On the theory of transforms associated with eigenvectors (III)

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Abstract

In this paper the author applies the theory of transforms developed in Refs. 1 and 2 to study the L^p-convergence of the eigenvector expansions associated with the differential system

$$(L - \lambda I)\,\phi = 0$$

in the finite as well as the singular case, where

$$L = \begin{pmatrix} -d^2/dx^2 + p(x) & r(x) \\ r(x) & -d^2/dx^2 + q(x) \end{pmatrix}$$

and ϕ is a two component column vector.

A property of transformations of L_0^p onto itself is first proved and a suitable inequality established. Asymptotic expansions of some vectors are then obtained and a suitable operator defined which leads to the L^p-convergence in the finite case. Finally some more asymptotic expansions are derived which under some specified conditions yield the following:

Theorem: The eigenvector expansion $O_\omega f$ of a vector f, in the singular case, converges in mean to the vector itself.

Some of the results obtained in this paper are generalisations of those of Rutovitz[3].

Key words: Transform, L^p-convergence, inverse-transformations, dense subset, asymptotic expansions, entire functions, contour integral, residue, convergence in mean.

1. Introduction

The object of this paper is to apply the theory of transforms developed in Refs. 1 and 2 to study the L^p-convergence of the eigenvector expansions associated with the differential system

$$(L - \lambda I)\,\phi = 0 \tag{1.1}$$

in the finite $[0, b]$ as well as the sigular case $[0, \infty)$, where

$$L = \begin{pmatrix} -d^2/dx^2 + p(x) & r(x) \\ r(x) & -d^2/dx^2 + q(x) \end{pmatrix}$$

and $\phi = \phi(x) = \{u(x), v(x)\}$ is a two component column vector function of x.

In order to avoid the repetition of the preliminaries, we have written this paper as an addendum to Refs. 1 and 2 and consequently we make free use of symbols, notations and results contained therein.

Let L_b^p and $\mathcal{L}_b^p$ be the spaces of column vectors (whose components are real valued functions of a real variable)

$$f(x) = \{f_1, f_2\} \qquad \text{and} \qquad F(t) = \{F_1, F_2\}$$

for which

$$|f|_{p,b} = \text{Max}\,[\int_0^b |f_r|^p\, dx]^{1/p} < \infty$$

and

$$|F, d\rho(b)|_p = \text{Max}\,[\int_{-\infty}^{\infty} |F_r|^p\, d\rho_{rs}(b, t)]^{1/p} < \infty$$

respectively, where $\rho_{rs}(b, t)$ $(r, s = 1, 2)$ are as defined in §3 of Ref. 1. Further, we set

$$L^p = L_\infty^p; \quad \mathcal{L}^p = \mathcal{L}_\infty^p; \quad |f|_p = |f|_{p,\infty}$$

and

$$|F, d\rho|_p = |F, d\rho(\infty)|_p$$

it being understood that, in the last expression, $b \to \infty$ through a suitable sequence. We assume that

$$1 < p \leq 2 \quad \text{and} \quad 1/p + 1/q = 1.$$

It follows from the arguments contained in §4 and §9 of Ref. 1 that

$$T_b f = \{T_{1b} f, T_{2b} f\},$$

where

$$T_{rb} f = \langle \phi_r(0 \mid x, t), f(x) \rangle$$

and

$$\mathfrak{T}_{w,b} F = \{\mathfrak{T}_{1w,b} F, \mathfrak{T}_{2w,b} F\} = \sum_{r=1}^{2} \int_{-w}^{w} \phi_r(0 \mid x, t)\,(F(t), d\rho_r(b, t))$$
$$= \{\langle U(x, t), F(t), d\rho(b, t)\rangle - w, w, \langle V(x, t), F(t), d\rho(b, t)\rangle - w, w\}$$

define transforms from L_b^2 onto $\mathcal{L}_b^2$ in one case and from a subset of $\mathcal{L}_b^2$ into L_b^2 in the other, where $U(x, t)$ and $V(x, t)$ are as defined in §1 of Ref. 2. Further, if $f \in L^2$, $T_b f$ converges in $\mathcal{L}^2$ to a vector Tf as $b \to \infty$ and if $F \in \mathcal{L}^2$, $\mathfrak{T}_{w,b} F$ converges in L^2 to a vector $\mathfrak{T} F$ as $w \to \infty$ and $b \to \infty$ through a suitable sequence. T_b and $\mathfrak{T}_b$ $(\equiv \mathfrak{T}_{\infty,b})$ are inverse transformations between L_b^2 and $\mathcal{L}_b^2$.

Since $\phi_r(0 \mid x, \lambda)$ $(r = 1, 2)$ are bounded uniformly for all eigenvalues λ, over each $[0, b]$, it follows that, for $b < \infty$, and at all eigenvalues λ

$$| T_b f | = \text{Max} \langle \phi_r(0 \mid x, \lambda), f(x) \rangle$$
$$\leq C(b) \, | f |_{1, b}$$

where $C(b)$ depends only on b.

2. A property of transformations in L_b^p

Lemma (2.1): Let a column vector $f \to O_w f$ be a linear transformation depending on the parameter w from L_b^p onto itself, such that

$$| O_w f |_{p, b} \leq C \, | f |_{p, b}, \tag{2.1}$$

where C is a constant independent of f and w, and

$$| f - O_w f |_{p, b} \to 0 \tag{2.2}$$

as $w \to \infty$ on a dense subset of L_b^p. Then

$$| f - O_w f |_{p, b} \to 0, \text{ as } w \to \infty$$

on L_b^p: $b = \infty$ being permissible.

PROOF: Let $f \in L_b^p$. Then it follows from the definition that for every $\varepsilon > 0$, there exist vectors $h = \{h_1, h_2\}$; $g = \{g_1, g_2\}$, such that

$$f = h + g,$$

where

$$| g |_{p, b} < \epsilon / 2 (1 + C)$$

and

$$| h - O_w h |_{p, b} \to 0 \quad \text{as} \quad w \to \infty.$$

It follows that there exists w_0 such that for all $w > w_0$

$$| f - O_w f |_{p, b} = | (h - O_w h) + g - O_w g |_{p, b}$$
$$\leq | h - O_w h |_{p, b} + | g |_{p, b} + | O_w g |_{p, b}$$

by Minkowski's inequality.

i.e.,

$$O_w f \to f \text{ in } L_b^p.$$

3. An inequality

Lemma (3.1) : Let

$$j_{m,b}(x, y) = 2b^{-1} \sum_{k=0}^{m} \cos kx \cos ky$$

and

$$J_{m,b}(f, g) = \int_0^b \int_0^b (f(y), g(y))\, j_{m,b}(x, y)\, dxdy.$$

Then there exists $K(p)$ depending only on the parameter shown such that

$$| J_{m,b}(f, g) | \leq K(p) \, | f |_{p,b} \, | g |_{q,b},$$

whenever

$$f \in L_b^p \quad \text{and} \quad g \in L_b^q.$$

This lemma is an immediate consequence of that of Rutovitz[3].

4. Asymptotic expansions in the finite case

In what follows we assume that the constants appearing in the boundary conditions (1.3) of Ref. 1 satisfy the following additional conditions:

At least two of the ratios a_{1j}/a_{2j} $(j = 1, 2, 3, 4)$ and also at least two of the ratios b_{1j}/b_{2j} $(j = 1, 2, 3, 4)$ are unequal, say

$$a_{12}/a_{22} \neq a_{14}/a_{24} \quad \text{and} \quad b_{12}/b_{22} \neq b_{14}/b_{24}.$$

The results of this section follow exactly in the same way as the corresponding results obtained by Titchmarsh[4] and Bhagat[5]. We, therefore, enunciate the relevant theorems and omit the details of the proof.

Theorem (4.1) : Let $\phi(\xi \mid x, \lambda) = \{u(\xi \mid x, \lambda), v(\xi \mid x, \lambda)\}$ be a solution of (1.1) such that

$$\phi(\xi \mid \xi, \lambda) = \{\alpha, \gamma\}; \quad \phi'(\xi \mid \xi, \lambda) = \{\beta, \delta\}.$$

Let

$$\lambda = s^2,$$

$$A = \begin{pmatrix} \alpha & \beta s^{-1} \\ \gamma & \delta s^{-1} \end{pmatrix}$$

and

$$B = \{\cos s(x - \xi), \sin s(x - \xi)\}.$$

Then

$$\phi(\xi \mid x, \lambda) = AB + s^{-1} \int_{\xi}^{x} (M(y)\, \varphi(\xi \mid y, \lambda)) \sin s(x-y)\, dy,$$

where $M(x)$ is the matrix defined in §5 of Ref. 2.

Theorem (4.2): Let $\phi_j(0 \mid x, \lambda)$ $(j = 1, 2)$ and $\phi_k(b \mid x, \lambda)$ $(k = 3, 4)$ be the boundary condition vectors for the system (1.1). Let

$$\lambda = s^2, \quad s = \sigma + i\tau.$$

Then, for $|s| \geqslant |s_0|$

(i) $\phi_j(0 \mid x, \lambda) = \{a_{j2}, a_{j4}\} \cos sx + O(e^{|\tau| x}/|s|)$

(ii) $\phi_k(b \mid x, \lambda) = \{b_{j2}, b_{j4}\} \cos s(b-x) + O(e^{|\tau|(b-x)}/|s|)$ $(j = 1, 2;\ k = 3, 4)$.

Theorem (4.3): Let the conditions of theorem (4.2) be satisfied. Then

(i) $[\phi_j(0 \mid x, \lambda), \phi_k(b \mid x, \lambda)] = -s(a_{j2}b_{r2} + a_{j4}b_{r4}) \sin sb + O(e^{|\tau| b})$,

where $j = 1, 2$; $r = 1$ when $k = 3$ and $r = 2$ when $k = 4$.

(ii) $D(b, \lambda) = s^2 (a_{12}a_{24} - a_{14}a_{22})(b_{12}b_{24} - b_{14}b_{22}) \sin^2 sb + O(se^{2|\tau| b})$.

We note that $[\phi_j, \phi_k]$ are not identically zero. In fact these are entire functions of s of order 1 and so entire functions of λ of order $\frac{1}{2}$.

We also note that $D(b, \lambda)$ is not identically zero and that it possesses an infinity of zeros. By arguments similar to those of Titchmarsh[4] (p. 19) it follows that the zeros of $D(b, \lambda)$ are asymptotic to the zeros of $s^2 \sin^2 sb$, that is to the points where $s = n\pi/b$ for large $|s|$ and n. For large values of n the eigenvalues are asymptotic to $n^2\pi^2/b^2$. It is also easy to see that $D(b, \lambda) \neq 0$ for $s = i\tau$ $(\tau > \tau_0)$, *i.e.*, for λ negative and sufficiently large.

Theorem (4.4): If $\psi_r(b, x, \lambda)$ be defined by (2.6) of Ref. 1 and $G(b, x, y, \lambda)$ as in 2 (iv) of Ref. 2 and the conditions of Theorem (4.2) be satisfied, then for $y \in [0, x)$,

$$G_{rr}(b; x, y, \lambda) = -\cos s(b-x) \cos sy/s \sin sb + O(e^{-\tau|y-x|}/|\lambda|)$$

$$G_{12}(b; x, y, \lambda) = O(e^{-\tau|y-x|}/|\lambda|) = G_{21}(b; x, y, \lambda)$$

and similar expressions for $y \in (x, b]$.

5. The matrix $h_{w,b}(x, y)$: The operator $O_{w,b}$

We define

$$h_{w,b}(x, y) = \begin{pmatrix} h^{11}_{w,b}(x, y) & h^{21}_{w,b}(x, y) \\ h^{12}_{w,b}(x, y) & h^{22}_{w,b}(x, y) \end{pmatrix}$$

$$= \sum_{r=1}^{2} \sum_{s=1}^{2} \int_{-w}^{w} (\varphi_r(0 \mid x, t)\, \varphi^T_s(0 \mid y, t))\, d\rho_{rs}(b, t) \tag{5.1}$$

and

$$O_{w,b} f(x) = \int_0^b h^T_{w,b}(y, x) f(y)\, dy. \tag{5.2}$$

Then

$$O_{w,b} f(x) = \sum_{r=1}^{2} \int_{-w}^{w} \phi_r(0 \mid x, t)\, (T_b f.\ d\rho_r(b, t))$$

$$= \mathfrak{I}_{w,b} T_b f(x)$$

[*cf.* (3.3) Ref. 1].

It is known from Theorem (4.3) that there exists $w = w_b$ for each b such that $D(b, t) \neq 0$ for $t < -w_b$. For $w > w_b$, let us denote by C the contour considered by Titchmarsh[4] (p. 13) which is symmetrical about the real axis and which corresponds in the upper half of the λ-plane to the boundary of the quarter-square in the s-plane

$$s = \sqrt{w_*} + i\tau \qquad (0 \leq \tau \leq \sqrt{w_*})$$

$$= \sigma + i\sqrt{w_*} \qquad (0 \leq \sigma \leq \sqrt{w_*}),$$

where $\lambda = s^2$, $s = \sigma + i\tau$ and w_* bisects the interval between the greatest eigenvalue not exceeding w and the succeeding one. Let $y \in [0, x)$. Then

$$\frac{1}{2\pi i} \int_c G_{11}(b : x, y, \lambda)\, d\lambda$$

$$= \frac{1}{2\pi i} \int_c (\psi_1(b ; x, \lambda), U(y, \lambda))\, d\lambda$$

$$= \frac{1}{2\pi i} \sum_{r=1}^{2} \int_c [\{l_{1r} u_r(0 \mid x, \lambda) + x_1(0 \mid x, \lambda)\}\, u_1(0 \mid y, \lambda)$$

$$+ \{l_{2r} u_r(0 \mid x, \lambda) + x_2(0 \mid x, \lambda)\}\, u_2(0 \mid y, \lambda)]$$

$$= \frac{1}{2\pi i} \sum_{r=1}^{2} \sum_{n} 2\pi i \; [R_{1r}(b, n)\, u_r(0 \mid x, \lambda_{nb})\, u_1(0 \mid y, \lambda_{nb})$$

$$+ R_{2r}(b, n)\, u_r(0 \mid x, \lambda_{nb})\, u_2(0 \mid y, \lambda_{nb})]$$

$$= \sum_{r=1}^{2} \sum_{s=1}^{2} \int_{-w}^{w} u_r(0 \mid x, t)\, u_s(0 \mid y, t)\, d\rho_{rs}(b, t)$$

$$= h^{11}_{w,b}(x, y),$$

where $\theta_k = \{x_k, y_k\}$ is as defined in § 2 of Ref. 1.

Similar results hold for contour integrals involving other $G_{ij}(b; x, y, \lambda)$ $(i, j = 1, 2)$ and accordingly we obtain

$$\frac{1}{2\pi i} \int_{c} G(b, x, y, \lambda)\, d\lambda = h_{w,b}(x, y). \tag{5.3}$$

The case when $y \in (x, b]$ can be dealt with in an identical manner.

Since $D(b, \lambda)$ has the same number of zeros inside the contour C as $s \sin sb$ [Theorem 4.3)], it follows by using the results of Theorem (4.4), the calculus of residues and (5.3), that

$$h^{rr}_{w,b}(x, y) = -j_{m,b}(x, y) + O\left(\textstyle\int_c (e^{-\tau \mid y-x \mid} / \mid \lambda \mid) \mid d\lambda \mid\right), \tag{5.4}$$

where $m = [w^{1/2}]$, the greatest integer not exceeding $w^{1/2}$, and

$$h^{rs}_{w,b}(x, y) = O\left(\int_{c} (e^{-\tau \mid y-x \mid} / \mid \lambda \mid) \mid d\lambda \mid\right). \quad (r \neq s) \tag{5.5}$$

Again, since

$$O\left(\int_c (e^{-\tau \mid y-x \mid} / \mid \lambda \mid) \mid d\lambda \mid\right)$$

$$= O\left(\frac{(1 - e^{-w^{1/2} \mid y-x \mid}}{w^{1/2} \mid y - x \mid}\right) + O(e^{-w^{1/2} \mid y-x \mid}), \tag{5.6}$$

we obtain

$$h^{rr}_{w,b}(x, y) = -j_{m,b}(x, y) + O(1), \quad (r = 1, 2) \tag{5.7}$$

and

$$h^{rs}_{w,b}(x, y) = O(1), \quad (r, s = 1, 2; r \neq s). \tag{5.8}$$

6. The L^p-convergence in the finite case

Theorem (6.1): The eigenvector expansion $O_{w,b} f$ of a vector $f(x) = \{f_1, f_2\}$ of class L_b^p converges in mean to the vector itself, *i.e.*,

$$\lim_{w \to \infty} | O_{w,b} f - f |_{p,b} = 0; \tag{6.1}$$

and

$$| O_{w,b} f |_{p,b} \leq K(p, b) | f |_{p,b}, \tag{6.2}$$

where $K(p, b)$ is independent of the vector $f(x)$.

PROOF: Let $g(x) = \{g_1, g_2\} \in L_b^q$.

Then from (5.2), (5.7), (5.8) and the lemma (3.1), we obtain

$$\int_0^b (O_{w,b} f(x), g(x)) \, dx$$

$$= \int_0^b \left(\int_0^b h_{w,b}^T (y, x) f(y) \, dy, g(x) \right) dx$$

$$\leqslant O\left(\int_0^b \int_0^b (f_1 g_1 + f_2 g_1 + f_1 g_2 + f_2 g_2) \, dx \, dy \right) + | J_{m,b}(f, g) |$$

$$\leqslant C |f|_{1,b} |g|_{1,b} + K(p) | f |_{p,b} | g |_{q,b}$$

$$\leqslant [Cb + K(p)] | f |_{p,b} | g |_{q,b}$$

which is (6.2) by the converse of Holder's inequality, (*cf.* Hardy, Littlewood and Polya[6], p. 142).

Thus $O_{w,b}$ satisfies the condition (2.1) of the lemma (2.1). Further, from the arguments contained in § 4 and § 9 of Ref. 1, it follows that (6.1) holds for $p = 2$. Also L_b^2-convergence implies L_b^p-convergence, and L_b^2 is dense in L_b^p $(1 < p \leq 2)$. The condition (2.2) of the lemma (2.1) is, therefore, satisfied and (6.1) follows.

7. Asymptotic expansions associated with $\phi_j (0/x.\lambda)$

Let

$$\tilde{M}(x) = \begin{pmatrix} \tilde{p}(x) & \tilde{r}(x) \\ \tilde{r}(x) & \tilde{q}(x) \end{pmatrix} = xM(x).$$

We assume that each element of $\tilde{M}(x) \in L[0, \infty)$ and is a function of bounded variation on $[0, \infty)$. It follows, therefore, that each element of $M(x) \in L[0, \infty)$ and that

$$\int_x^\infty |p(t)|\, dt,\ \int_x^\infty |q(t)\, dt,\ \int_x^\infty |r(t)|\, dt = 0\,(1/(1+x)) \tag{7.1}$$

as $x \to \infty$.

In what follows we assume that $a_{j1}a_{j3} \neq 0$ $(j = 1, 2)$. The analysis carried out below may be easily modified to cover the cases when $a_{j1}a_{j3} = 0$. We put $\lambda = s^2$ and assume s to be real.

Let

$$c_j(\lambda) = \{c_{j1}, c_{j2}\} = \{a_{j2}, a_{j4}\} - s^{-1}\int_0^\infty M(y)\,\phi_j(0 \mid y, \lambda) \sin sy\, dy$$

$$d_j(\lambda) = \{d_{j1}, d_{j2}) = \{a_{j1}, a_{j3}\}\, s^{-1} - s^{-1}\int_0^\infty M(y)\,\phi_j(0 \mid y, \lambda) \cos sy\, dy,$$

where $\phi_j(0 \mid x, \lambda) = \{u_j(0 \mid x, \lambda), v_j(0 \mid x, \lambda)\}$ $(j = 1, 2)$ are the boundary condition vectors at $x = 0$.

By arguments similar to those of Bhagat[5], it follows that (i) $u_j(0 \mid x, \lambda)$; $v_j(0 \mid x, \lambda)$ are bounded for $|s| > 0$, (ii) integrals involved in defining $c_{j1}(\lambda), c_{j2}(\lambda), d_{j1}(\lambda)$ and $d_{j2}(\lambda)$ converge uniformly, so that these are continuous functions of s for $|s| > 0$. We, therefore, obtain the following lemma from Theorem (4.1).

Lemma (7.1): Let M_{*r} denote the r^{th} row of the matrix M. Then

(i) $$u_j(0 \mid x, \lambda) = a_{j2}\cos sx - s^{-1} a_{j1}\sin sx + s^{-1}\int_0^x M_{*1}\,\phi_j(0 \mid y, \lambda)\sin s(x-y)\, dy \tag{7.2}$$

$$= a_{j2}\cos sx + O(s^{-1}), \text{ (uniformly in } x) \tag{7.3}$$

$$= c_{j1}\cos sx - d_{j1}\sin sx + O(1+x)^{-1} \tag{7.4}$$

$$\left.\begin{array}{l} = -s^{-1} a_{j1}\sin sx + O(s^{-2}) \\ = O(s^{-1}) \end{array}\right\} \quad a_{j2} = 0 \tag{7.5}$$

(ii) $$v_j(0 \mid x, \lambda) = a_{j4}\cos sx - s^{-1} a_{j3}\sin sx + s^{-1}\int_0^x M_{*2}(y)\,\phi_j(0 \mid y, \lambda) \times \sin s(x-y)\, dy \tag{7.6}$$

$$= a_{j4}\cos sx + O(s^{-1}), \text{ (uniformly in } x) \tag{7.7}$$

$$= c_{j2} \cos sx - d_{j2} \sin sx + O\left((1+x)^{-1}\right) \tag{7.8}$$

$$\left.\begin{aligned} &= -s^{-1} a_{j3} \sin sx + O(s^{-2}) \\ &= O(s^{-1}) \end{aligned}\right\} \quad a_{j4} = 0. \tag{7.9}$$

It may be noted that a_{j2} and a_{j4} cannot vanish simultaneously (*cf.* § 4).

Lemma (7.2):

(i) $$\frac{\partial}{\partial s} u_j(0 \mid x, \lambda) = O(1+x) \tag{7.10}$$

$$= -xa_{j2} \sin sx + O\left((1+x)/s\right) \tag{7.11}$$

$$\left.\begin{aligned} &= -s^{-1} xa_{j1} \cos sx + O\left((1+x)/s^2\right) \\ &= O\left((1+x)/s\right) \end{aligned}\right\} \quad a_{j2} = 0 \tag{7.12}$$

(ii) $$\frac{\partial}{\partial s} v_j(0 \mid x, \lambda) = O(1+x) \tag{7.13}$$

$$= -x\, a_{j4} \sin sx + O\left((1+x)/s\right) \tag{7.14}$$

$$\left.\begin{aligned} &= -s^{-1} xa_{j3} \cos sx + O\left((1+x)/s^2\right) \\ &= O\left((1+x)/s\right) \end{aligned}\right\} \quad a_{j4} = 0. \tag{7.15}$$

PROOF: Differentiating (7.2) partially with respect to s, we obtain

$$\begin{aligned} \frac{\partial}{\partial s} u_j(0 \mid x, \lambda) &= -xa_{j2} \sin sx - s^{-1} xa_{j1} \cos sx + s^{-2} a_{j1} \sin sx \\ &\quad - s^{-2} \int_0^x M_{*1}(y)\, \phi_j(0 \mid y, \lambda) \sin s(x-y)\, dy \\ &\quad + s^{-1} \int_0^x M_{*1}(y)\, \phi_j(0 \mid y, \lambda)(x-y) \cos s(x-y)\, dy \\ &\quad + s^{-1} \int_0^x M_{*1}(y) \frac{\partial}{\partial s} \phi_j(0 \mid y, \lambda) \sin s(x-y)\, dy \end{aligned} \tag{7.16}$$

$$= O(x) + O\left(s^{-1} \int_0^x \left[\, \left| p(y) \frac{\partial}{\partial s} u_j(0 \mid y, \lambda) \right| + \left| r(y) \frac{\partial}{\partial s} v_j(0 \mid y, \lambda) \right| \right] dy\right). \tag{7.17}$$

Similarly from (7.6)

$$\frac{\partial}{\partial s} v_j(0 \mid x, \lambda) = O(x) + O\left(s^{-1} \int_0^x \left[\, \left| r(y) \frac{\partial}{\partial s} u_j(0 \mid y, \lambda) \right|\right.\right.$$

$$+\left| q(y) \frac{\partial}{\partial s} v_j(0 \mid y, \lambda) \right| \Big] dy \Big). \tag{7.18}$$

Putting

$$\left.\begin{aligned} N_1(x) &= \operatorname{Sup} \left| \frac{\partial}{\partial s} u_j(0 \mid y, \lambda) \right| / (1+y) \\ N_2(x) &= \operatorname{Sup} \left| \frac{\partial}{\partial s} v_j(0 \mid y, \lambda) \right| / (1+y) \end{aligned}\right\} \quad (0 \leq y \leq x),$$

using (7.17) and (7.18), we obtain, as $s \to \infty$

$$(1+x) N_1(x) = O(1+x) + O((1+x)(N_1(x) + N_2(x))/s)$$

$$(1+x) N_2(x) = O(1+x) + O((1+x)(N_1(x) + N_2(x))/s).$$

Hence $N_1(x) + N_2(x) = O(1)$. uniformly in x, as $s \to \infty$.

Since $u_j(0 \mid x, \lambda)$, $v_j(0 \mid x, \lambda)$ are linearly independent, we obtain (7.10) and (7.13). (7.11) follows from (7.16) by using (7.10) and (7·13).

Similarly for (7.14).

Further, when $a_{j2} = 0$, (7.12) follows directly from (7.16).

A similar analysis yields (7.15) when $a_{j4} = 0$.

The lemma thus follows.

Lemma (7.3) :

(i) $c_{j1}(\lambda) = O(1), \quad a_{j2} \neq 0$
$\qquad = O(s^{-2}), \quad a_{j2} = 0$

(ii) $1/c_{j1}(\lambda) = O(1), \quad a_{j2} \neq 0$

(iii) $\dfrac{d}{ds} c_{j1}(\lambda) = O(s^{-2}), \quad a_{j2}a_{j4} \neq 0$
$\qquad = O(s^{-3}), \quad a_{j2}a_{j4} = 0$

(iv) $c_{j2}(\lambda) = O(1), \quad a_{j4} \neq 0$
$\qquad = O(s^{-2}), \quad a_{j4} = 0$

(v) $1/c_{j2}(\lambda) = O(1), \quad a_{j4} \neq 0$

(vi) $\frac{d}{ds} c_{j2}(\lambda) = O(s^{-2}), \quad a_{j2}\, a_{j4} \neq 0$

$\qquad\qquad\quad = O(s^{-3}), \quad a_{j2} a_{j4} = 0.$

PROOF: (i), (ii), (iv) and (v) are immediate consequences of the definitions of $c_{j1}(\lambda)$, $c_{j2}(\lambda)$ and the corresponding results of lemma (7.1).

From the definition of $c_{j1}(\lambda)$, we obtain

$$\frac{d}{ds} c_{j1}(\lambda) = s^{-2} \int_0^\infty M_{*1}(y)\, \varphi_j(0 \mid y, \lambda) \sin sy\, dy$$

$$- s^{-1} \int_0^\infty M_{*1}(y)\, \varphi_j(0 \mid y, \lambda)\, y \cos sy\, dy$$

$$- s^{-1} \int_0^\infty M_{*1}(y) \frac{\partial}{\partial s} \varphi_j(0 \mid y, \lambda) \sin sy\, dy$$

$$= O(s^{-2}) + s^{-1} \int_0^\infty (a_{j2} \tilde{p}(y) + a_{j4} \tilde{r}(y)) (\sin^2 sy - \cos^2 sy)\, dy$$

by (7.1), (7.3), (7.7), (7.11) and (7.14) if $a_{j2}\, a_{j4} \neq 0$.

Since $\tilde{p}(y), \tilde{r}(y)$ are functions of bounded variation, we obtain

$$\frac{d}{ds} c_{j1}(\lambda) = O(s^{-2}).$$

Similarly, if $a_{j2}\, a_{j4} = 0$, we get

$$\frac{d}{ds} c_{j1}(\lambda) = O(s^{-3}).$$

The relation (iii), therefore, follows.

A similar analysis yields (vi).

Lemma (7.4):

(i) $d_{j1}(\lambda) \quad = O(s^{-1})$

(ii) $1/d_{j1}(\lambda) \quad = O(s), \quad a_{j2} a_{j4} = 0$

(iii) $\frac{d}{ds} d_{j1}(\lambda) = O(s^{-1}), \quad a_{j2}a_{j4} \neq 0$

$O(s^{-2}), \quad a_{j2}a_{j4} = 0$

(iv) $d_{j2}(\lambda) = O(s^{-1})$

(v) $1/d_{j2}(\lambda) = O(s), \quad a_{j2}a_{j4} = 0$

(vi) $\frac{d}{ds} d_{j2}(\lambda) = O(s^{-1}), \quad a_{j2}a_{j4} \neq 0$

$= O(s^{-3}), \quad a_{j2}a_{j4} = 0.$

These results follow as in lemma (7.3) by using the definitions of $d_{j1}(\lambda)$ and $d_{j2}(\lambda)$.

8. Asymptotic expansions associated with $(K_{TS}(\lambda))$

As in Bhagat[5], we obtain fairly easily the following expressions for the functions $m_{rs}(\lambda)$ $(r, s = 1, 2)$ defined in § 6 of Ref. 1:

$$Im\,[\lim_{T\to 0} m_{rr}(\lambda)] = \sum_{k=1}^{2} (c_{jk}^2 + d_{jk}^2)/\lambda^{\frac{1}{2}}(A^2 + B^2) \tag{8.1}$$

(when $r = 1$, $j = 2$; and when $r = 2$, $j = 1$)

and

$$\begin{aligned} \operatorname{Im}\,[\lim_{T\to 0} m_{12}(\lambda)] &= \operatorname{Im}\,[\lim_{T\to 0} m_{21}(\lambda)] \\ &= -(c_{11}c_{21} + c_{12}c_{22} + d_{11}d_{21} + d_{12}d_{22})/\lambda^{\frac{1}{2}}(A^2 + B^2), \end{aligned} \tag{8.2}$$

where

$$\left.\begin{aligned} A = A(\lambda) &= c_{21}c_{12} - d_{21}d_{12} - c_{11}c_{22} + d_{11}d_{22} \\ B = B(\lambda) &= c_{21}d_{12} + c_{12}d_{21} - c_{11}d_{22} - c_{22}d_{11} \end{aligned}\right\}. \tag{8.3}$$

It follows quite easily that $A(\lambda)$ and $B(\lambda)$ both cannot vanish for any positive λ. If $a_{j2}a_{j4} \neq 0$, we obtain

$$\left.\begin{aligned} &A(\lambda) = O(1);\; 1/A(\lambda) = O(1);\; \frac{d}{ds} A(\lambda) = O(s^{-2}) \\ &B(\lambda) = O(s^{-1});\; \frac{d}{ds} B(\lambda) = O(s^{-1}) \end{aligned}\right\}, \tag{8.4}$$

and if $a_{j2}a_{j4} = 0$, we obtain

$$\left.\begin{aligned} &A(\lambda) = O(s^{-2});\; d/ds\, A(\lambda) = O(s^{-3});\; B(\lambda) = O(s^{-3}) \\ &d/ds\, B(\lambda) = O(s^{-4});\; 1/A(\lambda) = O(s^2) \end{aligned}\right\} \tag{8.5}$$

where

$$a_{11}a_{23} - a_{21}a_{13} \neq 0$$

in the last case, by making use of the definitions of the functions $c_{j1}(\lambda)$, etc., and their asymptotic expansions.

Now

$$K_{rr}(\lambda) = \lim_{\nu \to 0} \int_0^{\lambda} - \operatorname{Im} \, m_{rr}(\mu + i\nu) \, d\mu$$

and hence, for $\lambda = s^2$, $s \geqslant 0$, we get

$$K_{rr}(\lambda) = -2 \int_0^s E_{jj}(u) \, du, \tag{8.6}$$

where

$$E_{jj}(u) = \sum_{k=1}^{2} (c_{jk}^2(u^2) + d_{jk}^2(u^2))/(A^2(u^2) + B^2(u^2))$$

(when $r = 1$, $j = 2$; when $r = 2$, $j = 1$).

Therefore

$$E'_{ij}(u) = \frac{2 \sum_{k=1}^{2} \left[(A^2 + B^2) \left(c_{jk} \frac{d}{ds} c_{jk} + d_{jk} \frac{d}{ds} d_{jk} \right) - (c_{jk}^2 + d_{jk}^2) \left(A \frac{d}{ds} A + B \frac{d}{ds} B \right) \right]}{(A^2 + B^2)^2}$$

$$\begin{aligned} &= O(s^{-2}) \quad \text{if} \quad a_{j2} a_{j4} \neq 0 \\ &= O(s) \quad \text{if} \quad a_{j2} a_{j4} = 0 \end{aligned} \tag{8.7}$$

by using the relevant results obtained earlier. Similarly

$$K_{12}(\lambda) = K_{21}(\lambda) = 2 \int_0^s E_{12}(u) \, du = 2 \int_0^s E_{21}(u), \tag{8.8}$$

where

$$\begin{aligned} E_{12}(u) &= (c_{11}c_{21} + c_{12} c_{22} + d_{11}d_{21} + d_{12}d_{22})/(A^2 + B^2) \\ &= E_{21}(u) \end{aligned}$$

and

$$\begin{aligned} E'_{21}(u) = E'_{12}(u) &= O(s^{-2}) \quad \text{if} \quad a_{j2} a_{j4} \neq 0 \\ &= O(s) \quad \text{if} \quad a_{j2} a_{j4} = 0. \end{aligned}$$

The results of § 7 and § 8 yield, fairly easily, the following:

Lemma (8.1): There exists a number $s_0 > 0$ such that the functions $c_k(\lambda)\, c_{mn}(\lambda)\, E_{rs}(\lambda)$, $c_{jk}(\lambda)\, d_{mn}(\lambda)\, E_{rs}(\lambda)$ and $d_{jk}(\lambda)\, d_{mn}(\lambda)\, E_{rs}(\lambda)$ $(j, k, m, n, r, s, = 1, 2)$ are of bounded variation on (s_0, ∞) $(\lambda = s^2, s$ real).

9. L^p-convergence in the singular case

Let Q be the positive quadrant of the (x, y) plane, R be the closed region of Q bounded by the lines

$$y = 3^{-1/2} x, \quad y = 3^{1/2} x,$$

and

$$S = Q - R, \quad T \quad E\left\{\theta \mid 0 \leq \theta < \frac{\pi}{6} \ \text{ or } \ \frac{\pi}{3} < \theta < \frac{\pi}{2}\right].$$

(Rutovitz[3], p. 33)

We define

$$H_{c,d}(x, y) = \begin{pmatrix} H^{11}_{c,d}(x, y) & H^{21}_{c,d}(x, y) \\ H^{12}_{c,d}(x, y) & H^{22}_{c,d}(x, y) \end{pmatrix}$$

$$\sum_{r=1}^{2} \sum_{s=1}^{2} \int_c^d \varphi_r(0 \mid x, t)\, \varphi_s^T(0 \mid x, t)\, d\rho_{rs}(t), \tag{9.1}$$

where $\rho_{rs}(t) = \lim\limits_{b\to\infty} \rho_{rs}(b, t)$, ($b \to \infty$ through a suitable sequence). In our subsequent studies we closely follow Rutovitz[3] and, therefore, we simply enunciate the results, giving only those steps where we differ significantly from him.

Lemma. (9.1): If $f \in L^p$, $g \in L^q$, then

$$\iint_R [(f(x), g(y))/(x + y)]\, dxdy, \quad \iint_R [(f(x), g(y))/(x + 1)]\, dxdy$$

$$\iint_R [(f(x), g(y))/(y + 1)]\, dxdy = O(|f|_p\, |g|_q).$$

Lemma (9.2): If $f \in L^p$, $g \in L^q$, and $h(t)$ is a function of bounded variation on $[a, \infty)$, $a > 0$, then for $c > a$

$$\iint_R (f(x), g(y)) \int_a^c \frac{\cos}{\sin} xt \frac{\cos}{\sin} yt\, h(t)\, dt\, dx\, dy = O(|f|_p\, |g|_q).$$

Lemma (9.3): There exists a number $c > 0$ such that, for any $w > c$

$$\iint_R (f(x), g(y))\, H^{rs}_{c,w}(x, y)\, dxdy = O(|f|_p\, |g|_q),$$

where $f \in L^p$, $g \in L^q$.

PROOF: We show the calculations for $H^{11}_{c,w}(x, y)$. Writing in full, using (7.4) of Ref. 1 and substituting from (7.4), (7.8), (8.4) and (8.8), we get

$$H^{11}_{c,w}(x, y) = \frac{1}{2\pi} \int_{\sqrt{c}}^{\sqrt{w}} [- \{c_{11}^2 \cos sx \cos sy - c_{11}d_{11} \sin (sx + sy) + d_{11}^2 \sin sx \sin sy$$

$$+ O(1/(1 + x)) + O(1/(1 + y))\} E_{22}(s) + \{c_{11}c_{21} \cos sx \cos sy$$

$$- c_{11}d_{21} \cos sx \sin sy - d_{11}c_{21} \sin sx \cos sy + d_{11}d_{21} \sin sx \sin sy$$

$$+ O(1/(1 + x)) + O(1/(1 + y))\} E_{12}(s) + \{c_{21}c_{11} \cos sx \cos sy$$

$$- c_{21}d_{11} \cos sx \sin sy - c_{11}d_{21} \sin sx \cos sy + d_{21}d_{11} \sin sx \sin sy$$

$$+ O(1/(1 + x)) + O(1/(1 + y))\} E_{21}(s) - \{c_{21}^2 \cos sx \cos sy$$

$$- c_{21}d_{21} \sin (sx + sy) + d_{21}^2 \sin sx \sin sy + O(1/(1 + x))$$

$$+ O(1/(1 + y))\} E_{11}(s)]\, ds.$$

The required result for $H^{11}_{c,w}(x, y)$ now follows from lemmas (8.1), (9.1) and (9.2). Similarly for the other elements, and the lemma, therefore, follows.

Lemma (9.4): There exists a number $c > 0$ such that for any $w > c$ and $f \in L^p, g \in L^q$

$$\int\int_S (f(x), g(y))\, H_{c,w}(x, y)\, dxdy = O(|f|_p\, |g|_q).$$

(*cf.* previous results and Rutovitz[3] pp. 33–35).

Lemma (9.5): Under the conditions of lemma (9.4)

$$\int\int_Q (f(x), g(y))\, H_{c,w}(x, y)\, dxdy = O(|f|_p\, |g|_q).$$

Finally making use of the lemma (9.5), Fatou's lemma, lemma (2.1) and following closely the analysis of Rutovizt[3], pp. 33–35, we obtain the following:

Theorem (9.1): Let $1 < p \leq 2$, $f \in L^p$. Suppose that each element of the matrix $M(x) \in L[0, \infty)$ and (7.1) is satisfied. Then the eigenvector expansion $O_w f$ of a vector f, in the singular case, converges in mean to the vector itself, *i.e.*,

$$|O_w f - f|_p \to 0 \quad \text{as} \quad w \to \infty,$$

where $O_w f = \lim_{b\to\infty} O_{w,b} f$ ($b \to \infty$ through a suitable sequence). Further, there exists a constant c and a number $C(p)$ depending only on p, such that

$$|O_w f|_p \leq C(p)\, |f|_p.$$

for all $w > c$.

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