TORSIONAL OSCILLATIONS OF A DISK IN A VISCOUS FLUID

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ABSTRACT

The flow due to the torsional oscillations of a disk in an unbounded viscous fluid has been investigated by reducing the problem to a set of dual integral equations. A formal solution of these equations is obtained by a method of successive approximations. An expression for the torque on the disk is calculated.

Key Words: Torsional oscillations, dual-integral equations, torque.

1. INTRODUCTION

The slow rotation of axisymmetric bodies in an unbounded fluid has been studied by Jeffery [1] and he has shown that the solution can also be obtained by the dual integral equations method. Ray [2] has studied the slow rotation of a finite circular disk in an unbounded viscous fluid by constructing special integral solutions satisfying the boundary conditions. A physical quantity of interest namely the torque acting on the disk has been calculated.

The rotatory oscillations set up by axisymmetric bodies in an infinite mass of a viscous fluid has been discussed by Kanwal [3] and the results were given in terms of spheroidal wave functions of complex arguments whose numerical values are not available. Recently, Kanwal [4] has studied the slow rotation and rotatory oscillations of axisymmetric bodies in hydrodynamics and magnetohydrodynamics and presented the expressions for torque in various cases by a method due to Shail [5].

The aim of the present investigation is to study the flow due to torsional oscillations of finite disk in an unbounded viscous fluid by dual integral equations method. The problem has been reduced to a set of dual integral equations and a formal solution is obtained by a method of successive approximations due to King [6]. The rotational Reynolds number is assumed to be small in the analysis. The expression for torque obtained by this method agrees with the result of Kanwal [4].

2. FORMULATION OF THE PROBLEM

Consider the torsional oscillations of a circular disk of radius 'a' in the plane $z = 0$, about an axis passing through the centre and normal to the plane of the disk, in an infinite, incompressible viscous fluid. Neglecting the quadratic terms of inertia and the secondary flow, the equation of motion for the primary flow in cylindrical polar coordinates (r, θ, z) reduce to a single equation for azimuthal velocity given by:

$$\frac{1}{\nu}\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2} + \frac{\partial^2 V}{\partial z^2} \tag{1.1}$$

where ν is the kinematic coefficient of viscosity. The boundary conditions are:

$$\left.\begin{array}{l} V \to 0 \quad \text{as} \quad r \to \infty, \quad |z| \to \infty \\ V = 0 \quad \text{at} \quad r = 0 \quad \text{for} \quad |a| \geqslant 0 \end{array}\right\} \tag{1.2}$$

and

$$V = V_4 r e^{-i\omega t} \quad \text{for } 0 < r < a \quad \text{and} \quad z = 0, \tag{1.3}$$

$$\frac{\partial V}{\partial z} = 0 \quad \text{for } r > a \text{ and } z = 0. \tag{1.4}$$

The condition (1.2) follows from the fact that the fluid is at rest at infinity and the azimuthal velocity is zero along the axis. The condition (1.4) ensures the continuity of stress across the plane $z = 0$ and $r > a$.

The solutions of (1.1) satisfying the conditions (1.2) is taken as:

$$V(r, z) = e^{-i\omega t}\int_0^\infty A(p)\, p e^{-\alpha_1 |z|} J_1(pr)\, dp, \tag{1.5}$$

where $\alpha_1 = (p^2 - i\omega/\nu)^{\frac{1}{2}}$. The solution (1.5) satisfies the conditions (1.3) and (1.4), if the unknown function $A(p)$ is a solution of the following dual integral equations:

$$\left.\begin{array}{ll} \displaystyle\int_0^\infty \frac{B(p)}{a_1} pJ_1(pr)\, dp = V_0 r \; (0 < r < a), & \\ \displaystyle\int_0^\infty B(p)\, pJ_1(pr)\, dp = 0 & r > a, \end{array}\right\} \tag{1.6}$$

where

$$B(p) = a_1 A(p).$$

Introducing non-dimensional variables defined by $r = r/a$, $\eta = ap$, $f(p) = pB(p)$ and $f(\eta/a) = F(\eta)$, the equations (1.6) reduces to:

$$\int_0^\infty \frac{F(\eta)}{(\eta^2 - i\eta_1^2)^{1/2}} J_1(\eta\bar{r})\, d\eta = k\bar{r}, \qquad 0 < \bar{r} < 1,$$

$$\int_0^\infty F(\eta) J_1(\eta\bar{r})\, d\eta = 0 \qquad \bar{r} > 1, \tag{1.7}$$

Where $k = aV_0$ and $\eta_1^2 = \omega a^2/v$ is the rotational Reynolds number. Now the problem is to solve the dual integral equations (1.7) for $F(\eta)$. As closed from solution of (1.7) is not possible, a formal solution is obtained by the method successive approximations due to King [6].

3. First Approximation

As a first approximation we solve the following dual integral equations instead of (1.6),

$$\left.\begin{aligned} &\int_0^\infty \frac{F_{11}(\eta)}{\eta} J_1(\eta\bar{r})\, d\eta = k\bar{r}, \quad 0 < r < 1, \\ &\int_0^\infty F_{11}(\eta) J_1(\eta\bar{r})\, d\eta = 0, \qquad \bar{r} > 1. \end{aligned}\right\} \tag{2.1}$$

In writing down the solution of (2.1), the following results given for more general equations considered by Busbridge [7], are useful: For the system:

$$\begin{aligned} &\int_0^\infty p^\alpha f(p) J_\lambda(px)\, dp = g(x), \qquad 0 < x < 1 \\ &\int_0^\infty f(p) J_\lambda(px)\, dp = 0 \qquad x > 1 \end{aligned} \tag{2.2}$$

where $f(p)$ is the unknown function of p and $g(x)$ is a known function of x, the solution is given by:

$$f(p) = \frac{(2p)^{1-\alpha/2}}{\Gamma(\alpha/2)} \int_0^1 \beta^{1+\alpha/2} J_{\lambda+\alpha/2}(\beta p)\, d\beta \int_0^1 g(\beta y)\, y^{\lambda+1}(1 - y^2)^{\alpha/2-1} dr$$

$$\text{for } \alpha > 0 \tag{2.3}$$

and

$$I(p) = \frac{2^{-\alpha/2} p^{-\alpha}}{\Gamma(1+\alpha/2)} \Big[p^{\alpha/2+1} J_{\lambda+\alpha/2}(p) \int_0^1 y^{+1}(1-y^2)^{\alpha/2} g(y)\, dy$$

$$+ \int_0^1 u^{\lambda+1}(1-u^2)^{\alpha/2}\, du \int_0^1 g(uy)(py)^{\alpha/2+2} J_{\lambda+1+\alpha/2}(py)\, dy\Big]$$

$$\text{for } \alpha > -2 \tag{2.4}$$

and

$$-\lambda - 1 < \alpha - \tfrac{1}{2} < \lambda + 1.$$

Using the above results the solution of (2.1) is obtained as:

$$F_{11}(\eta) = 2k\sqrt{\frac{2}{\pi}}\, \eta^{1/2} J_{3/2}(\eta). \tag{2.5}$$

4. SUCCESSIVE APPROXIMATIONS

The successive approximations for the solutions of (1.7) are obtained starting with the first approximation in the following way. In equations (1.7), we take:

$$F(\eta) = F_{11}(\eta) + F_{12}(\eta), \tag{3.1}$$

where $F_{11}(\eta)$ satisfies the pair of equations (2.1). Substituting (3.7) in (1.7) leads to the following pair of equations for the determination of $F_{12}(\eta)$:

$$\left.\begin{aligned} &\int_0^\infty \frac{F_{12}(\eta)}{(\eta^2 - i\eta_1^2)^{1/2}} J_1(\eta\bar{r})\, d\eta = k\bar{r} - \epsilon_{11}(\bar{r}), && 0 < \bar{r} < 1, \\ &\int_0^\infty F_{12}(\eta) J_1(\eta\bar{r})\, d\eta = 0, && \bar{r} > 1, \end{aligned}\right\} \tag{3.2}$$

where

$$\epsilon_{11}(r) = \int_0^\infty \frac{F_{11}(\eta)}{(\eta^2 - i\eta_1^2)^{1/2}} J_1(\eta\bar{r})\, d\eta. \tag{3.3}$$

Let

$$F_{12}(\eta) = F_{22}(\eta) + F_{23}(\eta) \tag{3.4}$$

where $F_{22}(\eta)$ is the solution of the pair of equations:

$$\left.\begin{aligned} &\int_0^\infty \frac{F_{22}(\eta)}{\eta} J_1(\eta\bar{r})\, d\eta = k\bar{r} - \epsilon_{11}(\bar{r}), \quad 0 < \bar{r} < 1, \\ &\int_0^\infty F_{22}(\eta)\, J_1(\eta\bar{r})\, d\eta = 0 \qquad \bar{r} > 1. \end{aligned}\right\} \tag{3.5}$$

Subsituting (3.4) in (3.2), F_{23} can be obtained from the pair of equations:

$$\left.\begin{aligned} &\int_0^\infty \frac{F_{23}(\eta)}{(\eta^2 - i\eta_1^2)^{1/2}} J_1(\eta\bar{r})\, d\eta = k\bar{r} - \epsilon_{11}(\bar{r}) - \epsilon_{22}(\bar{r}), \quad 0 < \bar{r} < 1 \\ &\int_0^\infty F_{23}(\eta)\, J_1(\eta\bar{r})\, d\eta = 0, \qquad \bar{r} > 1, \end{aligned}\right\} \tag{3.6}$$

where

$$\epsilon_{22}(\bar{r}) = \int_0^\infty \frac{F_{22}(\eta)}{(\eta^2 - i\eta_1^2)^{1/2}} J_1(\eta\bar{r})\, d\eta. \tag{3.7}$$

Following the same procedure, we finally obtain $F(\eta)$ as:

$$F(\eta) = \sum_{n=1}^{\infty} F_{nn}(\eta), \tag{3.8}$$

where $F_{nn}(\eta)$ is to be determined from the pair of equations:

$$\left.\begin{aligned} &\int_0^\infty \frac{F_{nn}(\eta)}{\eta} J_1(\eta\bar{r})\, d\eta = k\bar{r} - \sum_{m=1}^{n-1} \epsilon_{mm}(\bar{r}), \quad 0 < \bar{r} < 1, \\ &\int_0^\infty F_{nn}(\eta)\, J_1(\eta\bar{r})\, d\eta = 0 \qquad \bar{r} > 1 \end{aligned}\right\} \tag{3.9}$$

and

$$\epsilon_{mm}(\bar{r}) = \int_0^\infty \frac{F_{mm}(\eta)}{(\eta^2 - i\eta_1^2)^{1/2}} J_1(\eta\bar{r})\, d\eta, \qquad (1 \leqslant m < n) \tag{3.10}$$

is a known function of $\bar{r}$. The success of this method mainly depends on the evaluation of the integrals for the expression $\epsilon_{mm}(\bar{r})$.

5. SECOND APPROXIMATION

In order to obtain the second approximation $F_{22}(\eta)$, we have to evaluate $\epsilon_{11}(\bar{r})$ given in (3.3). Writing $J_{3/2}(\eta)$ in terms cosine sine and taking the first two terms of the expansion for $J_1(\eta\bar{r})$, we get,

$$\epsilon_{11}(\bar{r}) = \int_0^\infty \frac{\psi(\eta)\, d\eta}{(\eta^2 - i\eta_1{}^2)^{1/2}}, \tag{4.1}$$

where

$$\psi(\eta) = \psi_S(\eta)\sin\eta + \psi_C(\eta)\cos\eta, \tag{4.2}$$

$$\psi_S(\eta) = \frac{2}{\pi}\, k\bar{r}\left[1 - \frac{\bar{r}^2}{8}\eta^2\right] \tag{4.3}$$

and

$$\psi_C(\eta) = -\frac{2}{\pi}\, k\bar{r}\eta\left[1 - \frac{\bar{r}^2}{8}\eta^2\right]. \tag{4.4}$$

By putting $\eta = \zeta e^{i\pi/4}$, in (4.1), we get

$$\epsilon_{11}(\bar{r}) = \int_0^\infty \frac{\psi(\zeta e^{i\pi/4})\, d\zeta}{(\zeta^2 - \eta_1{}^2)^{1/2}} \tag{4.5}$$

The branch points of the integrand in (4.5) now lie on the real axis ard b following the method given by Awojobi and Grootenhuis [8], the expression for $\epsilon_{11}(\bar{r})$ is obtained as:

$$\epsilon_{11}(\bar{r}) = k\bar{r}\left(1 + \frac{\eta_2{}^2}{4} + \frac{4}{9\pi}i\eta_2{}^3\right) - kr^3\frac{\eta_2{}^2}{16} + 0(\eta_2{}^4), \tag{4.6}$$

where

$$\eta_2 = \eta_1 e^{i\pi/4}.$$

Putting this value of $\epsilon_{11}(r)$ in (3.5) and making use of the results given in (2.4), we obtain the second approximation $F_{22}(\eta)$ as:

$$F_{22}(\eta) = k\sqrt{\frac{2}{\pi}}\left\{-\frac{\eta_2{}^2}{3}\left[\eta^{1/2}J_{3/2}(\eta) + \eta^{-1/2}J_{5/2}(\eta)\right] - \frac{i8}{9\pi}\eta_2{}^3\eta^{1/2}J_{3/2}(\eta) + 0(\eta_2{}^4)\right\}. \tag{4.7}$$

Subsequent approximations can be calculated from the equations (3.9). The expressions for $F(\eta)$ correct to the $0(\eta_2{}^4)$ is given by:

$$F(\eta) = F_{11}(\eta) + F_{22}(\eta). \tag{4.8}$$

6. Torque on the Disk

One of the physically interesting quantity, namely the torque on the disk (taking into account both sides) is:

$$M = -4\pi\mu \int_0^a r^2 \left[\frac{\partial V}{\partial z}\right]_{z=0} dr. \tag{5.1}$$

Substituting the expressions for V from (1.5) in (5.1), we obtain

$$M = M_0 \, e^{-iwt} \left(1 - \frac{\eta_2^{\,2}}{5} + \frac{4i\,\eta_2^{\,3}}{9\pi}\right) + 0\,(\eta_2^{\,4}), \tag{5.2}$$

where

$$M_0 = -\frac{32}{3}\,\mu V_0\, a^3, \tag{5.3}$$

is the expression for the torque for a steady rotating disk. The expression for the torque given in (5.2) agrees with the results obtained by Kanwal [4] up to the order $\eta_2^{\,3}$, by a different perturbation procedure.

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