CONTROL OF VIBRATION PATTERN IN STRETCHED STRINGS BY AUXILIARY APPLIED FORCES

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ABSTRACT

The problems encountered in the study of vibrating systems can be broadly classified into two main categories of analysis and synthesis. This paper is concerned with a third category, viz., *given a system and a prescribed set of forces acting on it, how can the spatial response of the system be altered in a desired manner by the application of additional (control) forces? In particular we consider the problem of controlling the energy of vibration of a driven string over a portion of its length by applying two control forces. Starting from graphical considerations an analytical method has been deduced. The results show that a good control is possible. The effect of varying the point of application of a single force is then discussed.*

Key words: Vibration of strings, vibration control, acoustics, noise control.

1. INTRODUCTION

There are many instances in which it is desirable to alter the spatial distribution of the amplitude or energy of vibration of a continuous system under excitation by a given force into a form that is more suitable for the purpose in hand. One such instance is the sound pressure distribution in an auditorium produced by a loudspeaker of a public address system. Even after a careful choice of the radiation characteristic of the loudspeaker, its position in the hall, etc., it may happen that the energy level at the microphone may be too high to permit the desired gain to be achieved elsewhere in the hall. To avoid the system from going into self-oscillation, several techniques have been tried before,[1, 2] but we might also ask if it is possible to create a region of decreased sound pressure level around the microphone by using an auxiliary set of loudspeakers excited in an appropriate manner. In vibration reduction practice and elsewhere it is worthwhile to consider whether it is simpler to control the vibration amplitude of a limited region rather than attempt to isolate the entire body from vibration. Therefore, we wish to enquire into the possibility of redistributing the energy in a vibrating system by the application of suitable control forces.

Before we attempt to solve such practical control problems, complicated by the varying character of the frequency of the source, lack of exact analytical expressions, etc., we must find suitable methods of controlling the energy of vibration in the case of simple systems which are analytically tractable.[3] The vibrating string has served in the past as an elegant model for understanding many phenomena. However, a large part of the present theory of a vibrating string deals with the analysis of its vibration under the action of prescribed forces. We wish to develop now methods of controlling its vibration.

2. Formulation of the Problem

Consider a stretched string AB (Fig. 1) of length l fixed at its ends and acted on by a force $F = fe^{j\omega t}$ at the point $x = \xi$. In the steady state, the energy of vibration will have a characteristic distribution along its length.

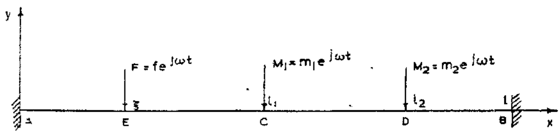

Fig. 1. Stretched string with applied forces.

We desire to alter this distribution in such a way that a relatively larger reduction is obtained in the energy of a certain part CD of the string than in the rest of the string, $(CD)'$. We seek to achieve this by applying two control forces $M_1 = m_1 e^{j\omega t}$ and $M_2 = m_2 e^{j\omega t}$ at C and D respectively. To determine m_1 and m_2 and the reductions p and q that are possible in the energies of the two parts CD and $(CD)'$, we proceed as follows:

Let $Y_F(x, t)$, $Y_{M_1}(x, t)$ and $Y_{M_2}(x, t)$ be the steady state response of the string to the forces F, M_1 and M_2 acting separately. Then, assuming linear behaviour,

$$y_C(x, t) = Y_F(x, t) + Y_{M_1}(x, t) + Y_{M_2}(x, t)$$

is the combined response due to the applied and controlling forces. Let $E_{CD}(f)$ and $E_{(CD)'}(f)$ denote respectively the energies of the parts CD and

$(CD)'$ of the string when the force F is acting alone. Similarly, let $E_{(CD)}(f\,;\, m_1, m_2)$ and $E_{(CD)'}(f\,;\, m_1, m_2)$ stand for the energies in these two parts when the forces F, M_1 and M_2 are acting.

Let us first require that

$$E_{CD}(f\,;\, m_1, m_2) \leq p\, E_{CD}(f) \tag{1}$$

and

$$E_{(CD)'}(f\,;\, m_1, m_2) \geq q\, E_{(CD)'}(f). \tag{2}$$

If we also require that $0 \leq p < 1$ and $q > p$ and determine a set of values m_1 and m_2 satisfying these requirements, we would achieve a larger reduction in the energy in the part CD of the string in comparison to that affected in the rest of the string.*

The response $Y_F(x, t)$ due to the force $F = fe^{j\omega t}$ acting at the point $x = \xi$ is given by[4]

$$Y_F(x, t) = \begin{cases} \dfrac{fe^{j\omega t} \sin \dfrac{\omega}{c}(l - \xi)}{\epsilon\omega c \sin \dfrac{\omega}{c} l} \sin \dfrac{\omega}{c} x & x \leq \xi \\[2ex] \dfrac{fe^{j\omega t} \sin \dfrac{\omega}{c} \xi}{\epsilon\omega c \sin \dfrac{\omega}{c} l} \sin \dfrac{\omega}{c}(l - x) & x \geq \xi \end{cases} \tag{3}$$

where ϵ is the linear mass density and c is the transverse wave-velocity of the string. Analogous expression can be written for the responses $Y_{M_1}(x, t)$ and $Y_{M_2}(x, t)$ due to forces M_1 and M_2 acting at l_1 and l_2 respectively $(f \to M_1, \xi \to l_1$, etc.).

Now, apart from a constant multiplying factor, the energy of vibration of a string due to a force acting on it is given by the integral of the square of the response due to the force. Accordingly we have

$$E_{CD}(f) = \int_{\text{over } CD} Y_F^2(x)\, dx \tag{4}$$

$$E_{(CD)'}(f) = \int_{\text{over } (CD)'} Y_F^2(x)\, dx \tag{5}$$

* While this particular choice of the criterion is suggested by the feedback problem referred to in the previous section, the formulation is general enough to admit other forms of redistribution of energy by a suitable choice of p and q.

$$E_{CD}(f; m_1, m_2) = \int_{\text{over } CD} y_C^2(x)\, dx \tag{6}$$

$$E_{(CD)'}(f; m_1, m_2) = \int_{\text{over } (CD)'} y_C^2(x)\, dx. \tag{7}$$

Clearly, the integrals (4) and (5) are independent of m_1 and m_2 while (6) and (7) depend quadratically on m_1 and m_2. For a given value of f, therefore one can write

$$E_{CD}(f) = g_0 \tag{8}$$

$$E_{(CD)'}(f) = h_0 \tag{9}$$

and

$$E_{CD}(f;\ m_1, m_2) = g(m_1, m_2)$$
$$= g_{11}m_1^2 + g_{12}\, m_1 m_2 + g_{22}m_2^2 + g_1 m_1 + g_2 m_2 + g_0 \tag{10}$$

$$E_{(CD)'}(f\ ;\ m_1, m_2) = h(m_1, m_2)$$
$$= h_{11}m_1^2 + h_{12}m_1 m_2 + h_{22}m_2^2 + h_1 m_1 + h_2 m_2 + h_0 \tag{11}$$

where $g_{11}, g_{12}, \ldots h_{11}, h_{12}, \ldots$, etc. (defined in Appendix A) are functions of ω, ξ, l_1 and l_2.

The requirements (1) and (2) imply therefore

$$g(m_1, m_2) \leq pg_0 \tag{12}$$

$$h(m_1, m_2) \geq qh_0. \tag{13}$$

3. Methods of Solution

The values of m_1 and m_2 satisfying (12) and (13) could be obtained graphically or by using Lagrange multiplier or by the more general techniques of optimization theory. Here we consider the first two methods as they provide an intuitive approach to the problem.

3.1. *Graphical Solution*

The quadratic forms (10) and (11) are positive definite (being energy functions) and therefore equations $g(m_1, m_2) =$ a constant and $h(m_1, m_2) =$ a constant represent ellipses in the $m_1 - m_2$ plane (These ellipses will henceforth be called the g-curve and the h-curve respectively). Thus for a given value of p and q ($0 \leq p < 1$ and $q > p$) the inequalities (12) and (13) imply that the acceptable values of m_1 and m_2 are those that lie inside the curve $g(m_1, m_2) = pg_0$ and outside the curve $h(m_1, m_2) = qh_0$. The shaded region in Fig. 2 represents these values.

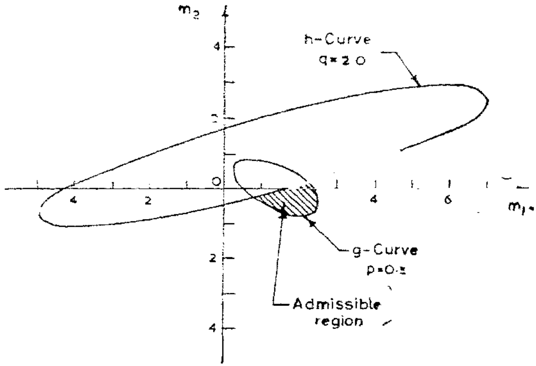

FIG. 2. g- and h-curves showing the admissible region.

The fact that there is a region in the $m_1 - m_2$ plane means that there is actually a large number of possible m_1 and m_2 values which meet the requirement. However, for a given q, the point which corresponds to a minimum p can be easily obtained from the graphical plot by drawing the g-curve which just touches the h-curve and noting the corresponding value of p. This minimum value depends upon the value of ξ, l_1, l_2 and ω.

An admissible region in $m_1 - m_2$ plane is obtained only when the g-curve intersects or lies completely outside the h-curve. Otherwise no admissible region exists and the value of p may have to be increased or the value of q decreased to get an admissible region. If the values of p and q required to get an admissible region are not compatible with the requirements then more control parameters may have to be introduced. However, the g- and h-curves do not, in general, lie in a completely arbitrary manner in the $m_1 - m_2$ plane for, these curves must pass through the origin in the case $p = 1$, $q = 1$. This fact enhances the possibility of the g- and h-curves intersecting each other for reasonable values of p and q.

3.2. *Analytical Method*

It was pointed out that, if the g-curve is shrunk as in Fig. 3 till it touches the h-curve for fixed q then the point of contact P (m_1', m_2') would correspond to a minimum p. This fact suggests an analytical method of solving the same problem.

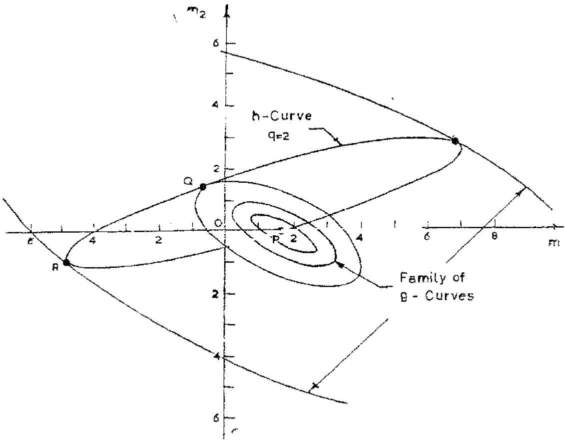

FIG. 3. Family of g-curves touching the h-curve at four points of tangency.

In order to find m_1, m_2 and p for a given q, the conditions existing at the common point P are first put in the mathematical form as follows:

(i) The point P must lie on both the g-curve and h-curve, *i.e.*,

$$h(m_1', m_2') = qh_0 \tag{14}$$

$$g(m_1', m_2') = pg_0. \tag{15}$$

(ii) At P, the g-curve and the h-curve must have the same tangent, *i.e.*,

$$\left.\frac{\partial g/\partial m_1}{\partial g/\partial m_2}\right|_{\substack{m_1=m_1'\\ m_2=m_2'}} = \left.\frac{\partial h/\partial m_1}{\partial h/\partial m_2}\right|_{\substack{m_1=m_1'\\ m_2=m_2'}} \tag{16}$$

Making use of (10) and (11), the condition (16) can be rewritten explicitly as

$$a_{11}m_1'^2 + a_{12}m_1'm_2' + a_{22}m_2'^2 + a_1m_1' + a_2m_2' + a_0 = 0 \tag{17}$$

where

$$
\begin{aligned}
a_{11} &= 2g_{11}\, h_{12} - 2h_{11}\, g_{12} \\
a_{12} &= 4h_{22}\, g_{11} - 4h_{11}\, g_{22} \\
a_{22} &= 2h_{22}\, g_{12} - 2h_{12}\, g_{22} \\
a_1 &= h_{12}\, g_1 + 2h_2\, g_{11} - 2h_{11}\, g_2 - h_1\, g_{12} \\
a_2 &= g_{12}\, h_2 + 2h_{22}\, g_1 - 2h_1\, g_{22} - h_{12}\, g_2 \\
a_0 &= h_2\, g_1 - h_1\, g_2.
\end{aligned} \tag{18}
$$

First the nonlinear simultaneous equations (14) and (17) are solved for m_1' and m_2' by Sylvester's method.[5] This leads to four pairs of values for m_1' and m_2' corresponding to the four possible points of tangency of the curve $h(m_1, m_2) = qh_0$ with the family of g-curves, as sketched out in Fig. 3. Each of these pairs of values of m_1' and m_2' when substituted in (15) leads to a different value of p. The smallest value obtained for p determines the maximum possible reduction in the energy of the part CD of the string. The corresponding values of m_1' and m_2' give the control forces required at C and D.

Similar arguments can be construed for the case where the energy over CD is specified (*i.e.*, p is given) and the energy of the remaining part is to be maximized.

The Lagrange multiplier technique: The above problem can be put in the format of a constrained minimization problem as follows:

Minimize $g(m_1, m_2)$ under constraint

$$h(m_1, m_2) = qh_0. \tag{19}$$

An application of Lagrange multiplier technique leads to

$$\partial g/\partial m_1 + \lambda\, \partial h/\partial m_1 = 0 \tag{20}$$

$$\partial g/\partial m_2 + \lambda\, \partial h/\partial m_2 = 0 \tag{21}$$

where λ is the Lagrange multiplier. The optimum values of m_1 and m_2 are obtained by solving equations (19), (20) and (21) for m_1, m_2 and λ. Eliminating λ, we get again equation (16).

3.3. *Example*

The method is now illustrated with a numerical example. With $l = 1$, $\xi = 0{\cdot}1l$, $\omega l/c = 40{\cdot}0$, $l_1 = 0{\cdot}85l$, $l_2 = 0{\cdot}95l$, $F = 1$ and $q = 2$, the pairs of values of m_1' and m_2' satisfying (14) and (17) are determined. Using these in (15) the minimum value of p is found. Equations (14), (15) and (17) now become respectively

$$0 \cdot 042\, m_1^2 - 0 \cdot 209\, m_1 m_2 + 0 \cdot 359\, m_2^2 + 0 \cdot 108\, m_1 - 0 \cdot 443\, m_2 - 0 \cdot 272 = 0 \tag{22}$$

$$0 \cdot 0136\, m_1^2 + 0 \cdot 0244\, m_1 m_2 + 0 \cdot 0338\, m_2^2 - 0 \cdot 0389\, m_1 - 0 \cdot 0350\, m_2 + (1 - p)\, 0 \cdot 0278 = 0 \tag{23}$$

$$0 \cdot 0077\, m_1^2 - 0 \cdot 0138\, m_1 m_2 + 0 \cdot 0316\, m_2^2 - 0 \cdot 0038\, m_1 - 0 \cdot 0526\, m_2 + 0 \cdot 0209 = 0. \tag{24}$$

The solution of (22) and (24) leads to

$$\begin{bmatrix} m_1' \\ m_2' \end{bmatrix} = \begin{bmatrix} 1 \cdot 43 \\ -0 \cdot 01 \end{bmatrix}, \quad \begin{bmatrix} 6 \cdot 26 \\ 3 \cdot 05 \end{bmatrix}, \quad \begin{bmatrix} -0 \cdot 76 \\ 1 \cdot 42 \end{bmatrix}, \quad \begin{bmatrix} -5 \cdot 10 \\ -1 \cdot 02 \end{bmatrix} \tag{25}$$

The pair of values of m_1' and m_2' when substituted in (23) leads to the values of p equal to $0 \cdot 001$, $35 \cdot 64$, $2 \cdot 07$ and $27 \cdot 97$ respectively. The pair $m_1' = 1 \cdot 43$ and $m_2' = -0 \cdot 01$ actually reduce the energy in the desired part of the string to almost zero value.

4. Control Using a Single Force

Since the required degree of control cannot always be achieved by varying m_1 and m_2 only, the effect of changing the points of application of forces is studied by considering a single control force whose magnitude and m position b can be chosen suitably.

As before we compute the energy of the parts CD and $(CD)'$ due to the applied force f acting at ξ and the control force m acting at b and write

$$E_{CD} = g\,(m, b)$$

$$E_{(CD)'} = h\,(m, b).$$

We now require the minimum value of $g\,(m, b)$ under constraint

$$h\,(m, b) = q h_0. \tag{26}$$

The functions $g\,(m, b)$ and $h\,(m, b)$ are given in Appendix B, and are seen to be quadratics in m for a given value of b.

Application of Lagrange multiplier technique for this case leads to the solution of transcendental equations which is not straight-forward.

A graphical analysis in $m - b$ plane is not straight-forward since m becomes complex for certain values of b. Therefore, we seek real values of m and b satisfying

$$h(m, b) = qh_0 \tag{27}$$

and

$$g(m, b) = pg_0 \tag{28}$$

The common root between (27) and (28) for suitable value of b would achieve the required degree of control. To consider this possibility, we let b vary in small steps and search for roots that are as nearly equal as possible. Table I lists the roots in the region $0{\cdot}10 \leq b \leq 0{\cdot}85$ at intervals of 0·01.

TABLE I

Average value of m *from the roots of the eqns* (27) *and* (28)

No.	b	Roots of equation (28)	Roots of equation (27)	Average value of m
1	0·40	0·622 and —2·397	—2·626 and —2·630	—2·513
2	0·48	2·154 and —0·551	2·204 and 2·204	2·179
3	0·56	0·502 and 1·967	—1·903 and —1·904	—1·935
4	0·64	1·820 and —0·465	1·680 and 1·680	1·750
5	0·85	1·560 and —4·148	1·430 and 1·429	1·500

Actual plotting of energy distribution diagrams show that it is not very sensitive to m, but depends strongly on the point of application b. This fact enhances the value of this method because an approximate m selected from this method gives good results. In fact, it turns out that with a single force the energy reduction takes place over the entire length from b to l instead of over the portion CD only (as was the case when two control forces weie available). This is because the standing wave patterns formed by the two forces so adjust themselves that a cancellation of energy takes place from b to l while some augmentation takes place from 0 to b. A promising modification of this result may be applied to the reduction of the transmission of sounds of particular frequencies in ventilation ducts. Since the m and b values calculated are applicable to only one frequency, that particular frequency component will be largely eliminated although the other frequency components will still be present.

5. Conclusions

We can expect much better control for the case of two control forces whose positions and magnitudes are adjustable. However, for such a formulation, the equations become very much complicated and the problem becomes one of minimizing a nonlinear function of several variables under a nonlinear constraint. Such problems cannot be solved by any of the methods suggested so far and more sophisticated techniques have to be used.

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Appendix A

Energy Expressions for the Two-Variable Case

$$\int_{l_1}^{l_2} y_c^2\,(x)\,dx = \int_{l_1}^{l_2} \Big[\frac{F}{\epsilon\omega c \sin(\omega l/c)} \sin(\omega\xi/c) \sin\{\omega(l-x)/c\}$$

$$+ \frac{m_1}{\epsilon\omega c \sin(\omega l/c)} \sin(\omega l_1/c) \sin\{\omega(l-x)/c\}$$

$$+ \frac{m_2}{\epsilon\omega c \sin(\omega l/c)} \sin\{\omega(l-l_2)/c\} \sin(\omega x/c)\Big]^2 dx$$

$$= g_{11}m_1^2 + g_{12}m_1m_2 + g_{22}m_2^2 + g_1m_1 + g_2m_2 + g_0 \tag{A1}$$

where (assuming $F = 1$)

$$g_{11} = KB_1 \sin^2(\omega l_1/c)$$

$$g_{22} = KB_2 \sin^2\{\omega(l-l_2)/c\}$$

$$g_{12} = 2KB_3 \sin(\omega l_1/c) \sin\{\omega(l-l_2)/c\} \tag{A2}$$

$$g_1 = 2KB_1 \sin(\omega\xi/c) \sin(\omega l_1/c)$$

$$g_2 = 2KB_3 \sin(\omega\xi/c) \sin\{\omega(l - l_2)/c\}$$

$$g_0 = KB_1 \sin^2(\omega\xi/c)$$

and

$$K = \frac{1}{2}\left[\frac{1}{\epsilon\omega c \sin(\omega l/c)}\right]^2$$

$$B_1 = (l_2 - l_1) + \frac{\sin\{2\omega(l - l_2)/c\}}{2\omega/c} - \frac{\sin\{2\omega(l - l_1)/c\}}{2\omega/c}$$

$$B_2 = (l_2 - l_1) - \frac{\sin(2\omega l_2/c)}{2\omega/c} + \frac{\sin(2\omega l_1/c)}{2\omega/c} \tag{A3}$$

$$B_3 = \frac{\sin\{\omega(l - 2l_1)/c\}}{2\omega/c} - \frac{\sin\{\omega(l - 2l_2)/c\}}{2\omega/c} - (l_2 - l_1)\cos(\omega l/c)$$

$$\int_{l_1}^{l_2} y_F^2(x)\,dx = \int_{l_1}^{l_2}\left[\frac{F}{\epsilon\omega c \sin(\omega l/c)} \sin(\omega\xi/c) \sin\{\omega(l - x)/c\}\right]^2 dx$$

$$= g_0. \tag{A4}$$

Similarly

$$\int_{\substack{\text{outside}\\(l_1,\, l_2)}} y_c^2(x)\,dx = 2K \int_0^{\xi} [\sin\{\omega(l - \xi)/c\} \sin(\omega x/c)$$

$$+ m_1 \sin\{\omega(l - l_1)/c\} \sin(\omega x/c)$$

$$+ m_2 \sin\{\omega(l - l_2)/c\} \sin(\omega x/c)]^2\, dx$$

$$+ 2K \int_{\xi}^{l_1} [\sin(\omega\xi/c) \sin\{\omega(l - x)/c\}$$

$$+ m_1 \sin\{\omega(l - l_1)/c\} \sin(\omega x/c)$$

$$+ m_2 \sin\{\omega(l - l_2)/c\} \sin(\omega x/c)]^2 dx$$

$$+ 2K \int_{l_2}^{l} [\sin(\omega\xi/c) \sin\{\omega(l - x)/c\}$$

$$+ m_1 \sin(\omega l_1/c) \sin\{\omega(l - x)/c\}$$

$$+ m_2 \sin(\omega l_2/c) \sin\{\omega(l - x)/c\}]^2\, dx$$

$$= h_{11}m_1^2 + h_{12}m_1m_2 + h_{22}m_2^2 + h_1m_1 + h_2m_2 + h_0 \tag{A5}$$

where

$$h_{11} = K[(D_1 + D_2)\sin^2\{\omega(l - l_1)/c\} + D_3 \sin^2(\omega l_1/c)]$$

$$h_{12} = 2K[(D_1 + D_2)\sin\{\omega(l - l_1)/c\}\sin\{\omega(l - l_2)/c\} + D_3 \sin(\omega l_1/c)\sin(\omega l_2/c)]$$

$$h_{22} = K[(D_1 + D_2)\sin^2\{\omega(l - l_2)/c\} + D_3 \sin^2(\omega l_2/c)]$$

$$h_1 = 2K[D_1 \sin\{\omega(l - \xi)/c\}\sin\{\omega(l - l_1)/c\} + D_3 \sin(\omega\xi/c)\sin(\omega l_1/c) + D_4 \sin(\omega\xi/c)\sin\{\omega(l - l_1)/c\}]$$

$$h_2 = 2K[D_1 \sin\{\omega(l - \xi)/c\}\sin\{\omega(l - l_2)/c\} + D_4 \sin(\omega\xi/c) \times \sin\{\omega(l - l_2)/c\} + D_3 \sin(\omega\xi/c)\sin(\omega l_2/c)]$$

$$h_0 = K[D_1 \sin^2\{\omega(l - \xi)/c\} + (D_3 + D_5)\sin^2(\omega\xi/c)] \qquad (A6)$$

and

$$D_1 = \xi - \frac{\sin(2\omega\xi/c)}{(2\omega/c)}$$

$$D_2 = (l_1 - \xi) - \frac{\sin(2\omega l_1/c)}{2\omega/c} + \frac{\sin(2\omega\xi/c)}{2\omega/c}$$

$$D_3 = (l - l_2) - \frac{\sin\{2\omega(l - l_2)/c\}}{2\omega/c}$$

$$D_4 = \frac{\sin\{\omega(l - 2\xi)/c\}}{2\omega/c} - \frac{\sin\{\omega(l - 2l_1)/c\}}{2\omega/c} - (l_1 - \xi\ \cos(\omega l/c) \qquad (A7)$$

$$D_5 = (l_1 - \xi) + \frac{\sin\{2\omega(l - l_1)/c\}}{2\omega/c} - \frac{\sin\{2\omega(l - \xi)/c\}}{2\omega/c}$$

$$\int_{\substack{\text{outside}\\(l_1,\ l_2)}} y_F^2(x)\,dx = 2K\int_0^{\xi}[\sin\{\omega(l - \xi)/c\}\sin(\omega x/c)]^2\,dx + 2K\int_{\xi}^{l_1}[\sin(\omega\xi/c)\sin\{\omega(l - x)/c\}]^2\,dx + 2K\int_{l_2}^{l}[\sin(\omega\xi/c)\sin\{\omega(l - x)/c\}]^2\,dx = h_0. \qquad (A8)$$

APPENDIX B

Energy Expressions for the Case of Single Control Force

$$g(m, b) = \int_{l_1}^{l_2} y_C^2(x)\,dx$$

$$= (B \sin^2 \omega b/c)\, m^2 + (B \sin \omega\xi/c \sin \omega b/c)\, m + B \sin^2 \omega\xi/c \tag{B1}$$

where

$$B = (l_2 - l_1) + \frac{\sin\{2\omega(l - l_2)/c\}}{2\omega/c} - \frac{\sin\{2\omega(l - l_1)/c\}}{2\omega/c} \tag{B2}$$

$$g_0 = \int_{l_1}^{l_2} Y_F^2(x)\,dx = KB \sin^2(\omega\xi/c) \tag{B3}$$

where

$$K = \frac{1}{2}\left[\frac{1}{\epsilon\omega c \sin(\omega l/c)}\right]^2.$$

Similarly,

$$h(m, b) = \int_{\substack{\text{outside}\\(l_1, l_2)}} y_C^2(x)\,dx$$

$$= [k_1(b) \sin^2\{\omega(l - b)/c\} + k_2(b) \sin^2(\omega b/c)]\, m^2$$
$$+ 2[k_3(b) \sin\{\omega(l - b)/c\} + k_2(b) \sin(\omega\xi/c) \sin(\omega b/c)]\, m$$
$$+ k_4(b) \tag{B4}$$

where

$$k_1(b) = \frac{1}{2}\left[b - \frac{\sin(2\omega b/c)}{2\omega/c}\right]$$

$$k_2(b) = \frac{1}{2}\left[(l_1 - b) + \frac{\sin(2\omega\{l - l_1)/c\}}{2\omega/c} - \frac{\sin\{2\omega(l - b)/c\}}{2\omega/c}\right.$$
$$\left.+ \frac{1}{2}\left[(l - l_2) - \frac{\sin\{2\omega(l - l_2)/c\}}{2\omega/c}\right]\right]$$

$$k_3(b) = \frac{1}{2} \sin\{\omega(l - \xi)/c\}\left[\xi - \frac{\sin(2\omega\xi/c)}{2\omega/c}\right]$$
$$+ \frac{1}{2} \sin(\omega\xi/c)\left[\frac{\sin\{\omega(l - 2\xi)/c\}}{2\omega/c} - \frac{\sin\{\omega(l - 2b)/c\}}{2\omega/c}\right.$$
$$\left. - (b - \xi) \cos(\omega l/c)\right]$$

$$k_4(b) = \frac{1}{2}\sin^2\{\omega(l-\xi)/c\}\left[\xi - \frac{\sin(2\omega\xi/c)}{2\omega/c)}\right]$$

$$+ \sin^2(\omega\xi/c)\left[\frac{1}{2}(b-\xi) + \frac{\sin\{2\omega(l-b)/c\}}{2\omega/c}\right.$$

$$\left. - \frac{\sin\{2\omega(l-\xi)/c\}}{2\omega/c} + k_2(b)\right] \tag{B5}$$

$$h_0 = \int_{\substack{\text{outside}\\(l_1,\,l_2)}} Y_F^2(x)\,dx = K[D_1\sin^2\{\omega(l-\xi)/c\}$$

$$+ (D_3 + D_5)\sin^2(\omega\xi/c)] \tag{B6}$$

where

$$D_1 = \xi - \frac{\sin(2\omega\xi/c)}{2\omega/c}$$

$$D_3 = (l - l_2) - \frac{\sin\{2\omega(l-l_2)/c\}}{2\omega/c}$$

$$D_5 = (l_1 - \xi) + \frac{\sin\{2\omega(l-l_1)/c\}}{2\omega/c} - \frac{\sin\{2\omega(l-\xi)/c\}}{2\omega/c} \tag{B7}$$

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