SPECTRUM OF A CLASS OF DIRAC TYPE DIFFERENTIAL OPERATOR

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Abstract

The paper deals with the nature of the spectrum of the differential operator derived from the pair of first order Dirac type equations

$$\begin{aligned} u'(x) - (\lambda + q_1)\, v(x) &= 0 \\ v'(x) + (\lambda + q_2)\, u(x) &= 0 \end{aligned} \qquad ' \equiv d/dx, \quad x \in [0, \infty),$$

where λ *is a non-zero complex parameter,* $0 < \arg \lambda < \pi$, $0 < \arg (\lambda + q_j)^p < p\pi$, $0 < p \leqslant 1$, *and* q_j, $j = 1, 2$, *are real-valued continuous functions of* x *such that* $q_j \notin L[0, \infty)$ *but* $q'_j \in L[0, \infty)$.

Key words: Differential operator, Homogeneous boundary conditions, Diracs relativistic wave equation, (Continuous, Discrete, Point) Spectrum, Meromorphic functions.

1. Introduction

Consider the differential operator

$$\begin{aligned} u'(x) - \big(\lambda + q_1(x)\big)\, v(x) &= 0 \\ v'(x) + \big(\lambda + q_2(x)\big)\, u(x) &= 0 \end{aligned} \quad (' \equiv d/dx),\ (0 \leqslant x < \infty), \tag{1.1}$$

where

(i) $\lambda = \mu + i\nu$, $\nu \neq 0$, $0 < \arg \lambda < \pi$, $0 < \arg (\lambda + q_j(x))^p < p\pi$, for each p, $0 < p \leqslant 1$, $j = 1, 2$.

and (ii) $q_j(x)$, $q'_j(x)$, $j = 1, 2$, denote real-valued functions of x such that $q_j \notin L[0, \infty)$ but $q'_j \in L[0, \infty)$.

Further let $q_j(x)$

(A) be positive and bounded below, *i.e.*, $0 < k \leqslant q_j(x) < \infty$ on $0 \leqslant x < \infty$;

or (B) tend to zero as x tends to infinity;

or (C) tend to non-zero finite limits c_j, say, as x tends to infinity.

The homogeneous boundary condition considered is

$$u(0) \cos \alpha + v(0) \sin \alpha = 0 \tag{1.2}$$

where α is a real parameter.

The system (1.1) is a special case of Dirac's relativistic wave equation where $q_j(x)$ are the potential functions.

In what follows a column vector $\binom{u}{v}$ is represented as $\{u, v\}$.

Our object in the present paper is to investigate the nature of the spectrum associated with the system (1.1) and (1.2).

The spectrum is defined as usual as the set of values of λ which contribute to the expansion formula and is characterised, following Everitt and Chaudhuri[1] by the properties of $m(\lambda)$, which occurs in a pair of L^2-solutions $\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda)$, satisfying the boundary conditions (1.2). It may be noted that $\phi(x, \lambda)$ and $\theta(x, \lambda)$ are the solutions of (1.1) with $\phi(0, \lambda) = \{-\sin\alpha, \cos\alpha\}$, $\theta(0, \lambda) = \{-\cos\alpha, -\sin\alpha\}$.

Thus

(i) μ does not belong to the spectrum iff $\lim_{\nu\to 0}\{im\, m(\lambda)\} = 0$; but μ belongs to the point spectrum iff $m(\lambda) \to \infty$, as $\lambda \to \mu$ and $im\, m(\lambda) \to 0$ as λ tends to any point in the neighbourhood of μ but excluding μ,

(ii) μ belongs to the continuous spectrum iff $im\, m(\lambda)$ does not tend to zero as λ tends to μ for any $\mu \in (\mu_1, \mu_2)$, say, and $\lim_{\nu\to 0}\{im\, m(\lambda)\}$ is a continuous, non-vanishing function bounded for all $\mu \in (\mu_1, \mu_2)$,

(iii) μ belongs to the point continuous spectrum iff $m(\lambda)$ tends to infinity as $\lambda \to \mu$ and $\lim_{\nu\to 0}\{im\, m(\lambda)\}$ is a non-vanishing function, continuous in a neighbourhood of μ but excluding μ.

Finally, the spectrum is said to be a pure point spectrum or a discrete spectrum iff $m(\lambda)$ is meromorphic.

It may be noted that Conte S. D. and Sangren W. C.[2] studied the nature of the spectrum associated with the system (1.1) under the boundary condition (1.2) with the condition that $q_j(x)$, $j = 1, 2$ belong to $L[0, \infty)$.

2. A Transformation of the Basic Equation

Let

$$\xi(x) = \lambda^{-1} \int_0^x \{(\lambda + q_1)(\lambda + q_2)\}^{\frac{1}{2}} dt$$

$$\eta_1(x) = (\lambda + q_1)^{-\frac{1}{4}} u, \qquad \eta_2(x) = (\lambda + q_2)^{-\frac{1}{4}} v, \tag{2.1}$$

where

$$0 < \arg \lambda < \pi, \quad 0 < \arg \left(\lambda + q_j(t)\right)^p < p\pi, \quad 0 < p \leqslant 1, \quad j = 1, 2.$$

Then

$$im\left(\lambda + q_j(t)\right)^{\frac{1}{2}} > 0, \quad im\,\xi(x) > 0.$$

[See § 22.26, p. 355 (ref. 5) and § 5.8, p. 120 (ref. 4)].

Since $d\eta_j/d\xi = d\eta_j/dx \,.\, dx/d\xi$, it follows by using (1.1) and (2.1) at relevant places that (1.1) transforms to

$$\frac{d\eta_1}{d\xi} - \lambda^{\frac{1}{2}}\eta_2 + Q_1(x)\,\eta_1 = 0$$

$$\frac{d\eta_2}{d\xi} + \lambda^{\frac{1}{2}}\eta_1 + Q_2(x)\,\eta_2 = 0 \tag{2.2}$$

where $Q_j = \frac{1}{2}\lambda^{\frac{1}{2}} q'_j/(\lambda + q_1)^l (\lambda + q_2)^n$, $j = 1$, with $l = 3/2$, $n = 1/2$ and $j = 2$, with $l = 1/2$, $n = 3/2$.

It is easy to verify that $Q_j \in L\,[0, \infty)$

(i) for all values of λ, $|\lambda| \neq k$,

(ii) for all non-zero values of λ,

and (iii) for all non-zero values of λ, $|\lambda| \neq |c_1|, |c_2|$, according as the conditions (A), (B) and (C), respectively, of § 1 are satisfied by $q_j, j = 1, 2$.

Finally, we note that when any of the conditions (A), (B), (C) of § 1 is satisfied, $Q_j\, d\xi/dt$, $j = 1, 2$, belong to $L\,[0, \infty)$, provided that $|\lambda|$ does not assume the value q_j at any point in (X_0, X), where X_0 is fixed and X is large enough.

3. On a Pair of L^2–solutions of the System (2.2)

Let $U \equiv \{u_1(x, \lambda),\ u_2(x, \lambda)\}$ be a solution of (2.2) such that $U(0, \lambda) \equiv \{-\sin\alpha, \cos\alpha\}$ and let

$$Q \equiv Q(t) \equiv \{Q_1(t), Q_2(t)\}$$

$$R \equiv R(t) \equiv \{R_1(t),\ R_2(t)\}$$

$$\equiv \{u_1(t)\cos\left(\lambda^{\frac{1}{2}}\,\Omega(x, t)\right), u_2(t)\sin\left(\lambda^{\frac{1}{2}}\,\Omega(x, t)\right)\},$$

$$S \equiv S(t) \equiv \{S_1(t), S_2(t)\}$$

$$\equiv \{u_1(t)\sin\left(\lambda^{\frac{1}{2}}\,\Omega(x, t)\right),\ -u_2(t)\cos\left(\lambda^{\frac{1}{2}}\,\Omega(x, t)\right)\},$$

where

$$\Omega(x, t) = \xi(x) - \xi(t)$$

and

$$(Q, R) = Q_1(t) R_1(t) + Q_2(t) R_2(t),$$

with a similar definition for (Q, S).

Then it follows from ref. 3 that $U(x, \lambda)$ satisfies the integral equation

$$U(x, \lambda) = \begin{pmatrix} u_1(x, \lambda) \\ u_2(x, \lambda) \end{pmatrix} = \begin{pmatrix} \sin\left(\lambda^{\frac{1}{2}} \xi(x) - \alpha\right) - \int_0^x (Q, R)\, d\xi/dt\, dt \\ \cos\left(\lambda^{\frac{1}{2}} \xi(x) - \alpha\right) + \int_0^x (Q, S)\, d\xi/dt\, dt \end{pmatrix} \tag{3.1}$$

Putting $U_j(x, \lambda) = P_j(x) \exp\left(im(\lambda^{\frac{1}{2}} \xi(x))\right)$, $j = 1, 2$, where P_j are continuous functions of x, it follows by the application of Conte and Sangren's lemma in ref. 2 that

$$|P_j| \leqslant M \exp\{ \int_0^x (|Q_1\, d\xi/dt| + |Q_2\, d\xi/dt|)\, dt, \tag{3.2}$$

where $M = O(1)$, for large x.

Since $Q_j\, d\xi/dt \in L[0, \infty)$, $j = 1, 2$, when one of the conditions (A), (B), (C) of § 1 is satisfied, it follows from (3.2) that $P_j, j = 1, 2$, are bounded for all x. Thus when any one of the conditions stated above is satisfied, we have

$$U_j(x, \lambda)\, O = \{\exp\left(im(\lambda^{\frac{1}{2}} \xi(x))\right)\}, \qquad j = 1, 2, \text{ for large } x.$$

A similar result holds for a second solution $V(x, \lambda)$ of (2.2) satisfying $V(0, \lambda) \equiv \{-\cos\alpha, -\sin\alpha\}$.

It may be noted that $U(x, \lambda)$ and $V(x, \lambda)$ are a pair of linearly independent solutions of (2.2), since the Wronskian

$$W(U, V) = U_1 V_2 - U_2 V_1 = 1.$$

Two cases arising with real and complex values of $\lambda^{\frac{1}{2}} \xi(x)$ are now to be distinguished.

Case I. Let $\lambda^{\frac{1}{2}} \xi(x)$ be real.

Putting

$$Y \equiv Y(t) \equiv \{U_1 \cos\left(\lambda^{\frac{1}{2}} \xi(t)\right), -U_2 \sin\left(\lambda^{\frac{1}{2}} \xi(t)\right)\},$$

$$Z \equiv Z(t) \equiv \{U_1 \sin\left(\lambda^{\frac{1}{2}} \xi(t)\right), \quad U_2 \cos\left(\lambda^{\frac{1}{2}} \xi(t)\right)\},$$

$$G \equiv G(t) \equiv \{ V_1 \cos(\lambda^{\frac{1}{2}} \xi(t)), \quad V_2 \sin(\lambda^{\frac{1}{2}} \xi(t)) \},$$
$$H \equiv H(t) \equiv \{ V_1 \sin(\lambda^{\frac{1}{2}} \xi(t)), \ - V_2 \cos(\lambda^{\frac{1}{2}} \xi(t)) \},$$

it follows as in Titchmarsh[4], as x tends to infinity, that the solution pair $U(x, \lambda)$, $V(x, \lambda)$ satisfying the boundary conditions stated before takes the form

$$U(x, \lambda) = \{ U_1(x, \lambda), \ U_2(x, \lambda) \}$$
$$= \begin{pmatrix} \cos(\lambda^{\frac{1}{2}} \xi(x)) & \sin(\lambda^{\frac{1}{2}} \xi(x)) \\ -\sin(\lambda^{\frac{1}{2}} \xi(x)) & \cos(\lambda^{\frac{1}{2}} \xi(x)) \end{pmatrix} \begin{pmatrix} \chi_1(\lambda) \\ \chi_2(\lambda) \end{pmatrix} + o(1) \qquad (3.3)$$

where

$$\begin{pmatrix} \chi_1(\lambda) \\ \chi_2(\lambda) \end{pmatrix} = \begin{pmatrix} -\sin\alpha - \int_0^\infty (Q, Y)\, d\xi/dt\, dt \\ \cos\alpha - \int_0^\infty (Q, Z)\, d\xi/dt\, dt \end{pmatrix} \qquad (3.3\,a)$$

and

$$V(x, \lambda) = \{ V_1(x, \lambda), \ V_2(x, \lambda) \}$$
$$= \begin{pmatrix} \cos(\lambda^{\frac{1}{2}} \xi(x)) & \sin(\lambda^{\frac{1}{2}} \xi(x)) \\ \sin(\lambda^{\frac{1}{2}} \xi(x)) & -\cos(\lambda^{\frac{1}{2}} \xi(x)) \end{pmatrix} \begin{pmatrix} \zeta_1(\lambda) \\ \zeta_2(\lambda) \end{pmatrix} + o(1) \qquad (3.4)$$

where

$$\begin{pmatrix} \zeta_1(\lambda) \\ \zeta_2(\lambda) \end{pmatrix} = \begin{pmatrix} -\cos\alpha - \int_0^\infty (Q, G)\, d\xi/dt\, dt \\ -\sin\alpha - \int_0^\infty (Q, H)\, d\xi/dt\, dt \end{pmatrix} \qquad (3.4\,a)$$

χ_1, χ_2, ζ_1, ζ_2 being continuous and bounded in λ.

Now

$$W(U, V) = U_1 V_2 - U_2 V_1 = \chi_1 \zeta_2 - \chi_2 \zeta_1 + o(1)$$

and from the boundary conditions $W(U, V) = 1$. Thus as $x \to \infty$,

$$\chi_1 \zeta_2 - \chi_2 \zeta_1 = 1. \qquad (3.4\,b)$$

It therefore follows that for the same λ, χ_1, χ_2 or ζ_1, ζ_2 cannot both vanish simultaneously.

Case II. Let $\lambda^{\frac{1}{2}} \xi(x)$ be complex.

Then it follows in a manner similar to that of Titchmarsh[4] that

$$U(x, \lambda) = \exp(-i\lambda^{\frac{1}{2}} \xi(x)) (M(\lambda) + o(1)), \qquad (3.5)$$

as x tends to infinity, where

$$M(\lambda)=\begin{pmatrix}M_1(\lambda)\\ M_2(\lambda)\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}\sin\alpha+i\cos\alpha-\int_0^\infty\{\exp\left(i\lambda^{\frac12}\xi(t)\right)(Q_1U_1-iQ_2U_2)\}\frac{d\xi}{dt}\,dt\\ \cos\alpha+i\sin\alpha+\int_0^\infty\{\exp\left(i\lambda^{\frac12}\xi(t)\right)(Q_2U_2-iQ_1U_1)\}\frac{d\xi}{dt}\,dt\end{pmatrix}$$

Similarly for the solution $V(x,\lambda)$, we have

$$V(x,\lambda)=\exp\{-i\lambda^{\frac12}\xi(x)\}(N(\lambda)+o(1)), \tag{3.6}$$

as x tends to infinity, where

$$N(\lambda)=\begin{pmatrix}N_1(\lambda)\\ N_2(\lambda)\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}\cos\alpha-i\sin\alpha-\int_0^\infty\{\exp\left(i\lambda^{\frac12}\xi(t)\right)(Q_1V_1-iQ_2V_2)\}\frac{d\xi}{dt}\,dt\\ -\sin\alpha+i\cos\alpha+\int_0^\infty\{\exp\left(i\lambda^{\frac12}\xi(t)\right)(Q_2V_2-iQ_1V_1)\}\frac{d\xi}{dt}\,dt\end{pmatrix}$$

Thus the solution $\psi(x,\lambda)=\{\psi_1(x,\lambda),\ \psi_2(x,\lambda\}=v(x,\lambda)+m(\lambda)u(x,\lambda)$ of (1.1) which belongs to $L^2\,[0,\infty)$, is given by

$$\exp\{-i\lambda^{\frac12}\xi(x)\}\left[\begin{pmatrix}N_1(\lambda) & M_1(\lambda)\\ N_2(\lambda) & M_2(\lambda)\end{pmatrix}\begin{pmatrix}(\lambda+q_2)^{\frac12}\\ m(\lambda)(\lambda+q_1)^{\frac12}\end{pmatrix}+o(1)\right] \tag{3.7}$$

4. BEHAVIOUR OF $\exp\{-i\lambda^{\frac12}\xi(x)\}$ WHERE x IS LARGE

Putting $\lambda=\mu+i\nu$, $\mu\neq 0$ and $\nu>0$ but sufficiently small, it follows from (2.1) that

$$\begin{aligned}&i\lambda^{\frac12}\xi(x)\\ &=\int_0^x[\{\nu^2+(-q_1-\mu)(\mu+q_2)\}^{\frac12}\\ &\qquad-\tfrac12 i\nu(2\mu+q_1+q_2)\{\nu^2+(-q_1-\mu)(\mu+q_2)\}^{-\frac12}\\ &\qquad+O\{\nu^2(2\mu+q_1+q_2)^2\left(\nu^2+(-q_1-\mu)(\mu+q_2)\right)^{-3/2}\}]\,dt\end{aligned}$$

when q_j, $j=1,2$, satisfy any of the conditions (A), (B) or (C) of § 1 and $(\mu+q_1)(\mu+q_2)$ is bounded away from zero for all x in $0\leqslant x<\infty$ and this is satisfied if $|\lambda|$ does not assume the values q_1, q_2 at any point in $0\leqslant x<\infty$.

It therefore follows after some easy calculations that as x tends to infinity, $\exp\{-i\lambda^{\frac{1}{2}}\xi(x)\}$

(i) tends to infinity, where $-k<\mu<k$;

(ii) tends to infinity, for all $\mu\neq 0$ in $(-\infty,\infty)$;

(iii) tends to zero when $-c_2<\mu<c_1$;

(iv) tends to infinity when $-\infty<\mu<-c_2$ or $-c_1<\mu<\infty$;

k and c_j being the same as defined in (A) and (C) of § 1.

It may be noted that q_j satisfy the condition (A) of § 1 for (i), the condition (B) of § 1 for (ii) and the condition (C) of § 1 for (iii) and (iv) respectively.

5. The Basic Theorem

We establish the following theorem.

Theorem : The system of first order differential equations (1.1) under the boundary conditions (1.2) has a purely continuous spectrum over the real λ-axis from $-k$ to k $(0<k<\infty$, k being defined as in (A) of §1), if $q_j(x)$, $j=1, 2$, are positive and bounded below; has a purely continuous spectrum over the entire real λ-axis (origin excluded) if $q_j(x)$ do not belong to L but tend to zero as x tends to infinity, and has purely continuous spectrum from $-\infty$ to $-c_2$ and $-c_1$ to ∞ with a discrete spectrum in $(-c_2, -c_1)$, (the points $-c_1$, $-c_2$ being excluded), when $q_j\to c_j\neq 0$, as $x\to\infty$. It is given that in every case $q'_j(x)\in L[0,\infty)$.

Proof: To study the nature of the spectrum as detailed in Titchmarsh[4] we study the properties of $m(\lambda)$, $0<\arg\lambda<\pi$, from the fact that $\psi(x,\lambda)$ determined by (3.7) belongs to $L^2[0,\infty)$. Two cases $\exp\{-i\lambda^{\frac{1}{2}}\xi(x)\}\to 0, \infty$, as x tends to infinity, have to be distinguished.

(*a*) When $\exp\{-i\lambda^{\frac{1}{2}}\xi(x)\}$ tends to infinity as x tends to infinity. It follows from (3.7) that

$$\begin{aligned} m(\lambda) &= -\lim_{x\to\infty}\{(\lambda+q_2)^{\frac{1}{2}}N_1(\lambda)/(\lambda+q_1)^{\frac{1}{2}}M_1(\lambda)\} \\ &= -\lim_{x\to\infty}\{(\lambda+q_2)^{\frac{1}{2}}N_2(\lambda)/(\lambda+q_1)^{\frac{1}{2}}M_2(\lambda)\}, \end{aligned} \tag{5.1}$$

where $\lambda=\mu+i\nu$, $\nu>0$.

Therefore, when $q_j\to 0$, as x tends to infinity,

$$m(\lambda)=-N_2(\lambda)/M_2(\lambda) \tag{5.2}$$

and when $q_j \to c_j \neq 0$, as x tends to ∞,

$$m(\lambda) = -(\lambda + c_2)^{\frac{1}{2}} N_2(\lambda)/(\lambda + c_1)^{\frac{1}{2}} M_2(\lambda) \tag{5.3}$$

When q_j are bounded below, q_j either tend to finite limits (zero or otherwise) or tend to infinity. When q_j tend to finite limits, $m(\lambda)$ has the form given in (5.2) or (5.3) according as the limits are zero or otherwise. When q_j tend to infinity, $Q_j\, d\xi/dt$ defined in (2.2) do not belong to $L[0, \infty)$ and this case is left out.

Now let λ tend to μ.

Then $\lim\limits_{\lambda \to \mu} M_2(\lambda) = \frac{1}{2}(\chi_2 - i\chi_1)$ and $\lim\limits_{\lambda \to \mu} N_2(\lambda) = \frac{1}{2}(\zeta_2 - i\zeta_1)$, where χ_j, ζ_j, $j = 1, 2$, are given by (3.3 *a*) and (3.4 *a*) respectively.

$$\lim_{\lambda \to \mu} m(\lambda) = -\left(\frac{\mu + \beta_2}{\mu + \beta_1}\right)^{\frac{1}{2}} \left\{\frac{(\chi, \zeta)}{|\chi|^2} + i\frac{1}{|\chi|^2}\right\}, \text{ by } (3.4\,b)$$

where as usual $(\chi, \zeta) = \chi_1\zeta_1 + \chi_2\zeta_2$ and $|\chi|^2 = \chi_1^2 + \chi_2^2$ and the same β_j represent the different limits to which q_j tend, as x tends to infinity, in cases (A), (B) and (C) of § 1. Thus

$$\lim_{\lambda \to \mu} im\, m(\lambda) = -\left(\frac{\mu + \beta_2}{\mu + \beta_1}\right)^{\frac{1}{2}} |\chi|^{-2},$$

which is a non-vanishing continuous, bounded function of μ. Therefore, $m(\lambda)$ does not tend to any real limit nor has it any pole for $\mu \in (-k, k)$ or $\mu \in (-\infty, \infty)$, $\mu \neq 0$ or $\mu \in (-\infty, -c_2)$ and $\mu \in (-c_1, \infty)$ according as the condition (A) or (B) or (C) is satisfied. The spectrum is continuous over each of these ranges.

(*b*) When $\exp\{-i\lambda^{\frac{1}{2}}\xi(x)\}$ tends to zero as x tends to infinity.

In this case it follows from (3.7) and the relation

$$im\, m(\lambda) = -\frac{1}{2i}\left(\psi_1(x, \lambda)\bar{\psi}_2(x, \lambda) - \psi_2(x, \lambda)\bar{\psi}_1(x, \lambda)\right)$$

that $\lim\limits_{\lambda \to \mu}\{im\, m(\lambda)\} = 0$ for sufficiently large x and that $m(\lambda)$ is a meromorphic function of λ.

Hence from § 4 (iii) and § 1, there exists a discrete spectrum on the real λ-axis in $(-c_2, -c_1)$. The theorem is therefore completely proved.

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