A STAR-SHAPED CRACK IN A COMPOSITE MATERIAL

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ABSTRACT

The stress and displacement fields are determined in the neighbourhood of a star-shaped crack in a composite material. It is assumed that a thin infinite plate has been formed by wedges of same vertical angle 2α *and are even in number. For the sake of simplicity it is assumed that the alternative wedges are elastic and rigid respectively. It is also assumed that there are cracks of unit length originating from the centre of the plate, and that all cracks are opened by the same pressure. By symmetry the problem has been reduced to a mixed boundary value problem for an elastic wedge. Mellin transform method has been used to reduce the mixed boundary value problem to a simultaneous set of dual integral equations involving inverse Mellin transform. The set of dual equations have been solved in the case when* $\alpha = \pi/2$ *by solving a pair of coupled Abel integral equations and the quantities of physical importance have been determined. Finally, it is shown that the dual equations of the present paper, for* $\alpha = \pi/2$*, through the Mellin transform reduce to those obtained by Chakrabarti*[4] *and Lowengrub*[3] *through Fourier transform by using the Convolution theorem for Mellin transform. The paper demonstrates further use of integral transforms to crack problems in composite materials.*

Key words: Star-shaped crack, Mellin transform, Dual integral equations.

1. INTRODUCTION

The problems of Griffith cracks at the interface of two dissimilar media have been the interest of Applied Mathematicians for a long time. In 1968, Erdogan[1] solved the problem of an even number of cracks at the interface of two bonded dissimilar materials by the method of Fourier transforms. The method of Erdogan does not need calculating the inverse Fourier transforms under consideration to compute the quantities of physical interest. England[2] considered the problem of a single Griffith crack opened by equal and opposite normal pressure between two dissimilar half planes. England used complex variable method to solve the problem and concluded that the solution is physically inadmissible since it predicts that the upper and lower surfaces of the crack should overlap at the ends. Lowengrub[3] has solved the problem of the stress distribution due to a Griffith crack at the interface of an elastic half plane and a rigid foundation. He obtains the solution through Fourier transforms which lead to a simultaneous set of dual integral equations with trigonometric Kernels. Lowengrub[3] uses almost the same method of ref. 1 to solve the set of dual integral equations and obtains an oscillatory character in a small region near the ends of the crack in the components of displacements and stresses. Recently Chakrabarti[4] has solved the problem considered by Lowengrub[3] in a completely different manner. First he[4] reduces the problem to the same system of dual integral equations as obtained by Lowengrub,[3] then reduces the system of dual integral equations to that of solving a pair of coupled Abel integral equations by making use of some important results given by Jones.[5] The simultaneous system of Abel integral equations have been reduced to that of solving simultaneous Rieman Hilbert problems for two sectionally holomorphic functions $\phi(z)$ and $\psi(z)$. In his technique, Chakrabarti[4] does not make any assumption as done by Lowengrub[3] to solve the simultaneous set of dual integral equations. Finally Chakrabarti[4] obtains the correct expressions for the displacement and the stress components.

2. Statement of the Problem and Formulation of the System of Dual Integral Equations

The problem is that of determining the stress and displacement fields in the neighbourhood of a star-shaped crack in a composite material. It is assumed that a thin infinite plate has been formed by wedges of same vertical angle (2α) and are even in number. For the sake of simplicity it is assumed that the alternative wedges are elastic and rigid respectively. It is also assumed that there are cracks of unit length originating from the centre of the plate, and that all the cracks are opened by the same pressure. By symmetry the problem has been reduced to a mixed boundary value problem for a single elastic wedge $(0 \leqslant \theta \leqslant \alpha,\ r \geqslant 0)$.

The boundary conditions on the boundary $\theta = 0$ of the elastic wedge with the crack $0 \leqslant r \leqslant 1$, can be expressed in the form

$$\begin{aligned} &\sigma_{r\theta}(r, 0) = 0, \quad 0 < r < 1, \\ &\sigma_{\theta}(r, 0) = -2\mu f(r), \quad 0 < r < 1, \\ &U_{\theta}(r, 0) = 0, \quad r > 1, \\ &U_{r}(r, 0) = 0, \quad r > 1, \end{aligned} \tag{2.1}$$

where $\sigma_{r\theta}$ and σ_θ are the shear and normal components of the stress while U_θ and U_r are the displacement components and $f(r)$ is a known function. On $\theta = \alpha$, we have the following boundary conditions

$$\begin{aligned} &U_\theta(r, \alpha) = 0, \\ &\sigma_{r\theta}(r, \alpha) = 0. \end{aligned} \tag{2.2}$$

The conditions (2.2) are symmetry conditions on the line $\theta = \alpha$.

It is well known[6] that the equations of equilibrium are satisfied if we assume that

$$\begin{aligned} \frac{\sigma_r}{2\mu} &= \frac{1}{r}\frac{\partial \chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \chi}{\partial \theta^2}, \\ \frac{\sigma_\theta}{2\mu} &= \frac{\partial^2 \chi}{\partial \theta^2}, \\ \frac{\sigma_{r\theta}}{2\mu} &= \frac{1}{r^2}\frac{\partial \chi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \chi}{\partial r \partial \theta}, \end{aligned} \tag{2.3}$$

where χ is the Airy stress function and μ is the rigidity modulus. The compatibility condition is fulfilled when χ is a solution of the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2}\right)^2 \chi(r, \theta) = 0. \tag{2.4}$$

Let

$$\bar{\chi}(r, \theta) = \int_0^\infty \chi(r, s)\, r^{s-1}\, dr \tag{2.5}$$

represent the Mellin transform of $\chi(r, \theta)$. Then, by Mellin's inversion

$$\frac{\sigma_r}{2\mu} = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \left(\frac{d^2\bar{\chi}}{d\theta^2} - s\bar{\chi}\right) r^{-(s+2)}\, ds, \tag{2.6}$$

$$\frac{\sigma_\theta}{2\mu} = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} s(s+1)\,\bar{\chi} \,.\, r^{-(s+2)}\, ds, \tag{2.7}$$

$$\frac{\sigma_{r\theta}}{2\mu} = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} (s+1)\frac{d\bar{\chi}}{d\theta} \,.\, r^{-(s+2)}\, ds, \tag{2.8}$$

$$U_r = -\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{1}{(s+1)}\left[(1-\eta)\frac{d^2\bar{\chi}}{d\theta^2} - s(1+\eta s)\bar{\chi}\right] r^{-(s+1)}\, ds \tag{2.9}$$

$$U_\theta = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{1}{(s+1)(s+2)} \Big[(1-\eta) \frac{d^3 \bar{\chi}}{d\theta^3}$$

$$+ \Big\{ (1-\eta) s^2 + (s+1)(s+2) \frac{d\bar{\chi}}{d\theta} \Big\} \Big] r^{-(s+1)} \, ds, \tag{2.10}$$

where η is the Poisson's ratio of the material.

Using (2.5) in (2.4) we get

$$\frac{d^4 \bar{\chi}}{d\theta^4} + \Big[s^2 + (s+2)^2 \Big] \frac{d^2 \bar{\chi}}{d\theta^2} + s^2 (s+2)^2 \bar{\chi} = 0. \tag{2.11}$$

The solution of (2.11) is given by

$$\bar{\chi}(s, \theta) = A \sin(s\theta) + B(s) \cos(s\theta) + C(s) \sin(s+2)\theta$$

$$+ D(s) \cos(s+2)\theta. \tag{2.12}$$

Using (2.2), (2.9), (2.10) and (2.12) we see that $\bar{\chi}(s, \theta)$ can be written in the form

$$\bar{\chi}(s, \theta) = B' \cos s(\theta - \alpha) + D' \cos(s+2)(\theta - \alpha), \tag{2.13}$$

where

$$B' = \frac{B}{\cos s\alpha} \quad \text{and} \quad D' = \frac{D}{\sin s\alpha}. \tag{2.14}$$

The boundary conditions (2.1), after using (2.7), (2.8), (2.9) (2.10), (2.13) and (2.14) may be expressed in the form

$$\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} [sB + sD] \, r^{-(s+1)} \, ds = \int_0^r f(u) \, du, \quad (0 < r < 1) \tag{2.15}$$

$$\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} [sB \tan s\alpha + (s+2) D \,.\, \tan(s+2)\alpha] \, r^{-(s+1)} \, ds = 0, \cdots$$

$$(0 < r < 1) \tag{2.16}$$

$$\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} [sB \tan s\alpha + D\{(s+2) - 4(1-\eta)\} \tan(s+2)\alpha] \, r^{-(s+1)}$$

$$\times \, ds = 0, \; (r > 1) \tag{2.17}$$

and

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}[sB+D\{s+4(1-\eta)\}]\,r^{-(s+1)}\,ds=0,\quad (r>1). \tag{2.18}$$

Now, let

$$sB+\{s+4\{1-\eta)\}D=F(s) \tag{2.19}$$

and

$$sB\tan s\alpha+\{(s+2)-4(1-\eta)\}\,D\tan(s+2)\,\alpha=G(s), \tag{2.20}$$

then equations (2.15) to (2.18) take the form

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}[F(s)-4(1-\eta)\,D(s)]r^{-(s+1)}\,ds=\int_0^r f(u)\,du,\quad (0<r<1), \tag{2.21}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}[G(s)+4(1-\eta)\,D(s)]\,r^{-(s+1)}\,ds=0,\quad (0<r<1), \tag{2.22}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}G(s)\,r^{-(s+1)}\,ds=0,\qquad (r>1), \tag{2.23}$$

and

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)\,r^{-(s+1)}\,ds=0,\qquad (r>1) \tag{2.24}$$

where, from (2.19) and (2.20) we get the expression for $D(s)$ in terms of $F(s)$ and $G(s)$

$$D(s)=\frac{F(s)\tan\alpha s-G(s)}{[s\tan s\alpha-(s+2)\tan(s+2)\,\alpha+4(1-\eta)\{\tan s\alpha+\tan(s+2)\,\alpha\}]} \tag{2.25}$$

3. Reduction of the System of Dual Integral Equations to Integral Equations of the Second Kind

In this section we reduce the system of dual integral equation (2.21) to (2.24) to integral equations of the second kind in the following way:

Let

$$F(s) = \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)} \int_0^1 t^s g_1(t)\, dt \tag{3.1}$$

and

$$G(s) = \frac{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} \int_0 t^s g_2(t)\, dt \tag{3.2}$$

where $g_1(t)$ and $g_2(t)$ are two unknown functions to be determined. Substituting the expressions for $G(s)$ and $F(s)$ from (3.1) and (3.2) in (2.23) and (2.24) respectively we see that they are satisfied automatically, while equations (2.21) and (2.22) reduce to

$$\int_r^1 \frac{G_1(t)}{\sqrt{t^2-r^2}}\, dt + \int_0^1 [g_1(t)\, k_1(r, t) + g_2(t)\, k_2(r, t)]\, dt$$

$$= \frac{r}{2} \int_0^r f(u)\, du, \quad (0 < r < 1) \tag{3.3}$$

and

$$\int_r^1 \frac{g_2(t)}{\sqrt{t^2-r^2}}\, dt + \int_0^1 [g_1(t)\, k_3(r, t) + g_2(t)\, k_4(r, t)]\, dt = 0,$$

$$(0 < r < 1), \tag{3.4}$$

where

$$G_1(t) = t g_1(t)$$

$$k_1(r, t) = -2(1-\eta)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\tan s\alpha}{P(s, \alpha)} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)} \left(\frac{t}{r}\right)^s ds, \tag{3.5}$$

$$k_2(r, t) = 2(1-\eta)\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{P(s, \alpha)} \frac{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} \left(\frac{t}{r}\right)^s ds, \tag{3.6}$$

$$k_3(r, t) = \frac{2(1-\eta)}{r} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\tan s\alpha \, . \, \tan(s+2)\alpha}{P(s, \alpha)}$$

$$\times \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)} \left(\frac{t}{r}\right)^s ds, \tag{3.7}$$

$$k_4(r, t) = -\frac{2(1-\eta)}{r} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\tan(s+2)\alpha}{P(s, \alpha)}$$

$$\times \frac{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} \left(\frac{t}{r}\right)^s ds, \tag{3.8}$$

$$P(s, \alpha) = s \tan s\alpha - (s+2)\tan(s+2)\alpha + 4(1-\eta)$$
$$\times \{\tan s\alpha + \tan(s+2)\alpha\}. \tag{3.9}$$

The following results were made use of in deducing the equations (3.3) and (3.4)

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)} \left(\frac{t}{r}\right)^s ds = \frac{2t}{\sqrt{t^2-r^2}} \quad (t > r)$$

$$= 0 \qquad (r > t) \tag{3.10}$$

and

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} \left(\frac{t}{r}\right)^s ds = \frac{2r}{\sqrt{r^2-t^2}} \quad (r > t)$$

$$= 0 \qquad (r < t)$$

Taking Abel inversion, the equations (3.3) and (3.4) reduce to integral equations of the second kind for the functions $G_1(t)$ and $G_2(t)$ in the following form:

$$G_1(t) + \int_0^1 [g_1(v) L_1(v, t) + g_2(v) L_2(v, t)] \, dv$$

$$= -\frac{2}{\pi} \frac{d}{dt} \int_t^1 \frac{r l(r)}{\sqrt{r^2-t^2}} \, dr$$

and

$$g_2(t) + \int_0^1 [g_1(v) L_3(v, t) + g_2(v) L_4(v, t)]\, dv = 0$$

where

$$L_i(v, t) = -\frac{2}{\pi} \int_t^1 \frac{r k_i(r, v)}{\sqrt{t^2 - r^2}}\, dr, \qquad (i = 1, 2, 3, 4)$$

and

$$l(r) = \frac{r}{2} \int_0^r f(u)\, du$$

4. THE CASE WHEN $\alpha = \pi/2$, REDUCTION TO A SYSTEM OF ABEL TYPE INTEGRAL EQUATIONS AND ITS SOLUTION

We now consider the case $\alpha = \pi/2$, which reduce the problem of a crack at the interface of an elastic half plane and a rigid foundation as considered by Chakrabarti,[4] Lowengrub[3] and others. In the case when $\alpha = \pi/2$, the values of $k_1(r, t)$, $k_2(r, t)$ $k_3(r, t)$, and $k_4(r, t)$ can easily be obtained as

$$\begin{aligned} k_1(r, t) &= -\frac{2(1-\eta)}{3-4\eta} \frac{t}{\sqrt{t^2 - r^2}} \qquad (r < t) \\ &= 0 \qquad (r > t), \end{aligned} \tag{4.1}$$

$$\begin{aligned} k_2(r, t) &= \frac{2(1-\eta)}{(3-4\eta)} \frac{r}{\sqrt{r^2 - t^2}} \qquad (r > t) \\ &= 0 \qquad (r < t), \end{aligned} \tag{4.2}$$

$$\begin{aligned} k_3(r, t) &= -\frac{2(1-\eta)}{3-4\eta} \frac{t}{r} \frac{1}{\sqrt{r^2 - t^2}} \qquad (r > t) \\ &= 0 \qquad (r < t), \end{aligned} \tag{4.3}$$

$$\begin{aligned} k_4(r, t) &= -\frac{2(1-\eta)}{3-4\eta} \frac{1}{\sqrt{t^2 - r^2}} \qquad (r < t) \\ &= 0 \qquad (r > t), \end{aligned} \tag{4.4}$$

we also find that

$$P\left(s, \frac{\pi}{2}\right) = 2(3 - 4\eta) \tan\frac{s\pi}{2},$$

$$D(s) = \frac{F(s) \tan\left(\frac{s\pi}{2}\right) - G(s)}{2(3 - 4\eta) \tan\frac{s\pi}{2}}.$$

In order to arrive at the final form of $k_i(r, t)$ $(i = 1, 2, 3, 4)$ we have made use of the following results:

$$\frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} \left(\frac{t}{r}\right)^s \cot\frac{s\pi}{2}\, ds = \frac{2r}{\sqrt{r^2 - t^2}} \quad (r > t)$$

$$= 0 \quad (r < t)$$

and (4.5)

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)} \left(\frac{t}{r}\right)^s \tan\frac{s\pi}{2}\, ds = -\frac{2t}{\sqrt{r^2 - t^2}} \quad (r > t)$$

$$= 0 \quad (r < t) \qquad (4.6)$$

Now, using (4.1), (4.2), (4.3) and (4.4) in (3.3) and (3.4) we arrive at

$$a_2 \int_r^1 \frac{G_1(t)}{\sqrt{t^2 - r^2}}\, dt + r a_1 \int_0^r \frac{g_2(t)}{\sqrt{r^2 - t^2}}\, dt = \frac{r}{2} \int_0^r f(u)\, du, \quad (0 < r < 1) \qquad (4.7)$$

and

$$r a_2 \int_r^1 \frac{g_2(t)}{\sqrt{r^2 - t^2}}\, dt - a_1 \int_0^r \frac{G_1(t)}{\sqrt{r^2 - t^2}}\, dt = 0 \quad (0 < r < 1) \qquad (4.8)$$

where

$$a_1 = \frac{2(1-\eta)}{(3-4\eta)} \qquad \text{and} \qquad a_2 = \frac{(1-2\eta)}{(3-4\eta)}\, .$$

We shall now obtain solution of the system of Abel type integral equations (4.7) and (4.8) for the functions $G_1(t)$, by a method similar to that of Chakrabarti.[4]

Assuming

$$\phi(z) = \int_0^1 \frac{G_1(t)}{\sqrt{z^2 - t^2}}\, dt, \qquad \psi(z) = \int_0^1 \frac{g_2(t)}{\sqrt{z^2 - t^2}}\, dt$$

the equations (4.7) and (4.8) can be converted into the following simultaneous Riemann-Hilbert problem for the sectionally analytic functions (with usual notations, see Chakrabarti[4]).

$$a_1 \left(\psi^+ (r) + \psi^- (r)\right) + i \frac{a_2}{r} \left(\phi^+ (r) - \phi^- (r)\right) = P (r) \; (-1 < r < 1) \tag{4.9}$$

$$a_2 \left(\psi^+ (r) - \psi^- (r)\right) + i \frac{a_1}{r} \left(\phi^+ (r) + \phi^- (r)\right) = 0 \; (-1 < r < 1) \tag{4.10}$$

where

$$P (r) = \int_0^r f (u) du, \qquad (0 < r < 1)$$

and

$$P (r) = - P (- r), \qquad (-1 < r < 0).$$

The Case of Constant Pressure

Here we restrict on calculations to the case when the crack is opened by constant pressure p_0, whereas the analysis remains valid for arbitrary integrable pressure distribution on the crack surfaces.

If we introduce two functions

$$\mu (z) = \psi (z) + i \frac{\phi (z)}{z} \tag{4.11}$$

and

$$\lambda (z) = \psi (z) - i \frac{\phi (z)}{z}, \tag{4.12}$$

we may write the equations equivalent to the equations (4.9) and (4.10) in the case when $f (x) =$ constant (p_0), as:

$$(3 - 4\eta) \lambda^+ (r) + \lambda^- (r) = (3 - 4\eta) p_0 r \; (-1 < r < 1) \tag{4.13}$$

and

$$\mu^+ (r) + (3 - 4\eta) \mu^- (r) = (3 - 4\eta) p_0 \, r \; (-1 < r < 1) \tag{4.14}$$

The solutions of the two Hilbert problems (4.13) and (4.14) are obtained by the technique of Muskhelishvili[9] in the form

$$\lambda (z) = \frac{3 - 4\lambda}{2\pi i} p_0 \, Y (z) \int_{-1}^{+1} \frac{t}{Y^+ (t) (t - z)} \, dt \tag{4.15}$$

and

$$\mu(z) = \frac{p_0}{2\pi i} X(z) \int_{-1}^{+1} \frac{t}{X^+(t)(t-z)} dt \tag{4.16}$$

where

$$Y(z) = \sqrt{z^2 - 1}\left(\frac{z+1}{z-1}\right)^{-ik} \tag{4.17}$$

$$X(z) = \sqrt{z^2 - 1}\left(\frac{z-1}{z+1}\right)^{-ik} \tag{4.18}$$

are the solutions of the homogencous problems (4.13) and (4.14) and

$$k = \frac{1}{2\pi} l_n (3 - 4\eta).$$

The functions $\phi(z)$ and $\psi(z)$ can then be calculated by using (4.11) and (4.12).

5. Determination of Stresses and Displacements

From (2.23), (2.24), (4.14), (4.17) and (4.18) we find that the displacement components on the crack surface are given by ($0 < r < 1$)

$$U_\theta(r, 0) = \frac{i}{2}\left[\left(\lambda^+(r) - \lambda^-(r)\right) + \left(\mu^+(r) - \mu^-(r)\right)\right] \tag{5.1}$$

and

$$U_r(r, 0) = \frac{1}{2}\left[\left(\mu^+(r) - \mu^-(r)\right) - \left(\lambda^+(r) - \lambda^-(r)\right)\right]. \tag{5.2}$$

Now, by the method of Muskhelishvili,[8] we find that from (4.15) and (4.16)

$$\lambda(z) = \frac{p_0(3 - 4\eta)}{4(1-\eta)}[z - Y(z)] \tag{5.3}$$

and

$$\mu(z) = \frac{p_0(3 - 4\eta)}{4(1-\eta)}[z - X(z)]. \tag{5.4}$$

Hence using (5.3) and (5.4) in (5.1) and (5.2) we get the expressions for the components of displacements in the range $0 < r < 1$ as:

$$U_\theta = \frac{p_0}{\sqrt{3 - 4\eta}} \sqrt{1 - r^2} \cos\left[k \ln\left|\frac{r+1}{r-1}\right|\right] \tag{5.5}$$

and

$$U_r = \frac{p_4}{\sqrt{3 - 4\eta}} \sqrt{1 - r^2} \sin\left[k \ln\left|\frac{r+1}{r-1}\right| \right]. \tag{5.6}$$

The stresses, on the line $y = 0$, for $|r| > 1$, can be calculated by means of (2.15), (2.16), (4.13), (4.17), (4.18), (5.3) and (5.4) as

$$\sigma_\theta (0, r) \big|_{|r| > 1} = p_0 \frac{d}{dr}\left[\sqrt{r^2 - 1} \cos\left\{k \ln\left|\frac{r+1}{r-1}\right|\right\} - r\right]$$

and

$$\sigma_{r\theta} (0, r) \big|_{|r| > 1} = - p_0 \frac{d}{dr}\left[\sqrt{r^2 - 1} \sin\left\{k \ln\left|\frac{r+1}{r-1}\right|\right\}\right].$$

These results are comparable with those obtained by Chakrabarti.[4]

6. Analogy Between the Equations (2.21) to (2.24) of the Present Paper (in the Case when $\alpha = \pi/2$) and the Equations (2.7) to (2.10) of Chakrabarti[4]

Let

$$F(s) = F'(s) \cos\frac{s\pi}{2} \qquad \text{and} \qquad G(s) = G'(s) \sin\frac{s\pi}{2}, \tag{6.1}$$

then, in the case when $\alpha = \pi/2$, we get from equations (2.21) to (2.24) the following equations in terms of $F'(s)$ and $G'(s)$:

$$a_1 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F'(s) \cos\frac{s\pi}{2} r^{-s} \, ds + a_2 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G'(s) \cos\frac{s\pi}{2} r^{-s} \, ds = rp(r), \qquad (0 < r < 1) \tag{6.2}$$

$$a_1 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G'(s) \sin\frac{s\pi}{2} r^{-s} \, ds + a_2 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F'(s) \sin\frac{s\pi}{2} r^{-s} \, ds = 0, \qquad (0 < r < 1) \tag{6.3}$$

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G'(s) \sin\frac{s\pi}{2} r^{-s} \, ds = 0 \qquad (r > 1) \tag{6.4}$$

and

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} F'(s)\cos\frac{s\pi}{2}\; r^{-s}\; ds = 0 \qquad (r > 1) \tag{6.5}$$

Now, let

$$F'(s) = a^*(s)\,\Gamma(s) \quad \text{and} \quad G'(s) = b^*(s)\,\Gamma(s), \tag{6.6}$$

where

$$a^*(s) = \int_0^\infty a'(r)\,r^{s-1}\,dr \quad \text{and} \quad b^*(s) = \int_0^\infty b'(r)\,r^{s-1}\,dr, \tag{6.7}$$

Then using the following results (*cf.* Sneddon[6])

$$\int_0^\infty \cos r\,.\,r^{s-1}\,dr = \Gamma(s)\cos\frac{s\pi}{2} \tag{6.8}$$

$$\int_0^\infty \sin r\,.\,r^{s-1}\,dr = \Gamma(s)\sin\frac{s\pi}{2} \tag{6.9}$$

and

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} f^*(s)\,g^*(s)\,r^{-s}\,ds = \int_0^\infty f\left(\frac{r}{u}\right) g(u)\,\frac{du}{u} \tag{6.10}$$

equations (6.2) to (6.5) take the form

$$a_1\int_0^\infty a(\xi)\cos r\xi\,d\xi + a_2\int_0^\infty b(\xi)\cos r\xi\,d\xi = r\,P(r), \quad (0 < r < 1) \tag{6.11}$$

$$a_1\int_0^\infty b(\xi)\sin r\xi\,d\xi + a_2\int_0^\infty a(\xi)\sin r\xi\,d\xi = 0, \qquad (0 < r < 1) \tag{6.12}$$

$$\int_0^\infty b(\xi)\sin r\xi\,d\xi = 0, \qquad (r > 1) \tag{6.13}$$

$$\int_0^\infty a(\xi)\cos r\xi\,d\xi = 0 \qquad (r > 1) \tag{6.14}$$

where

$$a(\xi) = \frac{1}{\xi}\,a'\left(\frac{1}{\xi}\right) \quad \text{and} \quad b(\xi) = \frac{1}{\xi}\,b'\left(\frac{1}{\xi}\right).$$

Equations (6.11) to (6.14) are similar to equations (2.7) to (2.10) of Chakrabarti.[4]

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