Short Communication

Steady flow in a porous region between two slowly rotating spheres

V. NARASIMHACHARYULU AND N. CH. PATTABHI RAMACHARYULU
Department of Mathematics, Regional Engineering College, Warangal 506 004, (A.P.), India.

Received on February 7, 1978; Revised on July 19, 1978

Abstract

In this paper, the flow of a viscous liquid through a porous region bounded by two rotating concentric spheres is examined by employing the generalized Darcy's Law proposed by Brinkman.

Key words: Porous, generalized Darcy's Law, rotating spheres.

1. Introduction

The classical Darcy's Law was improved recently by several investigators. Bevers *et al.*[1] gave a slip-condition at the interface of the clear region with porous region, in addition to the Darcy's Law. Brinkman[2] proposed a generalized Darcy's Law to study the flow through highly porous media which takes into account the shearing viscous stresses generated in the medium.

In this paper, the flow through porous region contained between two slowly rotating concentric spheres is completely investigated employing Brinkman's generalized Darcy's Law. Flow in a porous region around a rotating sphere and that in a rotating sphere are obtained as limiting cases.

2. Formulation and solution of the problem

We choose spherical co-ordinate system r, θ, ϕ such that the $\theta = 0$ axis lies along the axis of rotation of the two rotating spheres of radius a, b with the angular velocities Ω_a and Ω_b respectively and r stands for the distance from the origin. All the physical quantities are independent of ϕ due to axial symmetry.

The generalized Darcy's Law for the steady flow of a viscous liquid in a porous medium as proposed by Brinkman[2] is given by

$$0 = -\nabla p + \mu \nabla^2 \vec{V} - \left(\frac{\mu}{k}\right) \vec{V}. \tag{1}$$

Here p stands for the pressure field and μ, k are the coefficients of viscosity and permeability respectively. $\vec{V}$ is the velocity of the fluid and satisfies the equation of continuity

$$\operatorname{div} \vec{V} = 0. \tag{2}$$

It can be observed in the case of slow rotary flow between the two spheres that the pressure can be taken to be uniform throughout the creeping flow field. The velocity components in such a flow may be chosen to be $\{0, 0, w, (r, \theta)\}$ which satisfy the equation of continuity and the equation of motion (1) reduces to

$$\nabla^2 w - \frac{w}{r^2 \sin^2 \theta} - \frac{w}{k} = 0 \tag{3}$$

together with the boundary conditions

$$w(a, \theta) = \Omega_a \,.\, a \sin \theta \tag{4}$$

$$w(b, \theta) = \Omega_b \,.\, b \sin \theta. \tag{5}$$

Then from the equations (3) to (5), we get

$$w(r, \theta) = [a^{3/2}\, \Omega_a \,.\, T(r, b) - b^{3/2}\, \Omega_b T(r, \theta)] \sin \theta / [\sqrt{r}\, T(a, b)] \tag{6}$$

with

$$T(x, y) = I_{3/2}(x/k^{1/2})\, K_{3/2}(y/k^{1/2}) - I_{3/2}(y/k^{1/2})\, K_{3/2}(x/k^{1/2}) \tag{7}$$

where $I_{3/2}$, $K_{3/2}$ are the modified Bessel functions.

2.1. *Skin friction and moment on the spheres*

Skin friction on the sphere $r = a$ is

$$S_a' = [(T_{r\phi})]_{r=a} = \mu \left(\frac{\partial w}{\partial r} - \frac{w}{r} \right)_{r=a} - \frac{3}{2}\, \Omega_a \mu \sin \theta$$

$$\begin{aligned} = -\mu \frac{3}{2} [\Omega_a a \langle \{a^{-2} b^{-1} - (ka)^{-1} + (2/3)(kb)^{-1}\} \sinh (a - b)/\sqrt{k} \\ + \{a^{-2} k^{-1/2} - k^{-1/2} (ab)^{-1} + (2/3) k^{-3/2}\} \cosh (a - b)/k^{1/2}\rangle \\ - 2/3\, a^{-1} b^2 k^{-3/2}\, \Omega_b] \, [\{k^{-1} - (ba)^{-1}\} \sinh (b - a)/\sqrt{k} \\ + \{k^{-1/2} (b - a)/ab\} \cosh (b - a)/k^{1/2}]^{-1} \sin \theta. \end{aligned} \tag{8}$$

Moment on the sphere $r = a$ is

$$M_a = 4\pi\mu a^3\, \Omega_a - [\langle\{a^{-2} b^{-1} - (ka)^{-1} + 2/3\, (kb)^{-1}\} \sinh (a - b)/\sqrt{k}$$

$$+ (k^{-1/2} a^{-2} - k^{-1/2} (a - b)^{-1} + (2/3) k^{-3/2}\} \cosh (a - b)/\sqrt{k})\, a\Omega_a$$
$$- 2/3\, a^{-1} b^2 k^{-3/2} \Omega_b]\, 4\pi\mu a^3$$
$$[\{k^{-1} - (b)^{-1}\} \sinh b - a/\sqrt{k} + \{k^{-1/2} (b - a)/ab\} \cosh (b - a)/\sqrt{k}]^{-1}. \tag{9}$$

These parameters on the other sphere $r = b$ can be computed in a similar way.

2.2. *Flow in a rotating sphere*

The flow in a porous sphere with impermeable boundary can be realized from the equations (6) and (7), by taking $a \to 0$, $\Omega_a \to 0$

$$w = b^{3/2} \Omega_b r^{-1/2} \sin \theta \cdot \left.\frac{T(r, a)}{T(b, a)}\right|_{a=0} \tag{10}$$

$$= b^2 \Omega_b r^{-1} \{\cosh r/\sqrt{k} - (\sqrt{k}/r) \sinh r/\sqrt{k}\}$$
$$\times \{\cosh b/\sqrt{k} - (\sqrt{k}/b) \sinh b/\sqrt{k}\}^{-1} \sin \theta. \tag{11}$$

It can be observed that the flow in this case is not rigid unlike in the case of Navier-Stokes flow in a rotating shell[3]. Skin Friction in this case is given by

$$T_{r\phi}\big|_{r=b},\ a \to 0 = -3\Omega_b\mu \sin \theta - \frac{b\Omega_b\mu \sin \theta}{k^{3/2}(1 - k^{3/2}/b)} \tag{12}$$

Moment in this case is obtained as

$$M_b\big|_{a\to 0} = -8\pi b^3 \Omega_b\mu - 8/3\pi\, \frac{\Omega_b\mu b^4}{k^{1/2}(1 - k^{1/2}/b)} \tag{13}$$

It can be deduced from these results as $k \to 0$ that

$$T_{r\phi} = -3\Omega_b\mu \sin \theta \quad \text{and} \quad M = -8b^2 \Omega_b\mu$$

which are the same as those for the case of classical viscous flow through a clear medium.

2.3. *Flow in a porous medium around a rotating sphere*

In this case, as $b \to \infty$, $\Omega_b = 0$ and from the equation (7)

$$T_{(b,r)} = \tfrac{1}{2} (k/br)^{1/2} [1 + k/r - k/br] \exp \{(b - r)/\sqrt{k}\}. \tag{14}$$

Then w from the equation (6) is

$$w = a^2 \Omega_a (1 + \sqrt{k}/r) \exp \{(a - r)/\sqrt{k}\} \sin \theta\, \{r (1 + \sqrt{k}/a)\}^{-1}. \tag{15}$$

The skin friction and moment in this case given by

$$S_a|_{b\to\infty} = -3\Omega_a\mu\sin\theta - \frac{\mu\Omega_a a\sin\theta}{\sqrt{k}\,(1+\sqrt{k/a})} \tag{16}$$

$$M_a|_{b\to\infty} = -8\pi\Omega_a\mu a^3 - \mu\, 8/3\,\frac{\Omega_a\pi a^4}{\sqrt{k}\,(1+\sqrt{k/a})} \tag{17}$$

As $k\to\infty$, it follows from the above equations that the corresponding results in the usual flow of a viscous liquid around a rotating sphere can be realised[3].

2.4. ***When the permeability coefficient of the medium k is very small, i.e., 1/k is very large***

From equation (7), we have

$$T(r,b) = -1/2\,(k/rb)^{1/2}\,[1+\sqrt{k}/\sqrt{(rb)} - k/br]\exp\{(b-a)/\sqrt{k}\} \tag{18}$$

and from the equation (6) we have

$$\begin{aligned} w = {} & [a^2\,\Omega_a\,\{1+\sqrt{k}\,(r^{-1}-b^{-1}) - k/br\}\exp\{-(b-a)/\sqrt{k}\} \\ & + b^2\,\Omega_b\,\{1+\sqrt{k}\,(a^{-1}-r^{-1}) - k/ar\}\exp\{-(b-r)/\sqrt{k}\}] \\ & \times r^{-1}\,[1+\sqrt{k}\,(a^{-1}-b^{-1}) - k/ab]^{-1}\sin\theta. \end{aligned} \tag{19}$$

If $\delta_1 = (b-a)/\sqrt{k}$ and $\delta_2 = (b-r)/\sqrt{k}$ become very large, we observe that the velocity component w vanishes. This suggests the existence of a thin layer around the boundaries far away from which the velocity vanishes.

When the permeability coefficient is large, *i.e.*, $1/k$ is very small, we have from equation (7)

$$T(r,a) = (ra)^{-1/2}\left[(r-a) + \frac{(r-a)^3}{3}\left(\frac{1}{k}+\frac{1}{ra}\right)\right] \tag{20}$$

$$\begin{aligned} w = r^{-1} & \left[a^2\,\Omega_a\left\{(r-b) + \frac{(r-b)^3}{3}\left(\frac{1}{2k}+\frac{1}{rb}\right)\right\}\right. \\ & \left. - b^2\,\Omega_b\left\{(r-a) + \frac{(r-a)^3}{3}\left(\frac{1}{2k}+\frac{1}{ra}\right)\right\}\right] \\ & \times\left[(a-b) + \frac{(a-b)^3}{3}\left(\frac{1}{2k}+\frac{1}{ab}\right)\right]^{-1}\sin\theta. \end{aligned} \tag{21}$$

As $k\to\infty$, $1/k\to 0$, we get

$$w = \Omega_a\left[\frac{b^3 - \frac{b}{a} - a^3}{b^3-a^3}\,r + a^3\,b^3\,\frac{1-\frac{b}{a}}{r^2}\right]\sin\theta \tag{22}$$

which is the same as that obtained in the Navier-Stokes flow between two rotating concentric spheres[3].

3. Discussion

The flow in a porous region contained between two slowly rotating concentric spheres is examined. It is found from the equation (11), that the flow in a porous sphere differs from the classical flow in a clear sphere. The flow is not rigid in the present case unlike the flow in the classical case ($k \to \infty$). The non-dimensional form of the equation (11) is

$$w^* = \frac{w}{b\Omega_b \sin\theta} = \frac{\cosh R/\sqrt{k_1} - (\sqrt{k_1}/R)\sinh R/\sqrt{k_1}}{\cosh 1/\sqrt{k_1} - \sqrt{k_1}\sinh 1/\sqrt{k_1}} \tag{23}$$

where $R = r/a$, $k_1 = k/b^2$.

If S_1 denotes the rate of increase in skin-friction in the case of flow in a porous sphere over what it would be if the flow were in an ordinary sphere

$$S_1 = \frac{S^p_{b,\,a\to 0} - S^0_{b,\,a\to 0}}{S^0_{b,\,a\to 0}} = \frac{1}{3\sqrt{k_1}\,(1+\sqrt{k_1})}. \tag{24}$$

Here S^p_b is the skin-friction in the present case and it is S^0_b in the classical flow in the ordinary sphere (*i.e.*, $k \to \infty$). The fractional rate of increase in the moment, M_1 in the case of flow in a porous sphere over what it would be if the flow were in an ordinary sphere

$$M_1 = \frac{M^p_{b,\,a\to 0} - M^0_{b,\,a\to 0}}{M^0_{b,\,a\to 0}} = \frac{1}{3\sqrt{k_1}\,(1+\sqrt{k_1})}. \tag{25}$$

It can be noticed that S_1 and $M_1 \to 0$ as $k \to \infty$ but they $\to \infty$ as $k \to 0$.

From the equation (16), if S_2 denotes fractional rate of increase in the skin-friction in the case of flow on a sphere surrounded by a porous medium over what it would be on a sphere rotating in an infinite fluid,

$$S_2 = \frac{S^p_{a,\,b\to\infty} - S^0_{a,\,b\to\infty}}{S^0_{a,\,b\to\infty}} = \frac{1}{3\sqrt{k_1}\,(1-\sqrt{k_1}}\;. \tag{26}$$

From the equation (17) if M_2 denotes the fraction rate of increase in the moment on a sphere surrounded by a porous medium over what it would be that in the case of flow around a rotating sphere in an infinite viscous liquid is

$$M_2 = \frac{M^p_{a,\,b\to\infty} - M^0_{a,\,b\to\infty}}{M^0_{a,\,b\to\infty}} = \frac{1}{3\sqrt{k_1}\,(1-\sqrt{k_1})}. \tag{27}$$

It can be noticed that S_2, $M_1 \to \infty$ as $k \to 0$ and S_2, $M_2 \to 0$ as $k \to \infty$.

From the equation (19), we observe the existence of a thin layer far away between the boundaries where the velocity $w = 0$. Thus rigid flow is found in a thin layer in the porous region contained between the two spheres. A similar observation is made earlier in the case of flow in a porous region between two rotating cylinders[4].

4. Acknowledgement

The authors thank the referees for their valuable comments on the paper.

References

1. BEVERS, S. G. AND JOSEPH, D. D. — *J. Fluid Mech.*, 1967, **30**, 197.

2. BRINKMAN, H. C. — *Apl. Sc. Res.*, A_1, 1947, pp. 27–34.

3. LANDAU, L. D. AND LIFSHTIZ, E. M. — *Fluid Mech.*, Pergamon Press, London, 1959, p. 69.

4. NARASIMHACHARYULU, V. AND PATTABHI RAMACHARYULU, N. CH. — *JIISc*, 1978, **60** (*B*), 37.