THE EFFECT OF A PENNY-SHAPED CRACK IN A SEMI-INFINITE CYLINDER

ALOKNATH CHAKRABARTI

(*Department of Applied Mathematics, Indian Institute of Science, Bangalore* 560 012)

Received on March 12, 1977

ABSTRACT

A procedure is presented for solving the problem of a penny-shaped crack, in a semi-infinite cylinder opened by a known pressure. The crack is assumed to be situated at a finite distance from the flat end, which is assumed stress-free and the curved boundary constrained. Expressions for the quantities of practical interest have been obtained and some of them have been compared with the results of a particular limiting case of this problem. Numerical results are tabulated at the end.

Key words: Penny-shaped crack, Dual series relations, Fredholm integral equations.

1. INTRODUCTION

Axisymmetric mixed boundary value problems in Elasticity have been considered by many authors.[1-7] These authors have handled axisymmetric problems associated with either a half-space or a long cylinder. Problems concerning half cylinders containing a single crack even have not been tackled so far, either because of the lack of techniques available or because of the complication it gives rise to mathematically.

In this paper, we have investigated the problem of a semi-infinite cylinder containing a penny-shaped crack at a finite distance $h\ (>0)$ from its flat end, through a system of Fredholm integral equations. The crack is assumed to be symmetrically situated around the axis of the cylinder. The cylinder is assumed to be deformed by the application of a known pressure on the crack surfaces, whereas the flat end of the cylinder is assumed to be stress free and the curved boundary constrained. The method followed in solving the problem here is similar to the second method of Sneddon and Tait[3] with the modification given by Chakrabarti.[8] The problem has been reduced to a system of dual series relations which ultimately have been reduced to a system of Fredholm integral equations of the second kind by a technique similar to that described in Chakrabarti.[8] In the limiting case, when $h \to \infty$ in our problem, the problem reduces to that of Sneddon and Tait.[3]

Numerical solutions of the system of Fredholm equations have been obtained for different values of h, a and the Poisson's ratio η. Using these numerical solutions, the non-dimensional quantities involving the stress-intensity factors have been tabulated for different values of η, a and h.

2. THE STATEMENT AND MATHEMATICAL FORMULATION OF THE CRACK PROBLEM

We consider the following mixed boundary value problem:

The curved boundary $r = a$ of the cylinder is assumed to be constrained in such a manner that the radial displacement and the shearing stress are zero on $r = a$. The flat end $z = -h\,(h > 0)$ of the cylinder is assumed to be stress-free, whilst the stresses are prescribed on the surface of the crack ($z = 0\pm$, $0 \leqslant r < 1$), where r, θ, z are cylindrical polar coordinates, the z-axis being taken along the axis of the cylinder. We have considered the radius of the crack to be unity and assumed that $a > 1$.

With usual notations, the boundary conditions of the problem are:

$$\sigma_z = 0 = \tau_{rz}, \quad \text{on} \quad z = -h, (0 \leqslant r \leqslant a), \tag{2.1}$$

$$u = 0 = \tau_{rz}, \quad \text{on} \quad r = a, \tag{2.2}$$

$$\sigma_z(r, 0^-) = -p(r), \quad \tau_{rz}(r, 0^-) = 0, \quad (0 \leqslant r < 1) \tag{2.3}$$

$$\sigma_z(r, 0^+) = -p(r), \quad \tau_{rz}(r, 0^+) = 0, \quad (0 \leqslant r < 1) \tag{2.4}$$

Also, the continuity of the stresses and the displacements across the plane $z = 0$, unoccupied by the crack, requires

$$\left.\begin{aligned} &\sigma_z(r, 0^-) = \sigma_z(r, 0^+), \quad \tau_{rz}(r, 0^-) = \tau_{rz}(r, 0^+), \; (1 < r \leqslant a) \\ &u(r, 0^-) = u(r, 0^+), \quad w(r, 0^-) = w(r, 0^+), \quad (1 < r \leqslant a), \end{aligned}\right\} \tag{2.5}$$

where $u(r, z)$ and $w(r, z)$ are the nonvanishing components of the displacement vector.

3. REDUCTION TO A SYSTEM OF DUAL SERIES RELATIONS

To solve the above-posed, axisymmetric mixed problem in section 2, we look first for the expressions for the displacements and stresses in the cylinder in terms of an axisymmetric bi-harmonic stress function $\chi(r, z)$

(*cf.* Love[9]). Throughout the paper, η is Poisson's ratio and μ is the rigidity modulus of the material of the isotropic elastic cylinder.

Solution for $z \geqslant 0$:

We assume a bi-harmonic function $\chi(r, z)$ satisfying the conditions of the vanishing of stresses and displacements for large z, in the following form,

$$\chi(r, z) = \sum_{n=1}^{\infty} (A_n + B_n z) J_0(\xi_n r) \exp(-\xi_n z), \tag{3.1}$$

where A_n and B_n are constants and ξ_n's are the positive zeros of $J_1(\xi a)$, $J_n(x)$ being the Bessel function of the first kind of order n.

Expressions for the displacements and stresses in the region $z \geqslant 0$ can be obtained by using the relations in Love.[9]

Solution for $-h < z < 0$

In this region, we assume the representation of $\chi(r, z)$ as:

$$\chi(r, z) = \sum_{n=1}^{\infty} [C_n \cosh \xi_n (z + h) + D_n \sinh \xi_n (z + h)$$
$$+ E_n (z + h) \cosh \xi_n (z+h) + F_n (z + h) \sinh \xi_n (z + h)]$$
$$\times J_0(\xi_n r), \tag{3.2}$$

where C_n, D_n, E_n, F_n are constants.

We observe that these forms of the stress-functions satisfy the conditions (2.2) automatically because of the choice of the ξ_n's. We can satisfy the conditions (2.1), by choosing

$$\xi_n C_n = -2\eta F_n, \qquad \xi_n D_n = (1 - 2\eta) E_n. \tag{3.3}$$

Reduction to a system of dual series relations

The conditions (2.3) and (2.4) require:

$$\sum_{n=1}^{\infty} \xi_n^3 [A_n - 2\eta G_n] J_1(\xi_n r) = 0$$
$$= \sum_{n=1}^{\infty} \xi_n^2 \sinh(\xi_n h) [E_n (1 + \xi_n h \coth \xi_n h) + F_n \xi_n h]$$
$$\times J_1(\xi_n r) \quad (0 \leqslant r < 1) \tag{3.4}$$

and

$$\sum_{n=1}^{\infty} \xi_n^3 [A_n + (1 - 2\eta) G_n] J_0(\xi_n r) = - p(r)$$

$$= \sum_{n=1}^{\infty} \xi_n^2 \sinh(\xi_n h) [- E_n \xi_n h + F_n (1 - \xi_n h \coth \xi_n h)]$$
$$\times J_0(\xi_n r), \ (0 \leqslant r < 1) \tag{3.5}$$

where

$$B_n = \xi_n G_n. \tag{3.6}$$

The continuity conditions (2.5) together with (3.4) and (3.5) require:

$$\left.\begin{aligned} [A_n - 2\eta G_n] - E_n' (1 + y_n \coth y_n) - F_n' y_n = 0, \\ A_n + (1 - 2\eta) G_n + E_n' y_n - F_n' (1 - y_n \coth y_n) = 0, \end{aligned}\right\} \tag{3.7}$$

and [for $1 < r \leqslant a$],

$$\sum_{n=1}^{\infty} [(- A_n + G_n) - E_n' \big(y_n + (2 - 2\eta) \coth y_n\big)$$
$$- F_n' (1 - 2\eta) + y_n \coth y_n] \xi_n^2 J_1(\xi_n r) = 0, \tag{3.8}$$

$$\sum_{n=1}^{\infty} [\big(A_n + 2G_n (1 - 2\eta)\big) + E_n' \big((1 - 2\eta) - y_n \coth y_n\big)$$
$$+ F_n' (2 - 2\eta) \coth y_n - y_n]$$
$$\times \xi_n^2 J_0(\xi_n r) = 0, \tag{3.9}$$

where we have written

$$E_n' = E_n (\sinh \xi_n h)/\xi_n, \quad F_n' = F_n (\sinh \xi_n h)/\xi_n,$$
$$y_n = \xi_n h. \tag{3.10}$$

Solving equations (3.7) for E_n' and F_n' in terms of A_n and G_n and substituting in (3.8) and (3.9), we obtain:

$$\sum_{n=1}^{\infty} [A_n M_n + G_n N_n] \xi_n^2 J_1(\xi_n r) = 0, \quad (1 < r \leqslant a) \tag{3.11}$$

and

$$\sum_{n=1}^{\infty} [A_n P_n + G_n Q_n] \xi_n^2 J_0(\xi_n r) = 0, \qquad (1 < r \leqslant a) \tag{3.12}$$

where

$$\left.\begin{aligned} M_n &= -(2-2\eta)(1-y_n\,\mathrm{cosech}^2 y_n + \coth y_n)/ \\ &\qquad (y_n^2\,\mathrm{cosech}^2 y_n - 1), \\ N_n &= -2\eta\, M_n - K_n, \quad Q_n = K_n + (1-2\eta)\,P_n, \\ K_n &= (2-2\eta)\, y_n^2\,\mathrm{cosech}^2 y_n/(y_n^2\,\mathrm{cosech}^2 y_n - 1), \\ P_n &= -(2-2\eta)(1+y_n\,\mathrm{cosech}^2 y_n + \coth y_n)/ \\ &\qquad (y_n^2\,\mathrm{cosech}^2 y_n - 1). \end{aligned}\right\} \tag{3.13}$$

Thus by means of (3.4), (3.5), (3.11) and (3.12), we see that the problem is reduced to a system of dual series relations given by:

$$\left.\begin{aligned} &\sum_{n=1}^{\infty} [A_n + (1-2\eta)\,G_n]\,\xi_n^3\,J_0(\xi_n r) = -p(r), \\ &\sum_{n=1}^{\infty} [A_n - 2\eta G_n)\,\xi_n^3\,J_1(\xi_n r) = 0; \end{aligned}\right\} (0 \leqslant r < 1) \tag{3.14}$$

and

$$\left.\begin{aligned} &\sum_{n=1}^{\infty} [P_n A_n + Q_n G_n]\,\xi_n^2\,J_0(\xi_n r) = 0, \\ &\sum_{n=1}^{\infty} [M_n A_n + N_n G_n]\,\xi_n^2\,J_1(\xi_n r) = 0. \end{aligned}\right\} (1 < r \leqslant a) \tag{3.15}$$

Writing

$$P_n A_n + Q_n G_n = R_n/\xi_n, \qquad M_n A_n + N_n G_n = S_n/\xi_n^2 \tag{3.16}$$

and using the expressions (3.13) for M_n, N_n, P_n, Q_n and K_n, we can easily reduce the above system (3.14) and (3.15), after some manipulations, to the following simple form:

$$\left.\begin{aligned} &\sum_{n=1}^{\infty} [(1-V_n)\,R_n + \xi_n^{-1}\,U_n S_n]\,\xi_n^2\,J_0(\xi_n r) = -p(r), \\ &\sum_{n=1}^{\infty} [U_n R_n \xi_n + (1-W_n)\,S_n]\,\xi_n\,J_1(\xi_n r) = 0; \end{aligned}\right\} (0 \leqslant r < 1) \tag{3.17}$$

and

$$\sum_{n=1}^{\infty} R_n \xi_n J_0(\xi_n r) = 0 = \sum_{n=1}^{\infty} S_n J_1(\xi_n r), \qquad (1 < r \leqslant a) \tag{3.18}$$

where

$$U_n = -[y_n^2 \exp(-2y_n)]/4(2-2\eta), \tag{3.19}$$

$$V_n = -y_n^2 \exp(-2y_n)[(1-(1+y_n)\exp(-2y_n))/ (1-(1-y_n)\exp(-2y_n))]/4(2-2\eta) \tag{3.20}$$

and

$$W_n = -y_n^2 \exp(-2y_n)[(1-(1-y_n)\exp(-2y_n))/ (1-(1+y_n)\exp(-2y_n))]/4(2-2\eta) \tag{3.21}$$

It is easily seen that in the limiting case, when $h \to \infty$, U_n, V_n, $W_n \to 0$, and the problem reduces to that considered by Sneddon and Tait.[3]

4. Reduction to a System of Fredholm Integral Equations

To reduce the above system of dual series relations to a system of Fredholm integral equations, we follow a technique given by Chakrabarti[8] in reducing dual series relations to a Fredholm integral equation.

We assume (*cf.* Chakrabarti[9]):

$$\sum_{n=1}^{\infty} R_n \xi_n J_0(\xi_n r) = \int_r^1 g_1(t)(t^2-r^2)^{-\frac{1}{2}} dt \; (0 \leqslant r < 1) \; (g_1(0) = 0) \tag{4.1}$$

and

$$\sum_{n=1}^{\infty} S_n J_1(\xi_n r) = -\frac{\partial}{\partial r} \int_r^1 g_2(t)(t^2-r^2)^{\frac{1}{2}} dt. \; (0 \leqslant r < 1) \tag{4.2}$$

Then, we obtain by the technique of finding the Dini and the Fourier-Bessel coefficients (see Chakrabarti[8]),

$$R_n \xi_n^2 = [2/a^2 J_2^2(a\xi_n)] \int_0^1 g_1(t) \sin(\xi_n t)\, dt \tag{4.3}$$

and

$$S_n = [(2\pi)^{\frac{1}{2}}/a^2 J_2^2(a\xi_n)]\, \xi_n^{-\frac{1}{2}} \int_0^1 t^{3/2} g_2(t) J_{3/2}(\xi_n t)\, dt. \tag{4.4}$$

We also note that, for $0 \leqslant r < 1$ (see Chakrabarti[8]),

$$\sum_{n=1}^{\infty} R_n \xi_n^2 J_0(\xi_n r) = \int_0^r g_1'(t)(r^2 - t^2)^{-\frac{1}{2}} dt$$

$$- (2/\pi) \int_0^1 g_1(t)\, dt \int_0^\infty \big(K_1(ay)/I_1(ay)\big) I_0(ry)\, y \sinh(ty)\, dy, \quad (4.5)$$

and

$$\sum_{n=1}^{\infty} S_n \xi_n J_1(\xi_n r) = \int_0^r G_2'(t)(r^2 - t^2)^{-\frac{1}{2}} dt$$

$$+ (2/\pi)^{\frac{1}{2}} \int_0^1 G_2(t)\, dt \,.\, \frac{d}{dt}\{t^{-\frac{1}{2}} K_{1,1,\frac{1}{2},-\frac{1}{2}}(r, t;\, a)\}, \quad (4.6)$$

where $G_2(t) = t^2 g_2(t)$, and $K_{\nu,\alpha,\beta,\gamma}(r, t, a)$ is defined by Chakrabarti.[8]

Substituting from (4.3), (4.4), (4.5) and (4.6) in (3.17), using some well-known results,[10] we obtain, after using Abel inversion formula:

$$g_1(r) - \int_0^1 [L_1(r, t)\, g_1(t) - L_2(r, t)\, G_2(t)]\, dt = h(r) \quad (4.7)$$

and

$$G_2(r) + \int_0^1 [M_1(r, t)\, g_1(t) + M_2(r, t)\, G_2(t)]\, dt = 0, \quad (4.8)$$

where

$$L_1(r, t) = H\big((r + t)/a\big) - H)\big((r - t)/a\big),$$

$$H(\lambda) = (2/\pi^2 a) \int_0^\infty \big(k_1(y)/I_1(y)\big)\big(\cosh(\lambda y) - 1\big) dy$$

$$+ (2/\pi a^2) \sum_{n=1}^{\infty} \big(V_n/\xi_n J_2^2(a\xi_n)\big) \cos(\lambda \xi_n),$$

$$L_2(r, t) = (4/\pi a^2 t) \sum_{n=1}^{\infty} \big(U_n/\xi_n J_2^2(a\xi_n)\big) \sin(\xi_n r)$$

$$\times \big(\sin(t\xi_n)/(t\xi_n) - \cos(t\xi_n)\big),$$

$$M_1(r, t) = (4/\pi a^2) \sum_{n=1}^{\infty} \big(U_n \xi_n^{-\frac{1}{2}}/J_2^2(a\xi_n)\big) \sin(\xi_n t)$$

$$\times \big(\sin(r\xi_n)/(r\xi_n) - \cos(r\xi_n)\big),$$

$$M_2(r,t) = (4r^{3/2}/\pi^2 t) \int_0^\infty \big(K_1(ay)/I_1(ay)\big)\, y^{\frac{1}{2}} \big(\cosh(ty) - \sinh(ty)/(ty)\big)\big(\cosh(ry) - \sinh(ry)/(ry)\big)\, dy$$
$$- 4r^{3/2}/\pi a^2 t) \sum_{n=1}^{\infty} \big(W_n \xi_n^{-\frac{1}{2}}/J_2^2(a\xi_n)\big)\big(\sin(r\xi_n)/(r\xi_n) - \cos(r\xi_n)\big)\big(\sin(t\xi_n)/(t\xi_n) - \cos(t\xi_n)\big),$$

and

$$h(r) = -(2/\pi) \int_0^r \rho p(\rho)\,(r^2-\rho^2)^{-\frac{1}{2}}\, d\rho$$

5. Quantities of Practical Interest

The stress-intensity factors: K_1 and K_2 are defined by

$$\left.\begin{aligned} K_1 &= \lim_{r\to 1+} [(r-1)^{\frac{1}{2}} \sigma_z \,|\, _{z=0}]\,, \\ K_2 &= \lim_{r\to 1+} [(r-1)^{\frac{1}{2}} \tau_{rz} \,|\, _{z=0}]\,. \end{aligned}\right\} \tag{5.1}$$

Using the values of A_n and G_n from (3.16) and making use of the values of R_n and S_n given by (4.3) and (4.4), we obtain (*cf.* Love[9])

As $r \to 1^+$

$$\sigma_z|_{z=0} = -g_1(1)\,(r^2-1)^{-\frac{1}{2}} + 0(1), \qquad K_1 = -2^{-\frac{1}{2}} g_1(1), \tag{5.2}$$

and

$$\tau_{rz}|_{z=0} = -G_2(1)\,(r^2-1)^{-\frac{1}{2}} + 0(1), \qquad K_2 = -2^{-\frac{1}{2}} G_2(1). \tag{5.3}$$

It is easily checked from (4.7), (4.8), (5.2) and (5.3) that $K_{1\infty} = 2^{\frac{1}{2}} p_0/\pi$ and $K_{2\infty} = 0$, where the quantities $K_{1\infty}$ and $K_{2\infty}$ are the stress intensity factors at the tip of a penny crack in an infinite solid opened by a constant pressure p_0.

6. The Numerical Solution of the System of Fredholm Equations. Crack opened by Constant Pressure p_0

To solve the system of equations (4.7) and (4.8) numerically, we have reduced the system to a system of Algebraic equations by a technique similar to that described by Srivastav and Narain,[12] we have written the integrals in (4.7) and (4.8) by means of an n-point quadrature formula, *e.g.*:

$$\int_0^1 L_1(r,t)\, g_1(t)\, dt = \sum_{j=1}^{n} K_j\, L_1(r,t_j)\, g(t_j),$$

where K_j's are the weights of the formula.

We have taken $n = 13$ and employed Simpson's 13 point formula. The integrals involved in the Kernels L and M have been evaluated by Weddle's rule and the series involved have been computed by taking 20 terms. We have thus obtained a system of 26 algebraic equations for the 26 unknowns $g_1(r_i)$ and $G_2(r_i)$, $i = 1, 2, \ldots 13$, where r_i's are the abscissae. Finally, we have obtained the solution of these algebraic equations for different values of η, a, and h. Using these numerial solutions, the non-dimensional quantities $K_{1\infty}/K_1$ and $\pi K_2/2^{\frac{1}{2}} p_0$ are tabulated for four sets of values of a and h and for values of η ranging from 0·05 to 0·45 (Tables 1–4).

The computations were carried out on the computer IBM 360/44, at the Indian Institute of Science, Bangalore 560 012.

7. Acknowledgements

I express may warm thanks to Dr. (Mrs.) S. Chakrabarti and Mr. V. V. S. S. Sastry of the Department of Applied Mathematics, for their help in the computational part of the work.

TABLE I

$h = 1\cdot 5$, $a = 3\cdot 5$					
η	0·05	0·15	0·25	0·35	0·45
$K_1/K_{1\infty}$	1·0005	1·0005	1·0004	1·0004	1·0003
$\pi K_2/2^{1/2} p_0$	0·0019	0·0025	0·0024	0·0027	0·0029

TABLE II

$h = 2\cdot 5$, $a = 3\cdot 5$					
η	0·05	0·15	0·25	0·35	0·45
$K_1/K_{1\infty}$	1·0007	1·0006	1·0006	1·0006	1·0006
$\pi K_2/2^{1\cdot 2} p_0$	0·0003	0·0004	0·0004	0·0005	0·0005

TABLE III

$h = 3 \cdot 5, \quad a = 3 \cdot 5$					
η	0·05	0·15	0·25	0·35	0·45
$K_1/K_{1\infty}$	1·0007	1·0007	1·0007	1·0007	1·0007
$\pi K_2/2^{1/2} p_0$	0·0006	0·0007	0·0008	0·0009	0·00011

TABLE IV

$h = 4.5, \quad a = 3.5$					
η	0·05	0·15	0·25	0·35	0·45
$K_1/K_{1\infty}$	1·0007	1·0007	1·0007	1·0007	1·0007
$\pi K_2/2^{1/2} p_0$	0·0001	0·0001	0·0001	0·0002	0·00002

REFERENCES

1. Lebdev, N. N. and Ufliand, Y. S. — Axisymmetric contact problem for an elastic layer. *J. Appl. Math. Mech.*, 1958, **22**, 442–450.
2. Kuzmin, I. N. and Ufliand, Y. S. — Axisymmetric problem in the theory of elasticity for a half-space weakened by a plane circular crack. *J. Appl. Math. Mech*, 1965, **29**, 1330–1336.
3. Sneddon, I. N. and Tait, R. J. — The effect of a penny-shaped crack on the distribution of stress in a long circular cylinder. *Int. J. Engg. Sc.*, 1963, **1**, 391–409.
4. Sneddon, I. N. and Welch, J. T. — A note on the distribution of stress in a cylinder containing a penny-shaped crack. *Int. J. Engg. Sc.*, 1963, **1**, 411–419.
5. Srivastava, K. N. and Singh, K. — The effect of a penny-shaped crack on the distribution of stress in a semi-infinite solid, *Int. J. Engg. Sc.*, 1969, **7**, 469–490.
6. Watanabe, K. and Atsumi, A. — Long circular cylinder having an infinite row of penny-shaped cracks. *Int. J. Engg. Sc.*, 1972, **10**, 159–171.

7. Dhaliwal, R. S. — The steady state thermoelastic mixed boundary value problem for the elastic layer. *SIAM J. Appl. Math.*, 1971, **20**, 131–140.

8. Chakrabarti, A. — Certain dual series relations involving Bessel functions arising in crack problems. *ZAMM*, 1974, **54**, 436–439.

9. Love, A. E. H. — *A Treatise on the Mathematical Theory of Elasticity* (4th edn.), Cambridge, 1934.

10. Erdelyi, A. (Ed.) — *Tables of Integral Transforms*, 1954, **2**, 48, McGraw-Hill, New York, London.

11. Cooke, J. C. and Tranter, C. J. — Dual Fourier-Bessel Series. *Quart. J. Mech. Appl. Math.*, 1959, **12**, 379–386.

12. Srivastav, R. P. and Narain, P. — Certain two-dimensional problems of stress distribution in wedge-shaped elastic solids under discontinuous load. *Proc. Camb. Phil. Soc.*, 1965, **61**, 945–954.