AMPLITUDE-FREQUENCY CHARACTERISTICS OF NON-LINEAR TRANSVERSE VIBRATIONS OF CIRCULAR PLATES OF VARIABLE RIGIDITY

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ABSTRACT

Large amplitude vibrations of an isotropic clamped circular plate with variable rigidity has been investigated by Galerkin procedure applied to Berger's approximate method for large deflections. The non-linear second order differential equation thus obtained for the unknown time function is solved in terms of Jacobian elliptic functions.

Results obtained from numerical calculations are presented graphically.

Key words: Amplitude-frequency, Galerkin procedure, Berger's method; transverse vibrations, Jacobian functions.

1. INTRODUCTION

The general theory of transverse vibrations of circular plates was obtained by Kirchoff.[1] Vibration problems of various members have also been investigated by Lord Rayleigh[2]. Non-linear vibration problems have been solved using the method of multiple time-scales by Nayfeh,[3] Kevorkian[4] and Atluri.[5]

Grigoliuk[6] investigated non-linear vibrations of beams and shallow axially symmetrical shells employing Galerkin procedure to a set of equations representing generalisation of von Kármán[7] equations in their dynamical form. Hu-nan-chu and Herrmann[8] applied the dynamic counterpart of von Kármán[7] equations to the study of large vibrations of rectangular plates supported freely along the boundary using perturbation procedure and the principle of conservation of energy. Nowinski[9] investigated large amplitude vibrations of circular plates using von Kármán[7] dynamic equations in combination with an orthogonalisation procedure.

For such moderately large deflections, the strain of the middle plane of the plate must be considered which, however, is ignored in the investiga-

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tion of small deflection problems. But analytical investigation of large deflections by including the strain of the middle plane of the plate, particularly having variable rigidity, seldom lend themselves to exact analysis.[10]

Recourse must then be had to an approximate method,[10] such as that of Berger.[11] Berger's method[11] is essentially based on the neglect of the second invariant, in comparison to the first invariant, of the middle surface strains in the expression for the total potential energy of the system. Thus the variation of potential energy with respect to the in-plane displacements leads to the drastic simplification that the first invariant of the middle surface strain is constant. Thus the resulting differential equations for deflection, though approximate, are still non-linear and may be decoupled in such a manner that they may be readily solved.[11] The fact that the first invariant is a constant[11] is consistent with the results of exact solutions,[12–13] though at present no completely satisfactory physical explanation is available.[11]

Nash and Modeer[14] extended Berger's[11] method to the investigation of non-linear behaviour of vibrating rectangular plates with hinged restrained edges, and of circular plates with periphery somewhat elastically restrained against rotation.[9] Berger's[11] method has been used by Nowinski[15] to the case of orthotropic plates.

The same approximate method of Berger[11] has been adopted by the present author too to investigate the amplitude frequency characteristics of large transverse vibrations of clamped isotropic elastic circular plates of variable rigidity.

2. ANALYSIS

The transverse deflections of the plate under investigation is assumed to be of the order of the plate thickness. For analysis of such moderately large deflections, we neglect the second invariant of the middle surface strain,[11] and thus the potential energy of the system, in polar co-ordinates, will be[14]:

$$V = \frac{1}{2} \int \int D \left[(\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 - \frac{2(1-v)}{r} \frac{dw}{dr} \frac{d^2 w}{dr^2} \right] r dr \qquad (1)$$

in which the θ co-ordinate does not occur due to circular symmetry, and where

$V =$ potential energy,

$D =$ flexural rigidity, variable,

h = plate thickness, variable,

w = deflection, normal to plate-plane,

r = any radius of the plate, ($0 < r < a$, a = radius),

v = Poisson's ratio,

e_1 = first invariant of the middle surface strain.

Transformed to polar co-ordinates, e_1 is given by,[14]

$$e_1 = \epsilon_r + \epsilon_\theta = \frac{du}{dr} + \frac{u}{r} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \qquad (2)$$

where

u = displacement along radial direction,

$\epsilon_r, \epsilon_\theta$ = unit elongations respectively along radial and cross-radial directions.

The kinetic energy of the plate, in polar co-ordinates, is given by[14]

$$T = \frac{\rho h}{2}\iint(\dot{u}^2 + \dot{w}^2)\,dr \qquad (3)$$

where ρ = density of the plate material, and the dots represent derivatives with respect to time.

The Lagrangian function may now be formed from the sum of the bending energies in conjunction with the expression for the kinetic energy and then application of Hamilton's principle and Euler's variational equations will yield[14]:

$$D\nabla^4 w - \alpha^2 D_0 f(t)\nabla^2 w + 2\frac{dD}{dr}\frac{d^3 w}{dr^3}$$
$$+ \frac{d^2 w}{dr^2}\left(\frac{3+v}{r}\frac{dD}{dr} + \frac{d^2 D}{dr^2}\right) - \frac{2}{r^2}\frac{dw}{dr}\frac{dD}{dr}$$
$$+ \frac{v}{r}\frac{d^2 D}{dr^2}\frac{dw}{dr} + \rho h \cdot \frac{d^2 w}{dt^2} = 0 \qquad (4)$$

with

$$he_1 = h\left[\frac{du}{dr} + \frac{u}{r} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2\right] = \frac{\alpha^2 h_0^3}{12}f(t) \qquad (5)$$

where ∇^2, ∇^4 are Laplacian operators and α is a real normalised constant of integration.

Let the rigidity be considered now to vary according to the equation[10]

$$D = D_0 \left[1 - \frac{\lambda r}{a} \right]^3 \tag{6}$$

where

$$D_0 = \frac{E h_0{}^3}{12 (1 - \nu^2)} \tag{7}$$

where

$\lambda = $ a positive constant determining the variation of plate thickness

$a = $ radius of the plate,

$E = $ Young's modulus,

$h_0 = $ maximum plate thickness at centre.

It is to be noted that such a variation in rigidity corresponds to a thickness variation given by,[10]

$$h = h_0 \left[1 - \frac{\lambda r}{a} \right] \tag{8}$$

where $0 \leq h \leq h_0$

The solution to the governing differential eq. (4) is sought, in conjunction with (5), in the form

$$w = w_0 (t) \left[1 - \frac{r^2}{a^2} \right]^2 \tag{9}$$

where $w_0{}^2 (t) = f(t)$ and $u = w_0{}^2 (t) f(r)$, for immovable clamped edges, the boundary conditions of which are given by

$$(w)_{r=a} = 0 = \left(\frac{dw}{dr} \right)_{r=a} \tag{10}$$

Now from (5) one gets

$$f(r) = \int \frac{a^2 h_0{}^2 r}{12 \left(1 + \frac{\lambda r}{a} \right)} \, dr - \frac{8 w_0{}^2}{a^4} \int r^3 \left(1 - \frac{r^2}{a^2} \right) dr + c \tag{11}$$

where c is a constant of integration which can be determined for $f(r) = 0$ at $r = a$. Thus $u(r)$ is determined. Also, since $f(r) = 0$ at $r = 0$, the relation determining α becomes

$$\alpha^2 = \frac{4 w_0{}^2 \lambda^2}{h_0{}^2 a^2 [\lambda - \log (1 + \lambda)]} \tag{12}$$

which in the limit $\lambda \to 0$ (*i.e.*, for uniform thickness of the plate) reduces to

$$a^2 = \frac{8w_0^2}{a^2 h_0^2}. \tag{13}$$

Now applying Galerkin procedure to eq. (4) and putting the value of a^2 from eq. (12), the equation for the time-function is obtained in the form

$$\bar{w}_0 + \gamma w_0 + \delta w_0^3 = 0 \tag{14}$$

where

$$\gamma = \frac{20\,D_0\,(b_1 + \tfrac{8}{3})}{\rho h_0\,a^4\,(b_2 + \tfrac{1}{2})} \tag{15}$$

$$\delta = \frac{40\,D_0}{3\rho h_0 a^4} \cdot \frac{\lambda^2}{(b_2 + \tfrac{1}{2})\,[\lambda - \log(1 + \lambda)]}\left(\frac{w_0}{h_0}\right)^2 \tag{16}$$

where again,

$$b_1 = \frac{256\lambda^3}{105} + \frac{(22 - \nu)\,\lambda^2}{4} + \frac{32\,(7 - \nu)\,\lambda}{35} - \frac{6\nu}{5}\left(\frac{\lambda}{8} + \frac{16}{35}\right)\lambda^2 \tag{17}$$

$$b_2 = \frac{128\lambda}{693}. \tag{18}$$

Thus from eqs. (15) and (16), the ratio

$$\frac{\delta}{\gamma} = \frac{2}{3} \cdot \frac{\lambda^2}{(b_1 + \tfrac{8}{3})\,[\lambda - \log(1 + \lambda)]}\left(\frac{w_0}{h_0}\right)^2 \tag{19}$$

is obtained, which in the limit $\lambda \to 0$ reduces to

$$\frac{\delta}{\lambda} = \frac{1}{2}\left(\frac{w_0}{h_0}\right)^2. \tag{20}$$

The solution to eq. (14) may be represented, as usual, in terms of the cosine-type Jacobian elliptic function, as

$$w_0(t) = cn\,(w^*\,t,\,k^*) \tag{21}$$

where *cn* is Jacobi's elliptic function,

and

$$w^{*2} = \gamma + \delta \tag{22}$$

$$k^{*2} = \frac{\delta}{2\,(\gamma + \delta)} \tag{23}$$

where again,

$w^* = $ fundamental frequency of non-linear free vibration,

$\gamma^{\frac{1}{2}}$ = fundamental frequency of linear free vibration,

k^* = modulus of the elliptic function.

Now insertion for γ and δ from eqs. (15) and (16) respectively, into eq. (22) leads to the relation

$$\left(\frac{w_0}{h_0}\right)^2 = \frac{3}{40 \ D_0} \cdot \frac{[\lambda - \log(1 + \lambda)]}{\lambda^2}$$
$$\times \left[\rho h_0 \ a^4 \left(b_2 + \tfrac{1}{2}\right) w^{*2} - 20 \ D_0 \left(b_1 + \tfrac{8}{3}\right)\right] \tag{24}$$

which, in the limit $\lambda \to 0$, may be conveniently put in the form:

$$w^* = \frac{4 \cdot \sqrt{5}}{a^2 \ \sqrt{3}} \cdot \sqrt{\frac{D_0}{\rho h_0}} \cdot \sqrt{2 \left(\frac{w_0}{h_0}\right)^2 + 4}. \tag{25}$$

The period T^* of non-linear vibration is given by

$$T^* = \frac{4K}{w^*} = \frac{4K}{[\gamma + \delta]^{\frac{1}{2}}} \tag{26}$$

and the period T of linear vibration by

$$T = \frac{2\pi}{\gamma^{\frac{1}{2}}} \tag{27}$$

so that their ratio is

$$\frac{T^*}{T} = \frac{2K/\pi}{\left[1 + \dfrac{\delta}{\gamma}\right]^{\frac{1}{2}}} \tag{28}$$

where γ and δ are respectively given by eqs. (15) and (16), and K is the complete elliptic integral of the first kind.

For $\lambda \to 0$, eq. (28) reduces to

$$\frac{T^*}{T} = \frac{2K/\pi}{\left[1 + \dfrac{1}{2}\left(\dfrac{w_0}{h_0}\right)^2\right]^{\frac{1}{2}}} \tag{29}$$

The result, corresponding to eq. (29), obtained by Nowinski[9] under similar boundary conditions is

$$\frac{T_1^*}{T_1} = \frac{2K/\pi}{\left[1 + 0 \cdot 531 \left(\dfrac{w_1}{h}\right)^2\right]^{\frac{1}{2}}} \tag{30}$$

where T_1^*, T_1, w_1 and h are respectively the same as T^*, T, w_0 and h_0 of eq. (29).

Rejecting the non-linear term of eq. (14), the equation governing linear oscillation is obviously obtained; and thence the fundamental frequency[8] for a plate of constant thickness $(\lambda \to 0)$ is found to be

$$\gamma^{\frac{1}{2}} = \frac{8}{a^2} \frac{\sqrt{5}}{\sqrt{3}} \cdot \sqrt{\frac{D_0}{\rho h_0}} \tag{31}$$

Plates of constant thickness find extensive use in telephone industries;[2] whereas plates of variable thickness find application as parts of various machines.[16]

The numerical coefficient, corresponding to that in eq. (31), computed by Nowinski[9] is $10 \cdot 38$; and that by Timoshenko,[16] as a first approximation, is $10 \cdot 33$.

It may be noted that eq. (31) may also be had from eq. (25), by putting the amplitude equal to zero.

Fig. 1 displays the relative period against the relative amplitude according to eqs. (28) and (29). For the sake of comparison, the results of Nowinski[9] have also been shown in the same Fig. 1.

The amplitude frequency variation given by eqs. (24) and (25) are graphically presented in Fig. 2.
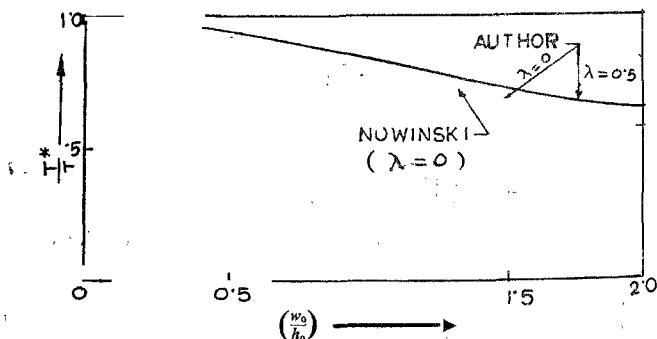

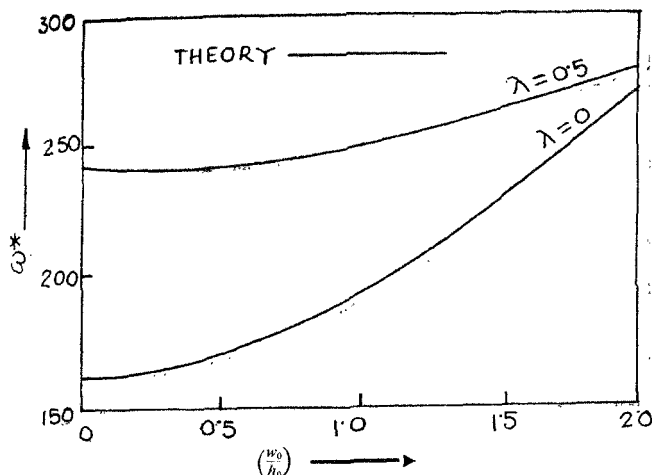FIG. 1. Relative period vs. relative amplitude.

FIG. 2. Frequency *vs.* amplitude relation.

NOTATIONS

The following symbols have been adopted:

a	radius of the plate,
D	flexural rigidity of the plate,
E	Young's modulus
e_1	first invariant of middle surface strains,
h_0	maximum plate thickness at centre,
u, v	displacements along radial and cross radial directions, respectively,
$\epsilon_r, \epsilon_\theta$	unit elongations along radial and cross-radial directions, respectively,
V	strain energy,
w	deflection, normal to plate plane,
v	Poisson's ratio,
λ	a constant determining the variation of plate thickness,
w^*	fundamental frequency of non-linear vibration.

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