Stability of nonlinear systems through energy-like Liapunov functions

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Abstract

The results derived in earlier works on the stability of linear vector equation of the type, $\ddot{x} + H\dot{x} + Gx = 0$, which were obtained through the use of energy-like Liapunov functions, are extended in this paper to nonlinear and time varying vector equations of the type, $\ddot{x} + h(t, x, \dot{x}) + g(t, x) = 0$. The derived results are useful only for certain types of $g(t, x)$. Stability analysis of a discretized partial differential equation through the method described can help in choosing a suitable Liapunov functional for the original equation.

Key words: Stability, Nonlinear system, Distributed parameter system.

1. Introduction

Suppose a nonlinear system is described by the vector differential equation

$$\dot{x} = Ax + f(x), \quad t \in [0, \infty) \tag{1}$$

where x is a $(m \times 1)$ vector, A is a $(m \times m)$ matrix, $\dot{x} = (dx/dt)$ and $f_i(\cdot)$ are nonlinear functions. The system can be treated as a feedback system, with the forward linear part described by

$$\dot{x} = Ax + u \tag{2}$$

and the nonlinear feedback part described by

$$u = f(x). \tag{3}$$

Methods are available to study the stability of such systems[1, 2].

Suppose a system is described by a set of second order equations of the form

$$\ddot{x} + h(t, x, \dot{x}) + g(t, x) = 0 \tag{4}$$

where x, h and g are $(n \times 1)$ vectors, $\dot{x} = (dx/dt)$, $\ddot{x} = (d\dot{x}/dt)$. It is possible to rewrite equation (4) as

$$\dot{y} = F(t, y) \tag{5}$$

where y is a $(2n \times 1)$ vector formed by $(x, \dot{x})$. Since F can be written as $By + (F - By)$, equation (4) can be converted to the form of (1). However, it is sometimes advantageous to study the stability of equation (4) as it is. For example, consider the following pair of scalar equations:

$$\ddot{y} + \dot{x} + \dot{y} + y = 0$$

$$\ddot{x} + \dot{x} + (1 + e^{-t})\, x = 0. \tag{6}$$

Theorem 1 of the following section is directly applicable to this system. It shows that the system is globally asymptotically stable. It is not necessary to convert the pair of equations (6) into the form of equation (1).

Results on the linear form of equation (4) are available[3, 4]. Stability conditions are obtained in this work through the use of energy-like Liapunov functions. Similar methods are used here.

Although the theorems presented here are applicable to ordinary differential equations only, an interesting application is to discretized partial differential equation. The result is interesting because in the limit of zero step size used for discretization the result yields a sufficient condition for the stability of the original partial differential equation.

Let $\| x \|$ represent the norm of x in R^n, and let $\| (x, \dot{x}) \|_{2n}$ represent the norm of $(x, \dot{x})$ in R^{2n}. Let $S_0(b)$ represent the open ball $\| (x, \dot{x}) \|_{2n} < b$. In the following theorems, by the statement " solution $x = 0$ of equation (4) is stable in $S_0(b) \subset R^{2n}$ " it is implied that the following conditions are satisfied.

(C1) If $(x(t_0), \dot{x}(t_0)) \in S_0(b)$, then there exists on $0 < a < \infty$ such that,

$$(x(t), \dot{x}(t)) \in S_0(ab), \text{ for all } t \geq t_0 \geq 0.$$

(C2) For every $\varepsilon > 0$ there is a $\delta > 0$ such that $(x(t_0), \dot{x}(t_0)) \in S_0(\delta) \rightarrow (x(t), \dot{x}(t)) \in S_0(\varepsilon)$ for all $t \geq 0$.

The solution $x = 0$ is said to be stable if condition (C2) is satisfied. It is asymptotically stable in $S(b)$, if it is stable in $S(b)$ and $\| (x, \dot{x}) \|_{2n} \rightarrow 0$ as $t \rightarrow \infty$ if $(x(t_0), \dot{x}(t_0)) \in S(b)$ for $t_0 \geq 0$. If a and δ in conditions (C1) and (C2) are independent of t_0 stability is uniform with respect to t_0.[5]

It is assumed that h and g in equation (4) are such that solution $x(t)$ and $\dot{x}(t)$ are continuous.

2. Results

Now we state the main theorems of the paper.

Theorem 1 *a* : If there are positive a, m_1 and m_2 such that in the region $S(a)$ of R^{2n}, the following conditions are satisfied for all $t \geq 0$;

(i) $h'(t, x, \dot{x}) \cdot \dot{x} \geq 0$

(ii) there is a scalar $p(t, x)$ continuous with respect to t and x such that

(a) $m_1 \| x \|^2 \leq p(t, x) \leq m_2 \| x \|^2$

(b) $\dfrac{\partial p(t, x)}{\partial x_i} = g_i(t, x), i = 1, 2, \cdots, n$

(c) $\dfrac{\partial p(t, x)}{\partial t}$ is non-positive and piecewise continuous with respect to x and t

then the trivial solution $x = 0$ of equation (4) is stable in $S_0(\bar{m}_1 a/\bar{m}_2)$, where

$$\bar{m}_1 = \min(\tfrac{1}{2}, m_1) \tag{7}$$

$$\bar{m}_2 = \max(\tfrac{1}{2}, m_2) \tag{8}$$

Theorem 1 *b*: If condition (ii) of Theorem 1 *a* is satisfied and there exist positive constants M, k, c_1 and c_2 such that for all $(x, \dot{x}) \in S(a)$ and all $t > 0$,

(iii) $h'(t, x, \dot{x}) \cdot \dot{x} \geq M \| \dot{x} \|^2$

(iv) $h(t, x, \dot{x}) \geq k \| \dot{x} \|$

(v) $\dfrac{\partial p(t, x)}{\partial t} \to 0$, as $t \to \infty$

(vi) for any fixed x_a, where $\| x_a \| \leq a$, $g(t, x_a) \to \bar{g}(x_a)$, as $t \to \infty$ where $\bar{g}$ is a time invariant vector function.

(vii) $c_1 \| x \| \leq \| \bar{g}(x) \| \leq c_2 \| x \|$

then, equilibrium $x = 0$ of equation (4) is asymptotically stable in $S(\bar{m}_1 a/\bar{m}_2)$.

Theorem 2: If there exist positive constants M_1, a and ε such that for all $t > 0$,

(i) $g(t, x) \leq M_1 \| x \|$, if $\| x \| \leq a$

(ii) $h'(t, x, \dot{x}) \cdot \dot{x} \geq \epsilon \| \dot{x} \|$, if $\| \dot{x} \| \leq a$

then the solution $x = 0$ of equation (4) is stable.

Theorem 3: Suppose equation (4) can be written as

$$\ddot{x} + h(t, x, \dot{x}) + C(t)x + N(x) = 0 \tag{9}$$

where $C(t)$ is a symmetric matrix for all $t \geq 0$, and $N(x)$ is a vector whose element n_i is a function of x_i alone.

If, in some region containing the origin as an interior point and for all $t \geq 0$

(i) $\frac{1}{2} x' Cx + \sum_{1}^{n} \int_{0}^{x_i} n_i(x_i)\, dx_i > 0, \; x \neq 0$

(ii) dC/dt is negative semi-definite

(iii) $(h' \cdot \dot{x}) \geq 0$

then the equilibrium $x = 0$ is stable.

3. Proofs

Proof of Theorem 1 *a* : Let a scalar $V(t, x, \dot{x})$ be introduced such that

$$V(t, x, \dot{x}) = p(t, x) + \tfrac{1}{2}(\dot{x}', \dot{x}). \tag{10}$$

It is seen from equations (7), (8) and (10) and condition (ii *a*) of Theorem 1 that

$$\bar{m}_1 \| (x, \dot{x}) \|_{2n}^2 \leq V(t, x, \dot{x}) \leq \bar{m}_2 \| (x, \dot{x}) \|_{2n}^2 . \tag{11}$$

Hence, V can be considered as a possible Liapunov function for establishing conditions for the stability of equation (4). If the vector x is considered analogous to a displacement vector, then $\dot{x}$ is analogous to velocity and $\frac{1}{2}(\dot{x} \cdot \dot{x})$ is similar to kinetic energy. Since, the scalar p can be expressed as a gradient of a vector g, p is similar to potential energy. Hence, Liapunov function V can be said to be an energy-like function.

Let $w(t)$ be introduced such that

$$w(t) = V\big(t, x(t), \dot{x}(t)\big). \tag{12}$$

Note that $w(t)$ is a continuous function of t because of the assumption that p and x are continuous. It is seen from equation (4) and (10) that

$$\dot{w} = \frac{\partial V}{\partial t} + \sum_{1}^{n} \left(\frac{\partial V}{\partial x_i} \dot{x}_i + \frac{\partial V}{\partial \dot{x}_i} \ddot{x}_i \right) \tag{13}$$

Substituting for $\ddot{x}_i$ from equation (4) in equation (13) and making use of condition (ii *b*) of Theorem 1, we get,

$$\dot{w} = \frac{dp}{dt} - h'(t, x, \dot{x}) \cdot \dot{x}. \tag{14}$$

It follows from conditions (i), (ii *c*) and equation (12) that

$$\dot{w}(t) > 0 \tag{15}$$

for all t for which $\big(x(t), \dot{x}(t)\big) \in S(a)$.

We shall now prove, by contradiction, that if

$$\| \big(x(t_0), \dot{x}(t_0)\big) \|_{2n} < \varepsilon < \bar{m}_1 a / \bar{m}_2 \leq a \tag{16}$$

then

$$\|(x(t), \dot{x}(t))\|_{2n} < \bar{m}_2\varepsilon/\bar{m}_1 < a \text{ for all } t \geq t_0. \tag{17}$$

Suppose, the assertion is not true. Then there must exist one or more values of $t > t_0$ for which $\|(x(t), \dot{x}(t))\|_{2n}$ is equal to $(\bar{m}_2\varepsilon/\bar{m}_1)$. Let the smallest of all such values of t be t_1. Then by definition of t_1,

$$\|(x(t_1), \dot{x}(t_1))\|_{2n} = \bar{m}_2\varepsilon/\bar{m}_1 < a \tag{18}$$

It also follows that

$$\|(x(t), \dot{x}(t))\|_{2n} < \bar{m}_2\varepsilon/\bar{m}_1 \text{ for } t \in [t_0, t_1). \tag{19}$$

It is seen from condition (11) that relations (16) and (18) imply that

$$w(t_0) < \bar{m}_2\varepsilon \tag{20}$$

$$w(t_1) > \bar{m}_2\varepsilon \tag{21}$$

At the same time, it is seen from equations (15) and (19) that

$$\dot{w}(t) \leq 0 \text{ for all } t \in (t_0, t_1). \tag{22}$$

Since, $w(t)$ is continuous, it is obvious that if inequalities (20) and (22) are satisfied then inequality (21) cannot be true. Hence, the assumption that there exists a t_1 for which equation (18) is satisfied is not true. It is therefore concluded that

$$\|(x(t), \dot{x}(t))\|_{2n} < \bar{m}_2\varepsilon/\bar{m}_1 < a \text{ for all } t > t_0. \tag{23}$$

and, furthermore from relations (10), (11), (15) and (23) it is concluded that

$$0 \leq w(t) < \bar{m}_2^2\, \varepsilon/m_1 \text{ for all } t > t_0 \tag{24}$$

and

$$\dot{w}(t) \leq 0 \text{ for all } t > t_0. \tag{25}$$

This proves Theorem 1 *a*.

Proof of Theorem 1 *b*: We shall now show that if conditions (i)–(vii) of Theorem 1 are satisfied, then for any initial condition which satisfied inequality (16) we have

$$\|(x(t), \dot{x}(t))\|_{2n} \to 0, \text{ as } t \to \infty. \tag{26}$$

Limit (26) along with inequality (17) establishes asymptotic stability.

It has been shown in the proof of Theorem 1 *a* that inequality (16) implies inequalities (24) and (25). It follows from inequality (24) that

$$\int_0^\infty \dot{w}(t)\, dt < \varepsilon\bar{m}_2^2/\bar{m}_1. \tag{27}$$

Since, $\dot{w}$ is piecewise continuous and nonpositive, inequality (27) can be satisfied only if

$$\dot{w}(t) \to 0, \text{ as } t \to \infty. \tag{28}$$

It is seen from equation (14) and conditions (iii), (iv) and (v) of Theorem 1 that limit (28) implies that

$$\| \dot{x}(t) \| \to 0, \text{ as } t \to \infty \tag{29}$$

$$\| h(t, x, \dot{x}) \| \to 0, \text{ as } t \to \infty \tag{30}$$

It follows from limit (29) that

$$x(t) \to \bar{x}, \text{ as } t \to \infty. \tag{31}$$

where $\bar{x}$ is a constant vector. Let us assume that $\bar{x} \neq 0$. Then, it is seen from condition (vii) that there is at least one i, say m, such that $\bar{g}_m(\bar{x}) \neq 0$.

From conditions (vi), (vii) and limits (30) and (31) it follows that given a pair of positive ε and δ, we can find a T such that

$$h_m(t, x, \dot{x}) \leq \delta \text{ for all } t \geq T \tag{32}$$

$$| g_m(t, x(t)) - \bar{g}_m(\bar{x}) | < \varepsilon \text{ for all } t \geq T. \tag{33}$$

Let us choose ε and δ such that

$$\bar{g}_m(\bar{x}) - (\varepsilon + \delta) > 0. \tag{34}$$

It can be seen from equation (34) and the mth component of equation (4) that

$$\ddot{x}_m(t)\, sgn\,(\bar{g}_m(\bar{x})) \leq (\varepsilon + \delta) - | \bar{g}_m(\bar{x}) | < 0, t \geq T \tag{35}$$

Inequality (35) imply that $sgn\,(\bar{g}_m(\bar{x}))\, \dot{x}_m(t) \to -\infty$. But we know that $\dot{x}_m \to 0$. Hence, the supposition that $\bar{x} \neq 0$ is not true, and we conclude that $x(t) \to 0$ and Theorem 1 b is proved.

Proof of Theorem 2: Let aV be chosen as

$$V(x, \dot{x}) = p(x) + \tfrac{1}{2} \dot{x}' \cdot \dot{x} \tag{36}$$

where $p(x)$ is positive definite in R^n and there are positive constants a and M_2 such that

$$\left\| \frac{dp}{dx} \right\| \leq M_2 \| x \| \text{ for all } (x, \dot{x}) \in S(a). \tag{37}$$

Function V is positive definite in $S(a)$.

From equations (4), (36) and condition (ii) of Theorem 2, it is found that

$$\dot{V} = \left(\frac{dp}{dx} - g\right)' \cdot \dot{x} - h' \cdot \dot{x} \leq \left(\left\| \frac{dp}{dx} - g \right\| - \varepsilon\right) \| \dot{x} \| \tag{38}$$

From conditions (i), (37) and inequality (38)

$$\dot{V} \leq ((M_1 + M_2) \| x \| - \varepsilon) \cdot \| \dot{x} \| \tag{39}$$

provided $(x, \dot{x}) \in S(a)$. If an $a > 0$ is chosen smaller than both a and $\varepsilon/(M_1 + M_2)$ then, it is seen from inequality (39) that $\dot{V} \leq 0$ in $S(a)$. Hence, solution $x = 0$ of equation (4) is stable.

Proof of Theorem 3: Let

$$V(t, x, \dot{x}) = \tfrac{1}{2} x' C x + \sum_{1}^{n} \int_{0}^{x_i} n_i(x_i)\, dx_i + \tfrac{1}{2} (\dot{x} \cdot \dot{x}). \tag{40}$$

It can be seen from condition (i) of Theorem 3 that V is positive definite. It is also seen from equations (4), (40) and conditions (ii) and (iii) of Theorem 3 that

$$\dot{V} = x' \frac{dC}{dt} x - h' \cdot \dot{x} \leq 0. \tag{41}$$

Hence solution $x = 0$ of equation (4) is stable.

4. Examples

Example 1 : Consider the scalar equation

$$\ddot{x} + h(t, x, \dot{x}) + C(t) x + f(x) = 0. \tag{42}$$

Application of Theorem 3 shows that equilibrium $x = 0$ is globally stable if for all x and all $t \geq 0$,

(i) $\frac{dC}{dt} \leq 0$

(ii) $h(t, x, \dot{x}) \dot{x} \geq 0$

(iii) $-\int_0^x f(p)\, dp \leq \frac{1}{2} C(t) x^2.$

Example 2 : Consider a scalar equation

$$\ddot{x} + h(\dot{x}) - kx = 0 \tag{43}$$

where $k > 0$. This system is obviously unstable if h does not satisfy condition (ii) of Theorem 2.

Suppose, $h(\dot{x})$ is given by

$$h(\dot{x}) = c\, sgn\,(\dot{x}) + b\dot{x}^3,\ \dot{x} \neq 0,\ c > 0. \tag{44}$$

Then

$$h(\dot{x})\,\dot{x} = c\,|\,\dot{x}\,| + b\dot{x}^4 \geq c\,|\,\dot{x}\,|. \tag{45}$$

Hence, it is seen that condition (ii) of Theorem 2 is satisfied.

Equation (43) is a special case of equation (4) where

$$g(x, t) = -\,kx. \tag{46}$$

It is obvious that g satisfies condition (i) of Theorem 2. Hence, the trivial solution $x = 0$ of equation (43) is stable for all positive c and all non-negative b and k.

Example 3: Consider the system described by

$$\ddot{x}_1 + h_1(t, x, \dot{x}) + C_1(t)\,x_1 + f_1(x_1) = x_2 \tag{47}$$

$$\ddot{x}_2 + h_2(t, x, \dot{x}) + C_2(t)\,x_2 + f_2(x_2) = x_1. \tag{48}$$

Equations (47) and (48) can be written in the form of equation (9) of Theorem 3 by defining

$$C(t) = \begin{bmatrix} C_1(t) & -1 \\ -1 & C_2(t) \end{bmatrix} \tag{49}$$

and

$$N(x) = \begin{bmatrix} f_1(x_1) \\ f_2(x_2) \end{bmatrix}. \tag{50}$$

Applying Theorem 3, we find that the system is stable if for all $t > 0$.

(i) $dC_1/dt \leq 0,\ dC_2/dt \leq 0$

(ii) $h_1\,\dot{x}_1 + h_2\,\dot{x}_2 \geq 0$

(iii) $\left(C_1 + \frac{1}{x_1^2}\int\limits_0^{x_1} f_1(y)\,dy\right)\left(C_2 + \frac{1}{x_2^2}\int\limits_0^{x_2} f_2(y)\,dy\right) \geq 1.$

Condition (iii) is satisfied if

$$C_1(t)\,C_2(t) \geq 1 \text{ for all } t \geq 0 \tag{51}$$

and integrals of $f_1(x_1)$ and $f_2(x_2)$, between zero and any positive number, are non-negative.

Example 4 : Consider a partial differential equation

$$\frac{\partial^2 x(t, z)}{\partial t^2} + h\left(t, z, x, \frac{\partial x}{\partial z}, \frac{\partial x}{\partial t}\right) - C^2 \frac{\partial^2 x(t, z)}{\partial z^2} = 0. \tag{52}$$

Let the boundary conditions at $z = 0$ and $z = L$ be such that

$$x(t, 0) \frac{\partial x(t, 0)}{\partial z} = x(t, L) \frac{\partial x(t, L)}{\partial z} = 0. \tag{53}$$

Liapunov's direct method can be used for a partial differential equation[6] but the choice of Liapunov function is not obvious.

It can be seen that $x(t, z) = 0$ is a solution of equation (52) which satisfies boundary condition (53).

In order to discretize equation (52), let Δ be the step size. Let $n\Delta = L$ and let x denote $x(t, i\Delta)$. Equation (52) can be approximated as

$$\ddot{x}_i + h_i(t, x, \dot{x}) - \frac{C^2}{\Delta^2}(x_{i+1} - 2x_i + x_{i-1}) = 0. \tag{54}$$

Boundary condition (53) becomes

$$x_0(x_1 - x_0) = x_n(x_n - x_{n-1}) = 0. \tag{55}$$

Let

$$V = \frac{C^2}{2\Delta} \sum_{1}^{n} (x_i^2 - 2x_i x_{i+1} + x_{i+1}^2) + \frac{1}{2} \sum_{1}^{n} \dot{x}_i^2 \Delta. \tag{56}$$

It can be seen that V is positive definite. It is found from equations (54) and (56) that

$$\dot{V} = - \sum_{1}^{n} h_i \dot{x}_i \Delta. \tag{57}$$

Hence, the trivial solution $x_i = 0$ of the discretized equation is stable if

$$\sum_{1}^{n} h_i \dot{x}_i \Delta > 0. \tag{58}$$

Condition (58) suggests that the partial differential equation is stable if

$$\int_{0}^{L} h(t, x, \dot{x}) \dot{x} \, dx \geq 0. \tag{59}$$

Since the properties of a discretized equation can differ from that of the original equation, we cannot conclude that inequality (59) is a sufficient condition for the stability of the partial differential equation. However, if h is equal to $a\partial x/\partial t$, equation (52) becomes linear, and through the use of Laplace transforms we find that condition (59) (at least in this case) is a sufficient condition for stability.

The study of discretized equation can help us in the choice of a Liapunov function for the partial differential equation. In some cases it may be possible to take the limit $\bar{V}$ of V as $\triangle$ tends to zero and express the resulting $\bar{V}$ as a function of the derivatives of x.

Liapunov function $\bar{V}$ can be used to determine the stability of the partial differential equation. For example the limit $\bar{V}$ of V given by equation (56) is

$$\bar{V} = \int_0^L \left[C^2 \left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial x}{\partial t}\right)^2 \right] dx \tag{60}$$

Differentiating with respect to t we get

$$\frac{d\bar{V}}{dt} = 2 \int_0^L C^2 \frac{\partial x}{\partial z} \frac{\partial^2 x}{\partial z \partial t}\, dx + 2 \int_0^L \frac{\partial x}{\partial t} \frac{\partial^2 x}{\partial t^2}\, dx \tag{61}$$

Integrating the first integral by parts we find

$$\frac{d\bar{V}}{dt} = 2 \int_0^L \left[- C^2 \frac{\partial^2 x}{\partial z^2} + \frac{\partial^2 x}{\partial t^2} \right] dx. \tag{62}$$

Combining equations (52) and (62) we find

$$\frac{d\bar{V}}{dt} = \int_0^L h \frac{\partial x}{\partial t}\, dx. \tag{63}$$

Since, $\dot{V} \leq 0$ is sufficient to establish stability we conclude that condition (59) is sufficient for the stability of trial solution $x = 0$ of equation (52).

6. Conclusions

The results presented here are useful only for certain class of the following vector equation

$$\ddot{x} + h(t, x, \dot{x}) + g(t, x) = 0.$$

If g can be expressed as a gradient of a positive definite scalar or as $G(x) \cdot x$ where G is symmetric, the results are applicable. The theorems are in general useful for cases where elements of g are non-increasing functions of t.

It is shown that discretization of a partial differential equation and study of its stability can help us in choosing a suitable Liapunov functional for the original partial differential equation.

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