Integrable-square solutions of certain vector-matrix differential equations

PRABIR KUMAR SEN GUPTA
Department of Pure Mathematics, University of Calcutta, Calcutta 700 019.

Received on July 15, 1978

Abstract

Here we establish three theorems regarding the number of square-integrable solutions of the differential systems of the type

$$M[\varphi] = \begin{pmatrix} -\dfrac{d^2}{dx^2} + p(x) & q(x) \\ q(x) & -\dfrac{d^2}{dx^2} + r(x) \end{pmatrix}$$

on $[0, \infty)$, φ is the column vector with elements $u(x)$ and $v(x)$, the coefficient $p(x)$, $q(x)$ and $r(x)$ are real valued on $[0, \infty)$ and

(i) $p(x)$, $q(x)$ and $r(x)$ belong to $L^2[0, \infty)$

(ii) $p(x)$, $q(x)$ and $r(x)$ are replaced by ax^{α}, bx^{β} and cx^{γ} respectively; $a, b, c, \alpha, \beta, \gamma$ are real constants with $a, c > 0$ and $\alpha, \gamma > 4\beta + 2$.

In the last theorem we prove that the differential system

$$L[\varphi] \equiv [-P(x)\varphi'(x)]' + Q(x)\varphi(x) = \lambda\varphi(x)$$

$$[\varphi(x, \lambda)\ \ \varphi_r(0/x, \lambda)] = 0 = [\varphi(x, \lambda)\ \ \varphi_j(b/x, \lambda)]\ (r = 1, 2; j = 3, 4)$$

is in the limit-2 case, by imposing certain restrictions on the elements of the 2×2 matrices $P(x)$ and $Q(x)$. $\varphi_r(0/x, \lambda)$, $r = 1, 2$; $\varphi_j(b/x, \lambda)$, $j = 3, 4$ are the boundary condition vectors at $x = 0$ and at $x = b$ respectively.

Key words: Integrable-square solutions, Deficiency index, Hilbert space, Self-adjoint, Boundary condition vector, Green's formula, Bilinear concomitant, Green's matrix, Eigenvalue, spectrum.

1. We discuss the differential expression determined by

$$M[\varphi] = \begin{bmatrix} u''(x) + p(x)\,u(x) + q(x)\,v(x) \\ v''(x) + r(x)\,v(x) + q(x)\,u(x) \end{bmatrix} \tag{1}$$

on $[0, \infty)$, where φ is the column vector with elements $u(x)$ and $v(x)$. The coefficients $p(x)$, $q(x)$ and $r(x)$ are real valued on $[0, \infty)$ and satisfy the basic conditions, which will be specified later in § 2. We denote by ν the number of linearly independent solutions belonging to $\mathcal{L}^2[0, \infty)$, (the Hilbert space of vector functions with integrable square) of the differential system

$$M[\varphi] = \lambda\varphi \text{ on } [0, \infty), \text{ where } im\,\lambda \neq 0. \tag{2}$$

It is known that ν does not depend on λ and that $2 \leqslant \nu \leqslant 4$ [See Chakravarty[2] and Chakravarty[3], Th. 2.1]. $M[.]$ is said to be in the limit -2, 3 or 4 according as (2) has 2, 3 or 4 linearly independent solutions in $\mathcal{L}^2[0, \infty)$.

2. Let $\mathcal{D}$ be the set of complex-valued vector functions $F(x) = \{f_1(x), f_2(x)\}^T$ ($T =$ Transpose) such that

(i) $F(x)$ is $\mathcal{L}^2[0, \infty)$, *i.e.*, $\int_0^\infty |F^T(x)\bar{F}(x)|\, dx$,

($\bar{F}(x)$ is the complex conjugate of $F(x)$.)

(ii) $f_1'(x), f_2'(x)$ are absolutely continuous in $[0, \infty)$

(iii) $f_1''(x) + p(x) f_1(x) + q(x) f_2(x)$ and $f_2''(x) + q(x) f_1(x) + r(x) f_2(x)$ belong to $L^2[0, \infty)$.

For any two vectors $F(x) = \{f_1(x), f_2(x)\}^T$ and $G(x) = \{g_1(x), g_2(x)\}^T$, let $[F G](.)$ denote the bilinear form on $\mathcal{D}$ defined by

$$[F G](x) = f_1'(x) g_1(x) - g_1'(x) f_1(x) + f_2'(x) g_2(x) - g_2'(x) f_2(x).$$

It follows by the method indicated in Sen Gupta[9] theorem 5.2. that $M[\cdot]$ is limit-2 at ∞ if and only if $\lim\limits_{x\to\infty} [F G](x) = 0$ for all F, G belonging to $\mathcal{D}$. Thus to establish that $M[\cdot]$ is not limit-2 at ∞ it is sufficient to produce one pair F, G of elements of $\mathcal{D}$ such that $\lim\limits_{x\to\infty} [F G] \neq 0$.

Theorem I: Let $a, b, c, \alpha, \beta, \gamma$ be real constants with $a > 0, c > 0$. Then, if $\alpha > 4\beta + 2$, $\gamma > 4\beta + 2$, $\beta \geqslant 0$, the differential system

$$\begin{pmatrix} \dfrac{d^2}{dx^2} + a x^\alpha & b x^\beta \\ b x^\beta & \dfrac{d^2}{dx^2} + c x^\gamma \end{pmatrix} \tag{3}$$

is not limit-2.

PROOF : [The method of proof follows Eastham[5, 6]]
We take $F = G$ and determine F by

$$f_j(x) = P_j(x)\, e^{i\int_0^x Q_j(t)\,dt}, \quad j = 1, 2 \tag{4}$$

where $P_j(x)$ and $Q_j(x)$ are real-valued and will be defined later. A calculation shows that

$$f_1'(x) = \{P_j'(x) + iP_j(x)\,Q_j(x)\}\, e^{i\int_0^x Q_j(x)\,dx}$$

$$f_j''(x) = \{P_j''(x) + 2iP_j'(x)\,Q_j(x) + iP_j(x)\,Q_j'(x) - P_j(x)\,Q_j^2(x)\}\, e^{i\int_0^x Q_j(x)\,dx}$$

From these results we obtain $[F F](x) = 2i\,(P_1{}^2 Q_1 + P_2{}^2 Q_2)$ and with details of the calculations omitted,

$$f_1''(x) + p(x) f_1(x) + q(x) f_2(x)$$

$$= \{(-Q_1^2 + p)\,P_1 + i\,(P_1 Q_1' + 2P_1' Q_1)\,P_1''\}\, e^{i\int_0^x Q_1(x)\,dx} + qP_2\, e^{i\int_0^x Q_2(x)\,dx}$$

$$f_2''(x) + q(x) f_1(x) + r(x) f_2(x)$$

$$= \{(-Q_2^2 + r)\,P_2 + i\,(P_2 Q_2' + 2P_2' Q_2)\,P_2''\}\, e^{i\int_0^x Q_2(x)\,dx} + qP_1\, e^{i\int_0^x Q_1(x)\,dx}$$

We choose $P_j(x)$ and $Q_j(x), j = 1, 2$ such that

$$-Q_1^2 + p = 0 \quad \text{and} \quad -Q_2^2 + r = 0 \tag{5}$$

$$P_1^2 Q_1 = A \quad \text{and} \quad P_2^2 Q_2 = B. \tag{6}$$

By (5) and the results obtained on differentiating (4) we find that

$$f_1''(x) + p(x) f_1(x) + q(x) f_2(x) = P_1''\, e^{i\int_0^x Q_1(x)\,dx} + q\,P_2\, e^{i\int_0^x Q_2(x)\,dx}$$

$$f_2''(x) + q(x) f_1(x) + r(x) f_2(x) = P_2''\, e^{i\int_0^x Q_2(x)\,dx} + qP_1\, e^{i\int_0^x Q_1(x)\,dx}$$

In order that F should be in $\mathcal{D}$, since $P_j(x)$, $Q_j(x)$, $j = 1, 2$ are real-valued, we should have

(1) $P_1(x)$ and $P_2(x)$ belong to $L^2[0, \infty)$

(2) qP_1 and qP_2 belong to $L^2[0, \infty)$

(3) $P_1''(x)$ and $P_2''(x)$ belong to $L^2[0, \infty)$

Solving (5) and (6), we find

$$P_1(x) = \{A^2/p(x)\}^{1/4}, \qquad P_2(x) = \{B^2/r(x)\}^{1/4},$$

$$Q_1(x) = \{p(x)\}^{1/2} \quad \text{and} \quad Q_2(x) = \{r(x)\}^{1/2}.$$

We now return to our original differential expression $M[\cdot]$ in which $p(x) = ax^{\alpha}$, $q(x) = bx^{\beta}$ and $r(x) = cx^{\gamma}$ from which we find

$$P_1(x) = O(x^{-\alpha/4}),\ P_2(x) = O(x^{-\gamma/4}),\ q(x)P_1(x) = O(x^{-\alpha/4+\beta})$$

$$q(x)P_2(x) = O(x^{-\gamma/4+\beta}).\ P_1''(x) = O(x^{-\alpha/4-2}),\ P_2''(x) = O(x^{-\gamma/4-2})$$

To satisfy the conditions we should have

$$\alpha > 4\beta + 2 \quad \text{and} \quad \gamma > 4\beta + 2$$

These establish the truth of the theorem.

In this case we note that (1) is not limit-2 if

(i) $p(x) \leqslant ax^{\alpha}$, $q(x) \leqslant bx^{\beta}$ and $r(x) = cx^{\gamma}$ in $[X, \infty)$ for some $X > 0$.

(ii) $p''(x) = O(x^{\alpha-2})$, $q''(x) = O(x^{\beta-2})$ and $r''(x) = O(x^{\gamma-2})$ as $x \to \infty$

where $a > 0$, $c > 0$, b, α, β, γ are real constants with $\alpha > 4\beta + 2$ and $\gamma > 4\beta + 2$, $\beta \geqslant 0$.

Theorem II : If $p(x)$, $q(x)$ and $r(x)$ belong to $L^2[0, \infty)$, then the differential system

$$\left.\begin{aligned} -u''(x) + p(x)u(x) + q(x)v(x) = 0 \\ -v''(x) + q(x)u(x) + r(x)v(x) = 0 \end{aligned}\right\} \tag{7}$$

is not limit-4.

PROOF : Let $y = \{y_1(x), y_2(x)\}^T$ be a solution of (7) belonging to $\mathcal{L}^2[0, \infty)$, then because of the conditions $p(x)$, $q(x)$ and $r(x)$ belong to $L^2[0, \infty)$ the functions

$$y_1''(x) = p(x)y_1(x) + q(x)y_2(x)$$

and

$$y_2''(x) = q(x)y_1(x) + r(x)y_2(x)$$

are summable, so that the limit

$$\lim_{x\to\infty} y_j'(x) = y_j'(x_0) + \int_{x_0}^{x} y_j''(x)\,dx, \ (j = 1, 2)$$

exists. Hence the functions $y_j'(x)$, $(j = 1, 2)$ are bounded as $x \to \infty$.

Let $U_j(x) = \{u_j(x), v_j(x)\}^T$, $(j = 1, 2, 3, 4)$ be the four linearly independent real solutions of the system (7), then the Wronskian of these solutions

$$[U_1U_2][U_3U_4] - [U_1U_3][U_2U_4] + [U_1U_4][U_2U_3] = \text{constant} \neq 0,$$

where $[F\,G]$ is the bilinear form $F^TG' - G^TF'$ for any two vectors F, G. If $U_j(x)$, $(j = 1, 2, 3, 4)$ all belong to $\mathcal{L}^2[0, \infty)$ then $u_j'(x)$, $v_j'(x)$ $(j = 1, 2, 3, 4)$ are bounded and hence

$$[U_j\, U_k] = u_j u_k' + v_j v_k' - u_j' u_k - v_j' v_k \text{ are } L^2[0, \infty),$$

so that the function

$$[U_1U_2][U_3U_4] - [U_1U_3][U_2U_4] + [U_1U_4][U_2U_3] = \text{constant} \neq 0,$$

also belongs to $L[0, \infty)$, which is impossible. Thus all the four solutions are not $\mathcal{L}^2[0, \infty)$.

3. We now discuss the $\mathcal{L}^2$-classification of the solutions of the differential system

$$L[\varphi] \equiv [-P(x)\,\varphi'(x)]' + Q(x)\,\varphi(x) = \lambda\varphi(x) \tag{8}$$

where $\varphi(x)$ is the column vector with elements $u(x)$, $v(x)$ and

$$P(x) = \begin{pmatrix} p_1(x) & p_2(x) \\ p_2(x) & p_3(x) \end{pmatrix}, \quad Q(x) = \begin{pmatrix} q_1(x) & q_2(x) \\ q_2(x) & q_3(x) \end{pmatrix};$$

elements of these matrices are real-valued, continuous and differentiable over $[0, b]$, $b > 0$.

Utilising the Green's formula, the bilinear concomitant of any two vectors F, G satisfying the system (8) turns out to be

$$[F\,G] = F^T PG' - G^T PF'.$$

The boundary conditions are given by

$$[\varphi(x, \lambda)\,\varphi_r(0/x, \lambda)] = 0 = [\varphi(x, \lambda)\,\varphi_j(b/x, \lambda)] \; (r = 1, 2;\; j = 3, 4) \tag{9}$$

where $\varphi(x, \lambda)$ is a solution for the system (8) and

$$\varphi_r(0/x, \lambda) = \{x_r(0/x, \lambda), y_r(0/x, \lambda)\}^T$$

and

$$\varphi_j(b/x, \lambda) = \{x_j(b/x, \lambda), y_j(b/x, \lambda)\}^T$$

are the boundary condition vectors at $x = 0$ and at $x = b > 0$, The Green's matrix $G\ (b, x, y, \lambda) \equiv (G_{kj}\ (b, x, y, \lambda))$

$$= \begin{pmatrix} G_{11}\ (b, x, y, \lambda) & G_{21}\ (b, x, y, \lambda) \\ G_{12}\ (b, x, y, \lambda) & G_{22}\ (b, x, y, \lambda) \end{pmatrix}$$

Details of construction of Green's matrix can be found in Bhagat[1] or in Chakravarty[2]
$G\ (b, x, y, \lambda) = G^T\ (b, x, y, \lambda)$

$$= \begin{pmatrix} \psi_{11}\ (x, \lambda) & \psi_{21}\ (x, \lambda) \\ \psi_{12}\ (x, \lambda) & \psi_{22}\ (x, \lambda) \end{pmatrix} \begin{pmatrix} x_1\ (y, \lambda) & y_1\ (y, \lambda) \\ x_2\ (y, \lambda) & y_2\ (y, \lambda) \end{pmatrix}, \qquad y \varepsilon\ [a, x)$$

$$= \begin{pmatrix} x_1\ (x, \lambda) & x_2\ (x, \lambda) \\ y_1\ (x, \lambda) & y_2\ (x, \lambda) \end{pmatrix} \begin{pmatrix} \psi_{11}\ (y, \lambda) & \psi_{12}\ (y, \lambda) \\ \psi_{21}\ (y, \lambda) & \psi_{22}\ (y, \lambda) \end{pmatrix}, \qquad y \varepsilon\ (x, b]$$

where,

$$\psi_1 \equiv \psi_1\ (x, \lambda) = \begin{pmatrix} \psi_{11}\ (x, \lambda) \\ \psi_{12}\ (x, \lambda) \end{pmatrix} = \frac{[\varphi_2\varphi_4]\ \varphi_3\ (b/x, \lambda) - [\varphi_2\varphi_3]\ \varphi_4\ (b/x, \lambda)}{W\ (\lambda)}$$

$$\psi_2 \equiv \psi_2\ (x, \lambda) = \begin{pmatrix} \psi_{21}\ (x, \lambda) \\ \psi_{22}\ (x, \lambda) \end{pmatrix} = \frac{[\varphi_1\varphi_3]\ \varphi_4\ (b/x, \lambda) - [\varphi_1\varphi_4]\ \varphi_3\ (b/x, \lambda)}{W\ (\lambda)}$$

and

$$W\ (\lambda) = [\varphi_1\varphi_2]\ [\varphi_3\varphi_4] - [\varphi_1\varphi_4]\ [\varphi_2\varphi_3] + [\varphi_1\varphi_3]\ [\varphi_2\varphi_4].$$

Following Chakravarty[2], we can extend the Green's matrix to the singular case, *i.e.*, when $b \to \infty$.

Definition: The differential expression (8) with (9) is said to be in the limit-2 case if it has only two linearly independent solutions in $\mathcal{L}^2\ [0, \infty)$ for all non-real values of the complex parameter λ.

Let $S\ (\lambda)$ denote the number of linearly independent $\mathcal{L}^2\ [0, \infty)$ solutions of the system (8)–(9).

Result 1: $S\ (\lambda)$ does not depend on the complex parameter λ.

Let $\varphi\ (x, \lambda_1)$ be a non-real solution of (8)–(9) belonging to $\mathcal{L}^2\ [0, \infty)$ for a given complex $\lambda = \lambda_1$, say.

Then

$$L\ [\varphi\ (x, \lambda_1)] = \lambda_1 \varphi\ (x, \lambda_1).$$

We consider

$$\Phi\ (x, \lambda) = \int_0^\infty G^T\ (x, y, \lambda)\ \varphi\ (y, \lambda_0)\ dy,$$

where $\lambda = \sigma + i\tau$, $\tau > 0$, obviously $\Phi(x, \lambda)$ is not null. Now following Chakravarty[3] the result follows. Also $2 \leqslant S(\lambda) \leqslant 4$.

From now on we designate the system (8) as the Q-system and the system obtained from (8) by replacing $Q(x)$ by

$$Q_0(x) = \begin{pmatrix} q_{10}(x) & q_{20}(x) \\ q_{20}(x) & q_{30}(x) \end{pmatrix}$$

as the Q_0-system.

Let $S_Q(\lambda)$ represent the number of square-integrable linearly independent solutions of the Q-system and $S_{Q_0}(\lambda)$ the same for the Q_0-system.

Result 2: For the bounded elements of the matrix $(Q - Q_0)$,

$$S_Q(\lambda) = S_{Q_0}(\lambda)$$

where λ be a given complex parameter.

Let $\psi(x) = \{\psi_1(x), \psi_2(x)\}^T$ be a non-null solution of the Q_0-system belonging to $\mathcal{L}^2[0, \infty)$, then

$$\begin{aligned} L[\psi] - \lambda\psi &= [-P(\psi)']' + Q_0\psi = 0 \\ &= [-P(\psi)']' + Q\psi + (Q_0 - Q)\psi = 0 \end{aligned}$$

Now,

$$\Phi(x, \lambda) = \int_0^\infty G^T(x, y, \lambda)\,(Q(y) - Q_0(y))\,\psi(y)\,dy$$

satisfies

$$L\Phi - \lambda\Phi = -(Q - Q_0)\psi.$$

Now following Chakravarty[3], Th. 2.2 the result follows.

Therefore, we conclude that the addition of a matrix function $Q(x)$ to

$$[-PU']' = \lambda U \tag{10}$$

does not alter $S(\lambda)$ for the system (8), *i.e.*, the $S(\lambda)$ are the same for the systems (8) and (10).

We define L_0 as the minimal closed symmetric linear differential operator associated with (9) in the complex Hilbert space $\mathcal{L}^2[0, \infty)$, the domain of definition $\mathcal{D}_0$ of L_0 is the same as discussed in Sen Gupta[9]. Then

$$L_0 U(x) = [-P(x)\,U'(x)]'$$

and

$$U^k(x) = [0], (k = 0, 1) \tag{11}$$

the superscript k denotes the k-th derivative.

For simplicity we write S for $S(\lambda)$. Let the deficiency indices of L_0 be (S, S) and in the case when λ is replaced by a real number λ_0, we denote it by (S_0, S_0). unlike S, S_0 may not be restricted to 2, 3 or 4 and it may vary with λ_0.

It can be proved following Dunford and Schwartz[4], [pt. II, pp. 1398, lemma 9] that $S_0 \leqslant S$, to prove $S_0 = S$ we require the opposite inequality

$$S_0 \geqslant S. \tag{12}$$

Let L_s be the self-adjoint extension of the operator L_0. The condition (11) holds if

(*a*) λ_0 is not an eigenvalue of L_0

and

(*b*) λ_0 is not in the continuous part of the spectrum of the self-adjoint extension L_s of L_0. [See Naimark[8], pp. 42–43, corollary 3; also Dunford and Schwartz[4], pp. 1398, Corollary 8].

Lemma 1: If

(i) $p_1(x) > 0, p_1(x)\, p_3(x) - p_2^2(x) > 0$

and

(ii) $\liminf p_1^{-1}(x)\, x^{-2}\, p_{1,2}(x) > K > 0$

$(p_{1,2}(x) = \min\{p_1(x), p_2(x)\})$.

Then the continuous part of the spectrum of the operator L_s spreads over the interval $[\frac{1}{4}K, \infty)$.

PROOF: Let ε be any arbitrary real number in $(0, K)$ and let X be such that

$$p_{1,2}^2(x) > (K - \varepsilon)\, p_1(x)\, x^2 \text{ in } [X, \infty).$$

Now let $f(x) = (f_1(x), f_2(x))^T$ be any vector function in D_0 whose support is compact and lies in $[x, \infty)$.

[The least closed domain outside which a given finite vector function identically vanishes is called its support.]

Then we have

$$
\begin{aligned}
(L_s f, f)\,dx &= \int_0^\infty \{-(p_1 f_1' + p_2 f_2')f_1 - (p_2 f_1' + p_3 f_2')f_2\}\,dx \\
&= \{(-p_1 f_1' - p_2 f_2')f_1 + (-p_2 f_1' - p_3 f_2')\,f_2\}_0^\infty + \\
&\quad + \int_0^\infty \{(p_1 f_1' + p_2 f_2')f_1' + (p_2 f_1' + p_3 f_2')f_2'\}\,dx \\
&\geqslant \int_0^\infty \{p_1(x)\,|f_1'(x)|^2 + 2p_2(x)\,|f_1'(x)|\,|f_2'(x)| + \\
&\quad + p_3(x)\,|f_2'(x)|^2\}\,dx \\
&\geqslant \int_0^\infty \frac{1}{p_1(x)}\,\{p_1(x)\,|f_1'(x)| + p_2(x)\,|f_2'(x)|\}^2\,dx \\
&\geqslant (K-\varepsilon)\int_0^\infty x^2\,(\,|f_1'(x)| + |f_2'(x)|\,)^2\,dx \\
&\geqslant \frac{(K-\varepsilon)}{4}\int_0^\infty (\,|f_1(x)| + |f_2(x)|\,)^2\,dx. \qquad (13)
\end{aligned}
$$

Applying the inequality [Glazman[7], pp. 83]

$$
\int_0^x x y^2\,dx \leqslant \frac{4}{(a+1)^2}\int_0^\infty x^{a+2}\,y'^2\,dx
$$

for any real function $y(x) \in C^1[0,\infty)$, $a > -1$.

[See also Eastham[6], lemma of § 2].

The lemma now follows from (13) and Glazman[7] [pp. 34, Th. 28].

We are now in a position to prove the following theorem.

Theorem III: If

(i) $p_1(x), p_3(x) > -p_2(x) > 0$ in $[0,\infty)$

(ii) $q_j(x)$, $j = 1, 2, 3$ are essentially bounded in $[0,\infty)$ and

(iii) $\liminf_{x\to\infty} x^{-2}\,p_1^{-1}(x)\,p_{1,2}^2(x) > 0$

then (8)–(9) is in the limit–2 case.

PROOF: Since $q_j(x)$, $j = 1, 2, 3$ are bounded in $[0, \infty)$ we only consider the equation (10) to determine the number of $\mathcal{L}^2$-solutions of the equation (8). The boundary conditions are the same in both the cases.

From (11) we see that L_0 has no eigenvalues; further if we take $\lambda_0 = 0$, then the conditions (*a*) and (*b*) of § 3 are satisfied. Therefore, if S_0 refers to the equation

$$[P(x)\,U'(x)]' = 0 \tag{14}$$

then (12) holds. We now prove that the equation (14) has two solutions not in $\mathcal{L}^2$ $[0, \infty)$, *i.e.*, $S_0 \leqslant 2$, then (12) and the inequality $S \geqslant 2$ would imply $S = 2$. Let $\varphi(x) = (\varphi_1(x), \varphi_2(x))^T$ and $\psi(x) = (\psi_1(x), \psi_2(x))^T$ be the two solutions of (14) of which no non-trivial linear combination is $\mathcal{L}^2$ $[0, \infty)$.

For $\varphi(x)$, we choose $\varphi(x) = (1, 1)^T$. Next let X_0 be such that

$$\frac{p_3(x)\,p_1(X_0) - p_2(x)\,p_2(X_0)}{p_1(x)\,p_3(x) - p_2^2(x)},\quad \frac{p_1(x)\,p_2(X_0) - p_2(x)\,p_1(X_0)}{p_1(x)\,p_3(x) - p_2^2(x)} > 0 \tag{15}$$

in $[X_0, \infty)$. These hold by condition (i).

We define $\psi(x)$ to be the solution of (10) which satisfies the initial conditions

$$(\psi_1(X_0), \psi_2(X_0), \psi_1'(X_0), \psi_2'(X_0)) = (0, 0, 1, 0) \tag{16}$$

at $x = X_0$.

On integrating (14) over the interval (X_0, X_1) with $U(x) = \psi(x)$ and using (16) we obtain

$$\psi_1'(x) = \frac{p_3(x)\,p_1(X_0) - p_2(x)\,p_2(X_0)}{p_1(x)\,p_3(x) - p_2^2(x)}$$

$$\psi_2'(x) = \frac{p_1(x)\,p_2(X_0) - p_2(x)\,p_1(X_0)}{p_1(X)\,p_3(x) - p_2^2(x)}$$

i.e., $\psi_1'(x), \psi_2'(x) > 0$ in (X_0, X_1) [by (15)].

Thus $\psi(x)$ is increasing in (X_0, X_1) and it follows that $\psi(x)$ is non-null at X_1 in (X_0, ∞), therefore, $\psi(x) \to \infty$ as $x \to \infty$.

These solutions are linearly independent and neither is square integrable. Further no non-trivial linear combination of $\varphi(x)$ and $\psi(x)$ can be square-integrable. Hence the theorem follows.

4. Acknowledgements

The author is grateful to Dr. N. K. Chakravarty, Professor and Head of the Dept. and to Dr. (Mrs.) Jyoti Das (*neę* Choudhuri), Reader, both of the Department of

Pure Mathematics, Calcutta University, for their kind help throughout the preparation of this paper. Thanks are also due to the referee for pointing out certain omissions.

The financial support was provided, in part, by the Education Directorate, Government of West Bengal.

References

1. BHAGAT, BIKAN — Some problems on a pair of singular second-order differentia equations. *Proc. Nat. Inst. Sci., India*, 1969, 35, 232–244.

2. CHAKRAVARTY, N. K. — Some problems in eigenfunction expansions III. *Quart. J. Math. (Oxford)*, 1968, **19** (2), 397–415.

3. CHAKRAVARTY, N. K. — Some problems in eigenfunction expansions IV. *Ind. J. Pure Appl. Math.*, 1970, 1, 347–353.

4 DUNFORD, N AND SCHWARTZ, S. T — Linear operators, Part I and II, Interscience. New York, 1963.

5. EASTHAM, M. S. P. — The limit-2 case of fourth-order differential equations. *Quart. J. Math. (Oxford)*, 1971, **22** (2), 131–134.

6, EASTHAM, M. S. P. — On the L^2-classification of fourth-order differential equations. *J. London Math. Soc.*, 1971. **3** (2), 297–300.

7. GLAZMAN, I. M. — Direct methods for the qualitative spectral analysis of singular differential operators, *Eng. Trans. Israel Programme for Sci, Trans. Jerusalem*, 1965.

8. NAIMARK, M. A. — Linear differential operators, Part II, Ungar, New York, 1968.

9. SEN GUPTA, PRABIR KUMAR — On a self-adjoint extension of a type of a matrix differential operator. *JIISc.*, 1978, **60** (*B*), 225–234.