_0} & \frac{-\theta_3}{4^2-\theta_0} & \frac{-\theta_4}{4^2-\theta_0} & \\
-1 & -\frac{\theta_1}{2^2-\theta_0} & \frac{(\beta-2)^2-\theta_0}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} & \frac{-\theta_2}{2^2-\theta_0} & \frac{-\theta_3}{2^2-\theta_0} & \\
0 & \frac{-\theta_2}{0^2-\theta_0} & \frac{-\theta_1}{0^2-\theta_0} & \frac{\beta^2-\theta_0}{0^2-\theta_0} & \frac{-\theta_1}{0^2-\theta_0} & \frac{-\theta_2}{0^2-\theta_0} & =0 \\
1 & \frac{-\theta_3}{2^2-\theta_0} & \frac{-\theta_2}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} & \frac{(\beta+2)^2-\theta_0}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} & \\
2 & \frac{-\theta_4}{4^2-\theta_0} & \frac{-\theta_3}{4^2-\theta_0} & \frac{-\theta_2}{4^2-\theta_0} & \frac{-\theta_1}{4^2-\theta_0} & \frac{(\beta+4)^2-\theta_0}{4^2-\theta_0} & \\
\hline
\end{array}$$

Hence the characteristic equation foı determining β of the modulated structure is given by

$$\nabla(\beta) = 0 \tag{34}$$

11. Evaluation of the Determinant

Let

$$\triangle(\beta) \equiv [A_{m,n}]$$

where

$$A_{mn} = \frac{-\theta_{m-n}}{4m^2-\theta_0} \quad (m \neq n)$$

i.e.,

$$A_{mn} = \frac{(\beta + 2n)^2 - \theta_0}{4m^2 - \theta_0}$$

where A_{mn} is only conditionally convergent. Form a new determinant by dividing the linear homogeneous equation

$$(\beta + 2n)^2 C_n - \sum_{m=-\infty}^{\infty} \theta_{n-m} C_n = 0$$

by $(\beta + 2n)^2 - \theta_0$ to secure convergence. Then the new determinant is

$$\triangle_1(\beta) \equiv [B_{m,n}]$$

where

$$\text{diag}\,[B_{m,n}] = B_{mm} = 1$$

and

$$\text{non-diag}\,[B_{m,n}] = B_{m,n} = \frac{-\theta_{m-n}}{(\beta + 2n)^2 - \theta_0}, \quad (n \neq m)$$

Hence the new determinant is

$$\begin{array}{c|ccccc|}
 & m \rightarrow = -2 & -1 & 0 & 1 & 2 \\
n\downarrow & & & & & \\
-2 & \boxed{1} & \dfrac{-\theta_1}{(\beta-4)^2-\theta_0} & \dfrac{-\theta_2}{(\beta-4)^2-\theta_0} & \dfrac{-\theta_3}{(\beta-4)^2-\theta_0} & \dfrac{-\theta_4}{(\beta-4)^2-\theta_0} \\
-1 & \dfrac{-\theta_1}{(\beta-2)^2-\theta_0} & \boxed{1} & \dfrac{-\theta_1}{(\beta-2)^2-\theta_0} & \dfrac{-\theta_2}{(\beta-2)^2-\theta_0} & \dfrac{-\theta_3}{(\beta-2)^2-\theta_0} \\
\triangle_1(\beta)= \quad 0 & \dfrac{-\theta_2}{(\beta)^2-\theta_0} & \dfrac{-\theta_1}{(\beta)^2-\theta_0} & \boxed{1} & \dfrac{-\theta_1}{(\beta)^2-\theta_0} & \dfrac{-\theta_2}{(\beta)^2-\theta_0} \\
1 & \dfrac{-\theta_3}{(\beta+2)^2-\theta_0} & \dfrac{-\theta_2}{(\beta+2)^2-\theta_0} & \dfrac{-\theta_1}{(\beta+2)^2-\theta_0} & \boxed{1} & \dfrac{-\theta_1}{(\beta+2)^2-\theta_0} \\
2 & \dfrac{-\theta_4}{(\beta+4)^2-\theta_0} & \dfrac{-\theta_3}{(\beta+4)^2-\theta_0} & \dfrac{-\theta_2}{(\beta+4)^2-\theta_0} & \dfrac{-\theta_1}{(\beta+4)^2-\theta_0} & \boxed{1}
\end{array} = (\qquad (35$$

The absolute convergence of $\Sigma\theta_n$ secures the convergence of $[B_{m,n}]$ except when β assumes a value such that the denominator of one of the element becomes zero.

By definition

$$\nabla(\beta) = \nabla_1(\beta) \lim_{p\to\infty} \prod_{n=-p}^{p} \left\{\frac{(\beta+2n)^2-\theta_0}{4n^2-\theta_0}\right\}$$

$$= \nabla_1(\beta) \prod_{-\infty}^{\infty} \frac{(\beta+2n-\sqrt{\theta_0})(\beta+2n+\sqrt{\theta_0})}{(2n)^2-(\sqrt{\theta_0})^2}$$

$$= \nabla_1(\beta) \frac{\beta^2-\theta_0}{-(\sqrt{\theta_0})^2} \prod_{\substack{-\infty \\ n\neq 0}}^{\infty} \frac{(2n)^2\left[1+\frac{\beta+\sqrt{\theta_0}}{2n}\right]\left[1+\frac{\beta-\sqrt{\theta_0}}{2n}\right]}{(2n)^2\left[1-\left(\frac{\sqrt{\theta_0}}{2n}\right)^2\right]}$$

$$= \nabla_1(\beta) \frac{\beta^2-\theta_0}{-\theta_0} \prod_{1}^{\infty} \frac{\left[1-\left(\frac{\beta+\sqrt{\theta_0}}{2n}\right)^2\right]\left[1-\left(\frac{\beta-\sqrt{\theta_0}}{2n}\right)^2\right]}{\left[1-\left(\frac{\sqrt{\theta_0}}{2n}\right)^2\right]^2} \tag{36}$$

The new determinant $\nabla_1(\beta)$ has a sequence of unity along the main diagonal flanked by a sequence of the form $\frac{-\theta_1}{(\beta-n)^2-\theta_0}$ on both neighbouring diagonals ($n = 2,\ 0,\ -2,$ etc.)

Since [33]

$$\frac{\sin z}{z} = \prod_{\substack{-\infty \\ n\neq 0}}^{\infty} \left[1-\frac{z}{n\pi}\right] e^{z/n\pi}$$

$$= \prod_{1}^{\infty} \left[1-\left(\frac{z}{n\pi}\right)^2\right]$$

therefore substituting for z, the following expressions

$$\frac{\pi(\beta+\sqrt{\theta_0})}{2},\ \frac{\pi(\beta-\sqrt{\theta_0})}{2},\ \frac{\pi\sqrt{\theta_0}}{2}$$

equation (36) reduces to

$$\nabla(\beta) = -\nabla_1(\beta) \frac{\sin\pi\left(\frac{\beta+\sqrt{\theta_0}}{2}\right)\sin\pi\left(\frac{\beta-\sqrt{\theta_0}}{2}\right)}{\sin^2\frac{\pi\sqrt{\theta_0}}{2}}$$

which can be simplified to

$$\nabla(\beta) = \nabla_1(\beta)\,\frac{\sin^2\frac{\pi\sqrt{\theta_0}}{2} - \sin^2\frac{\pi\beta}{2}}{\sin^2\frac{\pi\sqrt{\theta_0}}{2}} \tag{37}$$

The determinant $\nabla(\beta)$ involves the determinant $\nabla_1(\beta)$ which is periodic in β having simple poles at $\beta = -2n \pm \sqrt{\theta_0}$ due to off-diagonal elements $(\beta + 2n)^2 - (\sqrt{\theta_0})^2$. These are the only poles of $\nabla_1(\beta)$ and since $\nabla_1 \to 1$ as $\beta \to \infty$ the function is bounded at infinity. Substracting the poles from $\nabla_1(\beta)$ we get

$$K(\beta) = \nabla_1(\beta) - C\sum_{n=-\infty}^{\infty}\frac{1}{(\beta+2n)^2 - (\sqrt{\theta_0})^2}$$

$$= \nabla_1(\beta) - \frac{C}{2\sqrt{\theta_0}}\sum_{n=-\infty}^{\infty}\left[\frac{1}{(\beta+2n+\sqrt{\theta_0})} - \frac{1}{(\beta+2n-\sqrt{\theta_0})}\right] \tag{38}$$

where C equals the residue at each of the poles of ∇_1

$$\text{Since } \pi\cot x = \frac{1}{x} + \frac{2x}{x^2-1} + \frac{2x}{x^2-4} + \ldots = \sum_{n=-\infty}^{\infty}\frac{1}{x+n}$$

Hence the function $K(\beta)$ can be written as

$$K(\beta) = \nabla_1(\beta) - \frac{C\pi}{4\sqrt{\theta_0}}\left[\cot\pi\left(\frac{\beta+\sqrt{\theta_0}}{2}\right) - \cot\pi\left(\frac{\beta-\sqrt{\theta_0}}{2}\right)\right] \tag{39}$$

which has no poles for any value of β, so $K(\beta)$ is analytic for all values of β. $K(\beta)$ is also bounded at $\beta \to \infty$. Hence $K(\beta)$ is a constant (see Appendix A.1). As $\beta \to \infty$, $\nabla_1(\beta) \to 1$ and the second term under bracket in eq. (38) becomes zero. Hence $K(\beta)$ becomes unity. Consequently, from eq. (39) we obtain

$$\nabla_1(\beta) = 1 + \frac{C\pi}{4\sqrt{\theta_0}}\left[\cot\pi\left(\frac{\beta+\sqrt{\theta_0}}{2}\right) - \cot\pi\left(\frac{\beta-\sqrt{\theta_0}}{2}\right)\right]$$

Hence

$$\nabla(\beta) = -\left\{1 + \frac{C\pi}{4\sqrt{\theta_0}}\left[\cot\pi\left(\frac{\beta+\sqrt{\theta_0}}{2}\right) - \cot\pi\left(\frac{\beta-\sqrt{\theta_0}}{2}\right)\right]\right\}$$

$$\times \left\{ \frac{\sin \pi \left(\frac{\beta + \sqrt{\theta_0}}{2} \right) \sin \pi \left(\frac{\beta - \sqrt{\theta_0}}{2} \right)}{\sin^2 \pi (\sqrt{\theta_0}/2)} \right\}$$

$$= - \left\{ \frac{\sin \pi \left(\frac{\beta + \sqrt{\theta_0}}{2} \right) \sin \pi \left(\frac{\beta - \sqrt{\theta_0}}{2} \right)}{\sin^2 \pi (\sqrt{\theta_0}/2)} \right\}$$

$$+ \frac{C\pi}{4 \sqrt{\theta_0}} \left\{ \frac{- \sin \pi \sqrt{\theta_0}}{\sin^2 \pi \frac{\sqrt{\theta_0}}{2}} \right\}$$

$$= \frac{1 - \sin^2 (\pi\beta/2)}{\sin^2 \left(\pi \frac{\sqrt{\theta_0}}{2} \right)} - \frac{C\pi}{2 \sqrt{\theta_0}} \cdot \cot \left(\pi \frac{\sqrt{\theta_0}}{2} \right)$$

Let $\beta = 0$, then

$$\nabla (0) = 1 - \frac{C\pi}{2 \sqrt{\theta_0}} \cdot \cot \frac{\pi \sqrt{\theta_0}}{2}$$

$$\therefore \frac{C\pi}{2 \sqrt{\theta_0}} \cdot \cot \frac{\pi \sqrt{\theta_0}}{2} = 1 - \nabla (0)$$

the residue $C = [1 - \nabla (0)] \frac{2 \sqrt{\theta_0}}{\pi} \tan \frac{\pi \sqrt{\theta_0}}{2}$ (40)

where the determinant $\nabla (0)$ is given by

$$\triangle (0) = \begin{vmatrix} 1 & \frac{-\theta_1}{(4^2 - \theta_0)} & \frac{-\theta_2}{(4^2 - \theta_0)} & \frac{-\theta_3}{(4^2 - \theta_0)} & \frac{-\theta_4}{(4^2 - \theta_0)} \\ \frac{-\theta_1}{2^2 - \theta_0} & 1 & \frac{-\theta_1}{2^2 - \theta_0} & \frac{-\theta_2}{2^2 - \theta_0} & \frac{-\theta_3}{2^2 - \theta_0} \\ \frac{-\theta_2}{0^2 - \theta_0} & \frac{-\theta_1}{0^2 - \theta_0} & 1 & \frac{-\theta_1}{0^2 - \theta_0} & \frac{-\theta_2}{0^2 - \theta_2} \\ \frac{-\theta_3}{2^2 - \theta_0} & \frac{-\theta_2}{2^2 - \theta_0} & \frac{-\theta_1}{2^2 - \theta_0} & 1 & \frac{-\theta_1}{2^2 - \theta_0} \\ \frac{-\theta_4}{4^2 - \theta_0} & \frac{-\theta_3}{4^2 - \theta_0} & \frac{-\theta_2}{4^2 - \theta_0} & \frac{-\theta_1}{4^2 - \theta_0} & 1 \end{vmatrix}$$

(40 *a*)

which is independent of β and is convergent. Hence,

$$\triangle(\beta) = 1 - \frac{\sin^2 \frac{\pi\beta}{2}}{\sin^2 \frac{\pi\sqrt{\theta_0}}{2}} - [L - \triangle(0)] \frac{2\sqrt{\theta_0}}{\pi} \tan \frac{\pi\sqrt{\theta_0}}{2}$$

$$\frac{\pi}{2\sqrt{\theta_0}} \cot \frac{\pi\sqrt{\theta_0}}{2}$$

$$= \triangle(0) - \frac{\sin^2 \frac{\pi\beta}{2}}{\sin^2 \frac{\pi\sqrt{\theta_0}}{2}} \,. \tag{41}$$

In order that the coefficients C_n which represent the amplitudes of spatial harmonics, the determinant $\triangle(\beta) = 0$. Hence,

$$\triangle(0) = \frac{\sin^2 \frac{\pi\beta}{2}}{\sin^2 \frac{\pi\sqrt{\theta_0}}{2}} \tag{42}$$

which leads to the following expression for β

$$\beta = \frac{2}{\pi} \arcsin\left[\sqrt{\triangle(0)}\, .\, \sin \frac{\pi\sqrt{\theta_0}}{2}\right] \tag{43}$$

which characterises the propagation of E_0 waves in the artificially simulated co-sinusoidally modulated dielectric medium, since θ_0 involves the modulation index δ which is defined by the ratio of the difference to the sum of the maximum and minimum values or $\epsilon(z)$ which is a $f(s, b, a)$. The nature of β depends on the nature in which the physical parameters s, b or a of the corrugated structure is modulated. The propagating waves correspond to real values of β which is called the stable solution of the characteristic equation, $\nabla(\beta) = 0$.. Whereas, complex values of β correspond to growing or damped waves which correspond to unstable solution of $\triangle(\beta) = 0$. This concept of stability or unstability of modes is in conformity with the solution of Hill's equation which possesses two types of solutions, one of them being called stable and the other unstable.

12. Solution of $\triangle$ (0)

The fifth order determinant in $\triangle$ (0) can be reduced to the following fourth order by adding $\frac{\theta_4}{4^2 - \theta_0} R_5$ to R_1, $\frac{\theta_3}{2^2 - \theta_0} R_5$ to R_2, $\frac{\theta_2}{0^2 - \theta_0}$

R_5 to R_3 and $\frac{\theta_1}{2^2-\theta_0}$ R_5 to R_4 where R_n represents the respective rows in (40 *a*)

$$\triangle(0)=\begin{vmatrix} \frac{-\theta_4^2}{(4^2-\theta_0)^2}+1 & \frac{-\theta_3\theta_4}{(4^2-\theta_0)^2} & -\frac{\theta_1}{(4^2-\theta_0)} & \frac{-\theta_4\theta_2}{(4^2-\theta_0)^2} & -\frac{\theta_2}{4^2-\theta_0} & \frac{-\theta_1\theta_4}{(4^2-\theta_0)^2} & -\frac{\theta_3}{(4^2-\theta_0)} \\ \frac{-\theta_3\theta_4}{\lambda_1} & -\frac{\theta_1}{2^2-\theta_0} & \frac{-\theta_3^2}{\lambda_1}+1 & \frac{-\theta_2\theta_3}{\lambda_1} & -\frac{\theta_1}{2^2-\theta_0} & \frac{-\theta_1\theta_2}{\lambda_1} & -\frac{\theta_2}{2^2-\theta_0} \\ \frac{-\theta_2\theta_4}{\lambda_2} & -\frac{\theta_2}{0^2-\theta_0} & \frac{-\theta_2\theta_3}{\lambda_2} & -\frac{\theta_1}{0^2-\theta_0} & \frac{-\theta_2^2}{\lambda_2}+1 & \frac{-\theta_2\theta_1}{\lambda_2} & -\frac{\theta_1}{0^2-\theta_0} \\ \frac{-\theta_1\theta_4}{\lambda_1} & -\frac{\theta_3}{2^2-\theta_0} & \frac{-\theta_1\theta_3}{\lambda_1} & -\frac{\theta_2}{2^2-\theta_0} & \frac{-\theta_1\theta_2}{\lambda_1} & -\frac{\theta_1}{2^2-\theta_0} & \frac{-\theta_1^2}{\lambda_1}+1 \end{vmatrix}$$

where (43 *a*)

$$\lambda_1=(2^2-\theta_0)(4^2-\theta_0).$$
$$\lambda_2=(0^2-\theta_0)(4^2-\theta_0).$$

Expanding the determinant (43) and neglecting triple product terms such as $\theta_1\theta_2^2\theta_3$, etc., involving higher power of θ than the first $\triangle$ (0) can be further simplified to

$$\triangle(0)\cong 1+\frac{2\theta_1^2}{\theta_0(2^2-\theta_0)}\quad\frac{\theta_2^2}{(2^2-\theta_0)^2}-\frac{\theta_1^2}{(2^2-\theta_0)(4^2-\theta_0)}$$
$$+\frac{2\theta_1^2\theta_2}{\theta_0(2^2-\theta_0)(4^2-\theta_0)}+\frac{\theta_2^2}{\theta_0(4^2-\theta_0)}$$
$$+\frac{2\theta_1\theta_2\theta_3}{\theta_0(2^2-\theta_0)(4^2-\theta_0)}-\frac{\theta_3^2}{(2^2-\theta_0)(4^2-\theta_0)} \tag{44}$$

since θ's involve δ which satisfies the inequality $0 \leqslant \delta \leqslant 1$. Hence β can be evaluated in terms of δ which is a function of the structure parameters.

The study of mode stability is under progress.

13. Conclusions

The following conclusions which may be of practical interest may be drawn.

(i) The surface wave characteristics of a dielectric coated conductor if the dielectric is simulated by an artificial dielectric can be controlled by an appropriate modulation of either s, b or a parameter of the line.

(ii) Control of radiation characteristics if the structure is used as an antenna is also possible by suitable modulation of s, b, or a parameter.

Appendix A.1

The derivative $f'(a)$ of a function $f(a)$ which has isolated singularities may be written in the form of a contour integral

$$f'(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-a)^2} dz.$$

If the contour C is a circle of radius R centered at a and letting $|f(z)| \leqslant M$ by hypothesis, then

$$|f'(a)| \leqslant \left(\frac{M}{2\pi R^2}\right)(2\pi R) = \frac{M}{R}$$

Let $R \to \infty$, *i.e.*, assume that $f(a)$ is bounded for all values of z, then $f'(a) = 0$, hence $f(a) =$ constant which is Liouville's theorem.

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