ON THE THEORY OF TRANSFORMS ASSOCIATED WITH EIGENVECTORS (I)

S. TIWARI

(*Department of Mathematics, P.G. Wing of M.B.B. College, Agartala, Tripura 799004, India*)

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ABSTRACT

In this paper the author studies a transform theory based on the solutions of the differential system

$$(L - \lambda I)\,\phi = 0,$$

where

$$L = \begin{pmatrix} -d^2/dx^2 + p(x) & r(x) \\ r(x) & -d^2/dx^2 + q(x) \end{pmatrix}$$

and ϕ is a two component column vector function.

A pair of solutions of the above system in the interval $[0, b]$ containing scalars $l_{rs}(\lambda)\,(r, s = 1, 2)$ is obtained. A matrix $(\rho_{rs}(\lambda))$, $(r, s = 1, 2)$ consisting of step-functions is defined with the help of residues of $l_{rs}(\lambda)$. The expansion formula and Parseval formula are then expressed in the form of Stielte's integrals involving the functions ρ_{rs}. Further results are first obtained in the interval $[0, b]$ and then b is made to tend to infinity for the study of the singular case $[0, \infty)$. The transform $F(u) = \{F_1, F_2\}$ of $f(x) = \{f_1, f_2\}$ and the reverse transform $f(x)$ of $F(u)$ are obtained as

$$F_r = \int_0^\infty \phi^T{}_r(0 \mid x, \lambda) f(x)\,dx \qquad (r = 1, 2)$$

and

$$f(x) = \sum_{r=1}^{2} \int_{-\infty}^{\infty} \phi_r(0 \mid x, u)\, F^T(u)\, d\,\rho_r(u)$$

respectively, where $\phi_r(0 \mid x, \lambda)$, $(r = 1, 2)$ are the boundary condition vectors at $x = 0$ and ρ_r denotes the r^{th} column of $(\rho_{rs}(u))$. A good number of theorems are proved which ultimately lead to the following:

Theorem. *A necessary and sufficient condition that $f \in L^2$ is that $F \in \mathcal{L}^2$.*

Some of the results obtained are generalisations of those of Titchmarsh[3].

Key words: Boundary condition vectors, Bilinear concomitant, Wronskian, L^2-solution, residue, orthonormal, singular surface, transform, reverse transform, convergence in mean.

1. Introduction

The object of this paper is to develop a transform theory based on the solutions of the differential system

$$(L - \lambda I)\phi = 0, \tag{1.1}$$

where

$$L = \begin{pmatrix} -d^2/dx^2 + p(x) & r(x) \\ r(x) & -d^2/dx^2 + q(x) \end{pmatrix}, \tag{1.2}$$

$\phi = \phi(x) = \{u(x), v(x)\}$ is two component column vector; λ is a variable parameter real or complex; $p(x)$, $q(x)$ and $r(x)$ are all real valued and continuous functions of x throughout the interval $[\,||0, b||\,]$ and b will be ultimately made to tend to infinity. The boundary conditions are

$$\left.\begin{aligned} a_{j1}\, u(0) + a_{j2}\, u'(0) + a_{j3}\, v(0) + a_{j4}\, v'(0) = 0 \\ b_{j1}\, u(b) + b_{j2}\, u'(b) + b_{j3}\, v(b) + b_{j4}\, v'(b) = 0 \end{aligned}\right\} \tag{1.3}$$

$j = 1, 2$; accents denoting differentiation with respect to x, and the selfs adjointness conditions are given by

$$\left.\begin{aligned} a_{11}\, a_{22} - a_{12}\, a_{21} + a_{13}\, a_{24} - a_{14}\, a_{23} = 0 \\ b_{11}\, b_{22} - b_{12}\, b_{21} + b_{13}\, b_{24} - b_{14}\, b_{23} = 0 \end{aligned}\right\}. \tag{1.4}$$

2. Notations and Preliminaries

If $\phi_j = \{u_j, v_j\}$ and $\phi_k = \{u_k, v_k\}$ be two column vectors, then we define their 'Bilinear Concomitant' as

$$[\phi_j, \phi_k] = \begin{vmatrix} u_j & u_k \\ u'_j & u'_k \end{vmatrix} + \begin{vmatrix} v_j & v_k \\ v'_j & v'_k \end{vmatrix}.$$

We represent, after Chakrabarty[1], any vector $\phi(x)$ whose component together with their first derivatives assume prescribed values at $x = \xi$ by the symbol $\phi(\xi \mid x) = \{u(\xi \mid x),\ v(\xi \mid x)\}$. It follows, in usual manner, that there exist vectors $\phi_j(0 \mid x, \lambda), j = 1, 2$; $\phi_k(b \mid x, \lambda), k = 3, 4$, which are solutions of (1.1) and are such that

$$u_j(0 \mid 0, \lambda) = a_{j2}; u'_j(0 \mid 0, \lambda) = -a_{j1}; v_j(0 \mid 0, \lambda) = a_{j4};$$
$$v'_j(0 \mid 0, \lambda) = -a_{j3}, (j = 1, 2); u_k(b \mid b, \lambda) = b_{j2};$$
$$u'_k(b \mid b, \lambda) = -b_{j1}; v_k(b \mid b, \lambda) = b_{j4};$$
$$v'_k(b \mid b, \lambda) = -b_{j3} (k = 3, j = 1; k = 4, j = 2).$$

These vectors will be called the 'boundary condition vectors' at $x = 0$ and $x = b$ respectively.

If $\phi = \phi(\xi \mid x, \lambda)$ be any vector satisfying (1.3) and ϕ_j, ϕ_k be the boundary condition vectors then (1.3) and (1.4) respectively may be expressed in the following alternative 'Kodaira form'[2]:

$$[\phi, \phi_j] = 0, \qquad [\phi, \phi_k] = 0 \tag{2.1}$$

and

$$[\phi_1, \phi_2] = 0, \qquad [\phi_3, \phi_4] = 0. \tag{2.2}$$

If we denote by $D(\lambda)$ the Wronskian of the boundary condition vectors then

$$D(\lambda) = [\phi_1, \phi_3][\phi_2, \phi_4] - [\phi_1, \phi_4][\phi_2, \phi_3] \tag{2.3}$$

is an entire function of λ, independent of x and takes real values when λ is real.

For column vectors y and z; (y, z) denotes $y^T z$; $\langle y, z\rangle_{0,x}$ stands for $\int_0^x (y, z)\, dt$, and $\| y \|_{0,x}$ for $\langle y, y\rangle_{0,x} = \langle y, \bar{y}\rangle_{0,x}$ when y is complex. When $x = b$, $\langle y, z\rangle$ and $\| y \|$ stand for $\langle y, z\rangle_{0,b}$ and $\| y \|_{0,b}$ respectively. If $F(u) = \{F_1(u), F_2(u)\}$, $G(u) = \{G_1(u), G_2(u)\}$ and columns of

$$\begin{pmatrix} K_{11}(u) & K_{21}(u) \\ K_{12}(u) & K_{22}(u) \end{pmatrix}$$

are denoted by $K_r(u) = \{K_{r1}(u), K_{r2}(u)\}$, $r = 1, 2$, then $\langle F, G, dK\rangle_{c,d}$ stands for

$$\sum_{r=1}^{2} \sum_{s=1}^{2} \int_c^d F_r(u)\, G_s(u)\, dK_{rs}(u) = \sum_{r=1}^{2} \int_c^d F_r(u) \big(G(u), dK_r(u)\big)$$

and $\| F, dK \|_{c,d}$ for $\langle F, F, dK\rangle_{c,d}$.

Further $\langle F, G, dK\rangle_{-\infty,\infty}$; $\| F, dK \|_{-\infty,\infty}$ are denoted by $\langle F, G, dK\rangle$; $\| F, dK \|$ respectively. Let

$$\left.\begin{aligned} \psi_1(x, \lambda) &= \big([\phi_2, \phi_4)]\, \phi_3(b \mid x, \lambda) - [\phi_2, \phi_3]\, \phi_4(b \mid x, \lambda)\big)/D(\lambda) \\ \psi_2(x, \lambda) &= \big([\phi_1, \phi_3)]\, \phi_4(b \mid x, \lambda) - [\phi_1, \phi_4]\, \phi_4(b \mid x, \lambda)\big)/D(\lambda) \end{aligned}\right\}. \tag{2.4}$$

Corresponding to the boundary condition vectors $\phi_j(0 \mid x, \lambda), j = 1, 2$, let us choose two solutions $\theta_k = \theta_k(0 \mid x, \lambda)$ $(k = 1, 2)$ of (1.1) such that

$$[\phi_j, \phi_k] = \delta_{jk} \, (j, k = 1, 2) \qquad \text{and} \qquad [\theta_1, \theta_2] = 0. \tag{2.5}$$

Then

$$\psi_k(x, \lambda) = \sum_{r=1}^{2} l_{kr}(\lambda) \phi_r(0 \mid x, \lambda) + \theta_k(0 \mid x, \lambda), \tag{2.6}$$

where

$$[\psi_r(x, \lambda), \theta_s(0 \mid x, \lambda)] = l_{rs}(\lambda), \, (r, s = 1, 2). \tag{2.7}$$

$$\langle \psi_r(x, \lambda_1), \psi_s(x, \lambda_2) \rangle = \frac{l_{rs}(\lambda_2) - l_{rs}(\lambda_1)}{\lambda_1 - \lambda_2} \quad . \tag{2.8}$$

Also, $l_{rs}(\lambda)$ have an infinite number of simple poles at the zeros of $D(\lambda)$. If λ_n be a simple pole of $l_{rs}(\lambda)$ with residue $R_{rs}(n)$, then we have to consider the following cases:

Case I. Let λ_n be a simple zero of $D(\lambda)$, then

$$R_{11}(n) R_{22}(n) = R^2_{21}(n) = R^2_{21}(n) \tag{2.9}$$

and the corresponding normalised eigenvector, say $\psi_n(x)$, may be expressed as

$$\psi_n(x) = \sum_{r=1}^{2} R^{\frac{1}{2}}_{rr}(n) \phi_r(0 \mid x, \lambda_n). \tag{2.10}$$

Case II. Let λ_n be a double zero of $D(\lambda)$, then

$$R_{11}(n) R_{22}(n) - R_{12}{}^2(n) = 1 | (I_{11} I_{22} - I_{12}{}^2) > 0, \tag{2.11}$$

where

$$I_{rs} = \langle \phi_r(0 \mid x, \lambda), \phi_s(0 \mid x, \lambda) \rangle \quad (r, s = 1, 2)$$

and there are two orthogonal normalised eigenvectors, say $\psi_n{}^{(1)}(x)$ and $\psi_n{}^{(2)}(x)$, which may be expressed as

$$\psi_n{}^{(1)}(x) = R_{11}^{-\frac{1}{2}}(n) \sum_{r=1}^{2} R_{1r}(n) \phi_r(0 \mid x, \lambda_n)$$

$$\psi_n{}^{(2)}(x) = -R_{11}^{-\frac{1}{2}}(n) \{R_{11}(n) R_{22}(n) - R_{12}{}^2(n)\}^{\frac{1}{2}} \phi_2(0 \mid x, \lambda_n).$$

In this case, any suitable linear combination of $\psi_n{}^{(1)}(x)$ and $\psi_n{}^{(2)}(x)$ may be taken as the normalised eigenvector. We choose this vector as follows:

Let $f(x)$ be any two component column vector such that $(f(x), f(x)) \in L[0, b]$. Let

$$A_n = \langle \psi_n^{(1)}, f \rangle, \quad B_n = \langle \psi_n^{(2)}, f \rangle.$$

Then

$$\psi_n(x) = \{A_n/(A_n^2 + B_n^2)^{\frac{1}{2}}\} \ \psi_n^{(1)}(x) + \{B_n/(A_n^2 + B_n^2)^{\frac{1}{2}}\} \psi_n^{(2)}(x) \tag{2.12}$$

is our normalised eigenvector in this case.

The eigenvectors $\psi_n(x)$ given by (2.10) or (2.12) form an orthonormal system of vectors. If $f(x)$ possesses continuous derivatives upto the second order in $[0, b]$, satisfies the boundary conditions (2.1) and c_n, $\tilde{c}_n$ denote the Fourier coefficients of $f(x)$ and $Lf(x)$ respectively, then

$$\tilde{c}_n = \lambda_n c_n. \tag{2.13}$$

3. The Matrix $\rho(u)$

We now extend the finite interval $[0, b]$ to the infinite interval $[0, \infty)$, keeping in view that the functions $p(x)$, $q(x)$ and $r(x)$ in the operator L are well behaved at all points of the infinite interval $[0, \infty)$. We tackle the problem of this extension by considering the problem of the interval $[0, b]$ (to be referred to as the b-case) and then making $b \to \infty$. For this purpose, we assume that the conditions of the previous section remain valid for every $b > 0$ and we introduce b as a parameter in the entities of §2 to enable us to study the implications of making $b \to \infty$. For example, by $D(b, \lambda)$ we mean $D(\lambda)$ defined by (2.3) and similarly for other entities. Some of the results obtained here are generalisations of those of Titchmarsh in Chapter VI of Ref. 3.

Let λ_{nb} denote the eigenvalues for the b-case. Let us define a matrix

$$\rho(b, t) = (\rho_{rs}(b, t)) = \begin{pmatrix} \rho_{11}(b, t) & \rho_{21}(b, t) \\ \rho_{12}(b, t) & \rho_{22}(b, t) \end{pmatrix}$$

consisting of non-decreasing step-functions $\rho_{rs}(b, t)$, $(r, s = 1, 2)$ which satisfy the following conditions:

$\rho(b, 0) = 0$ and $\rho_{rs}(b, t)$ increases by $R_{rs}(b, n)$ when t increases through the value λ_{nb}; otherwise $\rho_{rs}(b, t)$ remains constant. The value at the discontinuity is given by

$$\rho_{rs}(b; \lambda_{nb}) = \tfrac{1}{2}[\rho_{rs}(b; \lambda_{nb} - 0) + \rho_{rs}(b; \lambda_{nb} + 0)].$$

Let $f(x) = \{f_1, f_2\}$ be integrable over $[0, b]$. Let

$$F(b; u) = \{F_1(b; u), F_2(b; u)\}$$

where

$$F_r(b; u) = \langle \phi_r(0 \mid x, u), f(x) \rangle \quad (r = 1, 2). \tag{3.1}$$

Let λ_{nb} be a simple zero of $D(b; \lambda)$, then the Fourier coefficients of $f(x)$ are given by

$$c_{nb} = \langle \psi_n(b; x), f(x) \rangle = \sum_{r=1}^{2} K_{rr}^{\frac{1}{2}}(b; n) F_r(b; \lambda_{nb}). \tag{3.2}$$

The expansion formula may be expressed as

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_{nb} \psi_n(b; x) \\ &= \sum_{n=-\infty}^{\infty} \sum_{r=1}^{2} \sum_{s=1}^{2} \phi_r(0 \mid x, \lambda_{nb}) F_s(b; \lambda_{nb}) R_{rs}(b; n) \\ &= \sum_{r=1}^{2} \sum_{s=1}^{2} \int_{-\infty}^{\infty} \phi_r(0 \mid x, u) F_s(b; u)\, d\rho_{rs}(b; u) \\ &= \sum_{r=1}^{2} \int_{-\infty}^{\infty} \phi_r(0 \mid x, u) \big((F(b; u), d\rho_r(b; u)\big). \end{aligned} \tag{3.3}$$

The Parseval formula may be written as

$$\begin{aligned} \| f \| &= \sum_{n=-\infty}^{\infty} c^2_{nb} = \sum_{n=-\infty}^{\infty} \sum_{r=1}^{2} \sum_{s=1}^{2} F_r(b; \lambda_{nb}) R_{rs}(b; n) F_s(b; \lambda_{nb}) \\ &= \| F(b; u),\ d\rho(b; u) \| \end{aligned} \tag{3.4}$$

The Parseval formula for $\tilde{f}(x) = Lf(x)$ becomes

$$\begin{aligned} \| \tilde{f} \| &= \sum_{n=-\infty}^{\infty} \overset{2}{\lambda}_{nb} \Big[\sum_{r=1}^{2} \sum_{s=1}^{2} F_r(b; \lambda_{nb}) F_s(b; \lambda_{nb}) R_{rs}(b; n) \Big] \\ &= \| uF(b; u),\ d\rho(b; u) \| \end{aligned} \tag{3.5}$$

If λ_{nb} is a double zero of $D(b, \lambda)$ and the corresponding normalised eigenvector is given by (2.12), then the Fourier Coefficients of $f(x)$ are given by

$$c_{nb} = (A_n^2 + B_n^2)^{\frac{1}{2}},$$

where

$$A_n = R_{11}^{-\frac{1}{2}}(b; n) \sum_{r=1}^{2} R_{1r}(b; n) F_r(b; \lambda_{nb})$$

$$B_n = -R_{11}^{-\frac{1}{2}}(b; n) [R_{11}(b; n) R_{22}(b; n) - R_{12}^2(b; n)]^{\frac{1}{2}} F_2(b; \lambda_{nb}).$$

It can be easily verified that even in this case the expansion formula, the Parseval formula and the Parseval formula for $\tilde{f}(x)$ reduce to (3.3), (3.4) and (3.5) respectively.

THEOREM (3.1). The functions $\rho_{rs}(b;u)$ $(r, s = 1, 2)$ are bounded over any fixed finite u-interval, independently of b.

Proof: Since $R^{\frac{1}{2}}_{rr}(b;n)/(\lambda-\lambda_{nb})$ and $R^{\frac{1}{2}}_{ss}(b;n)/(\bar{\lambda}-\lambda_{nb})$ are the Fourier coefficients of $\psi_r(b;x,\lambda)$ and $\psi_s(b;x,\bar{\lambda})$ $(r, s = 1, 2)$ respectively, we obtain

$$\langle \psi_r(b;x,\lambda),\ \bar{\psi}_s(b;x,\lambda)\rangle = \sum_{n=-\infty}^{\infty} R_{rs}(b;n)/\{(\mu-\lambda_{nb})^2+\nu^2\}$$

if $D(b;\lambda)$ has a simple zero at $\lambda=\lambda_{nb}$, $(\lambda=\mu+i\nu)$; and

$$\langle \psi_r(b;x,\lambda),\ \bar{\psi}_s(b;x,\lambda)\rangle > \sum_{n=-\infty}^{\infty} R_{rs}(b;n)/\{(\mu-\lambda_{nb})^2+\nu^2\}$$

if $D(b,\lambda)$ has a double zero at $\lambda=\lambda_{nb}$.

Therefore, from (2.8), we get

$$-\frac{I_m l_{rs}(b;\lambda)}{\nu} \geqslant \int_{-\infty}^{\infty} \frac{d\rho_{rs}(b;u)}{(\mu-u)^2+\nu^2}. \tag{3.6}$$

By arguments similar to those of Chakrabarty[4] and Titchmarsh[3] it follows that $l_{rs}(b;\lambda)$ are bounded as $b\to\infty$ through a suitable sequence if $\nu\neq 0$. Hence, putting $\mu=0$ and $\nu=1$ in (3.6), we obtain

$$\int_{-\infty}^{\infty} \frac{d\rho_{rs}(b;u)}{u^2+1} \leqslant K, \tag{3.7}$$

where K is independent of b. So

$$\int_{-U}^{U} \frac{d\rho_{rs}(b;u)}{u^2+1} \leqslant K \tag{3.8}$$

and

$$\rho_{rs}(b;U) = \int_{0}^{U} \rho_{rs}(b;u) \leqslant K(U^2+1) \tag{3.9}$$

which proves the theorem.

In view of the above theorem, we can apply Helly's selection theorem to define a set of functions $\rho_{rs}(u)$ $(r, s = 1, 2)$, $u \geqslant 0$, such that $\rho_{rs}(b; u) \to \rho_{rs}(u)$ as $b \to \infty$ through a suitable sequence, say W. Let (u_1, u_2) be any finite interval and $f(u) = \{f_1, f_2\}$ any continuous vector, then as $b \to \infty$ we obtain from Helly-Bray theorem

$$\int_{u_1}^{u_2} \big(f(u), d\rho_r(b; u)\big) \to \int_{u_1}^{u_2} \big(f(u), d\rho_r(u)\big). \tag{3.10}$$

Further, let $w_1 = \max(u_1, v_1)$ and $w_2 = \min(u_2, v_2)$, where w_1 and w_2 are the points of continuity of $\rho_{rs}(u)$. Then as $\nu \to 0$

$$\int_{u_1}^{u_2} d\rho_{rs}(u) \int_{v_1}^{v_2} \frac{\nu d\mu}{(\mu - u)^2 + \nu^2} \left.\begin{matrix} \to \pi\,[\rho_{rs}(w_2) - \rho_{rs}(w_1)]\ (w_1 < w_2) \\ \to 0 \qquad\qquad \text{otherwise} \end{matrix}\right\}. \tag{3.11}$$

4. THE TRANSFORM

Let $f(x) = \{f_1, f_2\}$ be the integral of an absolutely continuous vector and $(f''(x)), f''(x)) \in L[0, c]$. Let $f(x) = \{0, 0\}$ for $x \geqslant c$ and let $f(x)$ satisfy the boundary conditions of our problem at $x = 0$. Let

$$F(u) = \{F_1(u), F_2(u)\},$$

where

$$F_r(u) = \langle \phi_r(0 \mid x, u),\ f(x) \rangle_{0,\infty}. \tag{4.1}$$

Then, if $b > c$, we obtain

$$\| F(b; u), d\rho(b; u) \|_{-\infty, -U} + \| F(b; u), d\rho(b; u) \|_{U,\infty}$$
$$\leqslant U^{-2} [\| uF(b; u),\ d\rho(b; u) \|_{-\infty, -U} + \| uF(b; u), d\rho(b; u \|_{U,\infty}]$$
$$\leqslant U^{-2} \| uF(b; u), d\rho(b; u) \| \leqslant U^{-2} \| \tilde{f} \|_{0,\infty}$$

since (3.5) holds in this case. Also, for fixed U and $b > c$

$$\| F(b; u), d\rho(b; u \|_{-U, U} = \| F(u), d\rho(b; u) \|_{-U, U} \to \| F(u), d\rho(u) \|_{-U, U}$$

by making $b \to \infty$ through a suitable sequence. First making $b \to \infty$ for fixed U and then making $U \to \infty$, it follows that

$$\| F(b; u), d\rho(b; u) \| \to \| F(u), d\rho(u) \|.$$

Hence

$$\| f \|_{0,\infty} = \| F(u), d\rho(u) \| \tag{4.2}$$

for our special class of vectors $f(x)$.

Now, let $f(x)$ be any two component column vector such that $\big(f(x), f(x)\big) \in L[0, \infty)$. Then a sequence of vectors $f^{(n)}(x) = \{f_1^{(n)}(x), f_2^{(n)}(x)\}$ can be determined such that each $f^{(n)}(x)$ belongs to the special class and that

$$\lim_{n\to\infty} \| f - f^{(n)} \|_{0,\infty} = 0.$$

Let

$$F^{(n)}(u) = \{F_1^{(n)}(u),\ F_2^{(n)}(u)\},$$

where

$$F_r^{(n)}(u) = \langle \phi_r(0 \mid x, u), f^{(n)}(x) \rangle_{0,\infty}.$$

Then, from (4.2) we obtain

$$\| \big(F^{(m)}(u) - F^{(n)}(u)\big), d\rho \| = \| f^{(m)} - f^{(n)} \|_{0,\infty}$$

which tends to zero as m and n tend independently to infinity. Hence the sequence of vectors $F^{(n)}(u)$ converges in mean with respect to $\rho(u)$, say to $F(u)$, leading to

$$\| F(u), d\rho(u) \| < \infty$$

and

$$\lim_{n\to\infty} \| (F - F^{(n)}), d\rho \| = 0. \tag{4.3}$$

Further

$$\begin{aligned} &\big|\, \| F, d\rho \| - \| F^{(n)}, d\rho \| \,\big| \\ &\quad \leqslant \big| \langle F, F - F^{(n)} d\rho \rangle + \langle F^{(n)} F - F^{(n)}, d\rho \rangle \big| \\ &\quad \leqslant \{[\| F, d\rho \| \, \| F - F^{(n)}, d\rho \|]^2 \\ &\qquad\quad + [\, \| F^{(n)}, d\rho \| \, \| F - F^{(n)}, d\rho \| \,]^{\frac{1}{2}}\} \to 0 \end{aligned}$$

as $n \to \infty$, in view of the above results.
[*cf.* Hardy, Littlewood and Polya[5], § 29, p. 33]. Hence

$$\| F, d\rho \| = \lim_{n\to\infty} \| F^{(n)}, d\rho \| .$$

Therefore from (4.2), $\forall\ f(x) \in L^2[0, \infty)$, we obtain the Parseval formula

$$\| F(u), d\rho(u) \| = \| f(x) \|_{0,\infty}. \tag{4.4}$$

We call the vector $F(u)$ the Transform of $f(x)$.

If $g(x)=\{g_1(x), g_2(x)\}$ be another vector of $L^2[0, \infty)$ and $G(u)$ be its transform, then $F(u)+G(u)$ is the transform of $f(x)+g(x)$ and using (4.4) we obtain

$$\langle F, G, d\rho \rangle = \langle f, g \rangle_{0, \infty} \tag{4.5}$$

THEOREM (4.1). Let $f(x)=\{f_1(x), f_2(x)\} \in L^2[0, \infty)$, and let

$$F_a(u)=\{F_{1a}(u), F_{2a}(u)\},$$

where

$$F_{ra}(u)=\langle \phi_r(0 \mid x, u), f(x) \rangle_{0, a}, (r=1, 2). \tag{4.6}$$

Then $F_a(u)$ converges in mean with respect to $\rho(u)$ to $F(u)$, as $a \to \infty$, *i.e.*,

$$\| F(u) - F_a(u), d\rho(u) \| \to 0 \text{ as } a \to \infty. \tag{4.7}$$

Proof : We have

$$F_r(u) - F_{ra}(u) = \langle \phi_r(0 \mid x, u), f(x) \rangle_{a, \infty}.$$

Thus $F(u) - F_a(u)$ is the transform of $f(x)$ in $[a, \infty)$ and that of $\{0, 0\}$ in $[0, a]$. Hence we obtain from (4.4)

$$\| F(u) - F_a(u), d\rho(u) \| = \| f(x) \|_{a, \infty},$$

where the right hand side tends to zero as $a \to \infty$.

THEOREM (4.2). Let $F(u)$ be the transform of $f(x)$, where $\big(f(x), f(x)\big) \in L\,[0,\infty)$ and let

$$f_a(x)=\{f_{1a}(x), f_{2a}(x)\} = \sum_{r=1}^{2} \int_{-a}^{a} \phi_r(0 \mid x, u) \big(F(u), d\rho_r(u)\big). \tag{4.8}$$

Then as $a \to \infty$, $f(x)$ is the limit in man of $f_a(x)$;

i.e.,

$$\| f(x) - f_a(x) \|_{0, \infty} \to 0, \quad as \quad a \to \infty. \tag{4.9}$$

Proof: Let $G(u)$ be the transform of $g(x)$, where $\big(g(x), g(n)\big) \in L[0, X]$ and $g(x)=\{0, 0\}$ for $x > X$. Let $G_a(u)=\{G_{1a}(u), G_{2a}(u)\}$, where

$$G_{ra}(u) = \langle \phi_r(0 \mid x, u), g(x) \rangle_{0, a} = \langle \phi_r(0 \mid x, u), g(x) \rangle_{0, x},$$
$$(a > X), \ (r=1, 2)$$

If $G(u)=\{G_1(u), G_2(u)\}$, then we obtain from (4.6)

$$G_r(u) = \langle \phi_r(0 \mid x, u), g(x) \rangle_{0, x}, \ (r=1, 2).$$

Therefore,

$$\langle f_a(x), g(x)\rangle_{0,X} = \langle \sum_{r=1}^{2} \int_{-a}^{a} \phi_r(0 \mid x, u) \big(F(u), d\rho_r(u)\big), g(x)\rangle_{0,X}$$

$$= \langle F, G, d\rho\rangle_{-a,a}. \tag{4.10}$$

Now, from (4.10) and (4.5), we obtain

$$[\langle \big(f(x) - f_a(x)\big), g(x)\rangle_{0,X}]^2 = [\langle F, G, d\rho\rangle_{-\infty,-a} + \langle F, G, d\rho\rangle_{a,\infty}]^2$$

$$\leqslant [\| F, d\rho \|_{-\infty,-a} + \| F, d\rho \|_{a,\infty}] \| G, d\rho \|$$

$$\leqslant [\| F, d\rho \|_{-\infty,-a} + \| F, d\rho \|_{a,\infty}] \| g \|_{0,X} \tag{4.11}$$

(*cf.* Levinson,[6] p. 307).

Let $g(x) = f(x) - f_a(x)$ for $x \leqslant X$. Then

$$\| f(x) - f_a(x) \|_{0,X} \leqslant \| F, d\rho \|_{-\infty,-a} + \| F, d\rho \|_{a,\infty}.$$

Making X arbitrarily large

$$\| f(x) - f_a(x) \|_{0,\infty} \leqslant \| F, d\rho \|_{-\infty,-a} + \| F, d\rho \|_{a,\infty}$$

which yields the desired result.

5. Analogy with Fourier Transforms

(I) Let X be fixed. Then

$$\int_0^X f(x)\, dx = \lim_{a\to\infty} \int_0^X f_a(x)\, dx$$

$$= \lim_{a\to\infty} \sum_{r=1}^{2} \int_{-a}^{a} \big(F(u), d\rho_r(u)\big) \int_0^X \phi_r(0 \mid x, u)\, dx$$

$$= \lim_{a\to\infty} \sum_{r=1}^{2} \int_{-a}^{a} \tilde{\phi}_r(X, u) \big(F(u), d\rho_r(u)\big),$$

where

$$\tilde{\phi}_r(X, u) = \int_0^X \phi_r(0 \mid x, u)\, dx.$$

Hence

$$f(x) = d/dx \sum_{r=1}^{2} \int_{-\infty}^{\infty} \tilde{\phi}_r(x, u) \big(F(u), d\rho_r(u)\big) \tag{5.1}$$

almost everywhere.

$$\text{(II)} \int_0^U (F(u), d\rho_r(u))$$
$$= \lim_{n\to\infty} \sum_{s=1}^{2} \int_0^U F_{sn}(u)\, d\rho_{rs}(u)$$
$$= \lim_{n\to\infty} \sum_{s=1}^{2} \int_0^U d\rho_{rs}(u) \int_0^n (\phi_s(0 \mid x, u), f(x))\, dx$$
$$= \lim_{n\to\infty} \langle f(x),\ W_r(x, U)\rangle_{0,n},$$

where

$$W_r(x, U) = \sum_{s=1}^{2} \int_0^U \phi_s(0 \mid x, u)\, d\rho_{rs}(u). \tag{5.2}$$

Therefore

$$(F(u), \rho'_r(u)) = d/du\, \langle f(x), W_r(x, u)\rangle_{0,\infty} \tag{5.3}$$

at the points where $\rho'_r(u)$ exists.

6. The Vectors $\chi_r(x, \lambda)$, $r = 1, 2$.

By arguments similar to those of Chakrabarty[4], it follows from (2.6) by making $b \to \infty$ through a suitable sequence, that

$$\psi_k(x, \lambda) = \sum_{r=1}^{2} m_{kr}(\lambda)\, \phi_r(0 \mid x, \lambda) + \theta_k(0 \mid x, \lambda), \quad (k = 1, 2) \tag{6.1}$$

where

$$m_{kj}(\lambda) = \lim_{b\to\infty} l_{kj}(b, \lambda), \qquad m_{kj}(\lambda) = m_{jk}(\lambda),$$

the convergence to limits of various entities being uniform. Also

$$\| \psi_k(x, \lambda) \|_{0,\infty} \leqslant -\operatorname{Im} m_{kk}(\lambda)/v. \tag{6.2}$$

Thus $\psi_k(x, \lambda) \in L[0, \infty)$. Adopting the analysis of Everitt,[7] we obtain

$$m_{11}(\lambda)\, m_{22}(\lambda) - m_{12}^2(\lambda) \neq 0, \qquad (\operatorname{Im}(\lambda) \neq 0). \tag{6.3}$$

The following Lemma has been obtained by Bhagat.[8]

Lemma (6.1). The matrix

$$K(\lambda) = (K_{rs}(\lambda)) = \left(\lim_{\nu\to 0} \int_0^\lambda -\operatorname{Im} m_{rs}(\mu + i\nu)\, d\mu\right) \tag{6.4}$$

exists for all real λ; each $K_{rs}(\lambda)$ is a function of bounded variation and

$$K_{rs}(\lambda) = \tfrac{1}{2}\{K_{rs}(\lambda + 0) + K_{rs}(\lambda - 0)\}. \tag{6.5}$$

Also

$$\lim_{\nu \to 0} \int_0^\lambda - \operatorname{Im} \psi_r(x, \mu + i\nu)\, d\mu = \sum_{s=1}^{2} \int_0^\lambda \phi_s(0 \mid x, \mu)\, dK_{rs}(\mu). \tag{6.6}$$

Further we note from (6.2) that $-\operatorname{Im} m_{rr}(\mu + i\nu) > 0$ if $\nu > 0$ and therefore $K_{rr}(\lambda)$ are non-decreasing functions of $\lambda\,(r = 1, 2)$.

THEOREM (6.1). Let

$$\chi_r(x, \lambda) = \{\chi_{r1}(x, \lambda), \quad \chi_{r2}(x, \lambda)\} = \sum_{s=1}^{2} \int_0^\lambda \phi_s(0 \mid x, u)\, dK_{rs}(u), \tag{6.7}$$

where $r = 1, 2$ and λ is real. Then

$$\big(\chi_r(x, \lambda), \chi_r(x, \lambda)\big) \in L[0, \infty).$$

Proof: If λ_{nb} be an eigenvalue and $\psi_n(b; x)$ be corresponding eigen-vector in the b-case, then

$$\langle \psi_n(b; x), \quad \psi_r(b; x, \lambda) \rangle = R^{\frac{1}{2}}{}_{rr}(b; n) | (\lambda - \lambda_{nb}). \tag{6.8}$$

Hence, if $\lambda = \mu + i\nu$, the Parseval formula yields

$$\| \psi_r(b; x, \lambda) \| = \sum_{n=-\infty}^{\infty} R_{rr}(b; n) / \{(\mu - \lambda_{nb})^2 + \nu^2\}. \tag{6.9}$$

If $\lambda = i$, then the left hand side of (6.9) is bounded as $b \to \infty$ through a suitable sequence. Therefore

$$\sum_{n=-\infty}^{\infty} R_{rr}(b; n) | (\lambda^2{}_{nb} + 1) = 0\,(1). \tag{6.10}$$

If λ is real and lies in fixed interval, we obtain from (6.8)

$$\langle \psi_n(b; x), \quad \int_0^\lambda \operatorname{Im} \psi_r(b; x, \mu + i\nu)\, d\mu \rangle = 0\,\big(R^{\frac{1}{2}}{}_{rr}(b; n) | (\lambda^2{}_{nb} + 1)\big).$$

Hence using Parseval formula and then making $b \to \infty$ through a suitable sequence, we obtain

$$\| \int_0^\lambda \operatorname{Im} \psi_r(x, \mu + i\nu)\, d\mu \|_{0,\infty} = 0\,(1).$$

Finally, making $v \to 0$ and using (6.6), we have

$$\| \sum_{s=1}^{2} \int_{0}^{\lambda} \phi_s (0 \mid x, \mu)\, dK_{rs}(\mu) \|_{0,\infty} = 0\,(1)$$

which yields the desired result.

7. Relation between $\chi_r\,(x, u)$ and $W_r\,(x, u)$

Making $b \to \infty$ through a suitable sequence and then $U \to \infty$ in (3.8) it follows that

$$\int_{-\infty}^{\infty} d\rho_{rs}(u)|(u^2 + 1) \leqslant K. \tag{7.1}$$

By Green's theorem

$$(\lambda - \lambda_{nb}) \langle \phi_r (0 \mid x, \lambda_{nb}),\quad \psi_1 (b\,; x, \lambda)\rangle$$
$$= \langle \phi_r (0 \mid x, \lambda_{nb}), L\psi_1 (b\,; x, \lambda)\rangle - \langle \psi_1 (b\,; x, \lambda), L\phi_r (0 \mid x, \lambda_{nb})\rangle$$
$$= [\psi_1 (b\,; x, \lambda), \phi_r (0 \mid x, \lambda_{nb})]\,(b) - [\psi_1 (b : x, \lambda), \phi_r (0 | x, \lambda_{nb})]\,(0).$$

The second term on the right hand side

$$= -1, \qquad \text{if } r = 1$$
$$= 0, \qquad \text{if } r = 2.$$

The first term on the right hand side is zero because $\psi_1 (b\,; x, \lambda))$, $\psi_n (b; x)$, $\psi_n^{(1)} (b\,; x)$ and $\psi_n^{(2)} (b\,; x)$ satisfy the same boundary conditions at $x = b$ and it follows from the expressions for $\psi_n (b\,; x)$, $\psi_n^{(1)} (b\,; x)$ and $\psi_n^{(2)} (b; x)$ that $\phi_r (0 \mid x, \lambda_{nb})$ $(r=1, 2)$ also satisfy the same boundary conditions at $x=b$.

Hence

$$\begin{aligned} \langle \phi_r (0 \mid x, \lambda_{nb}), \psi_1 (b; x, \lambda)\rangle &= 1/(\lambda - \lambda_{nb}), \quad \text{if } r = 1 \\ &= 0 \qquad\qquad\quad, \quad \text{if } r = 2. \end{aligned} \tag{7.2}$$

Therefore, the transform of $\psi_1 (b\,; x, \lambda)$ in $[0, b]$ is $\{1/(\lambda - u), 0\}$.

Similarly the transform of $\psi_2 (b\,; x, \lambda)$ ib $[0, b]$ in $\{0, 1|(\lambda - u)\}$. The formula (4.5), therefore, yields

$$\langle \psi_r (b\,; x, \lambda_1), \psi_s (b\,; x, \lambda_2)\rangle = \int_{-\infty}^{\infty} d\rho_{rs} (b\,; u)|(\lambda_1 - u)(\lambda_2 - u),$$

$r, s = 1, 2$. Putting $\lambda = \lambda_1 = \mu + i\nu$, $\bar{\lambda} = \lambda_2 = \mu - i\nu$ and using (2.8), we obtain

$$- \frac{\operatorname{Im} l_{rs}(b, \lambda)}{\nu} = \int_{-\infty}^{\infty} d\rho_{rs}(b; u) | \{(\mu - u)^2 + \nu^2\}. \tag{7.3}$$

Therefore

$$\operatorname{Im} l_{rs}(b; i) - \operatorname{Im} l_{rs}(b; \lambda)/\nu$$
$$= \int_{-\infty}^{\infty} \left\{ \frac{1}{(\mu - u)^2 + \nu^2} - \frac{1}{u^2 + 1} \right\} d\rho_{rs}(b; u)$$

Making $b \to \infty$ through a suitable sequence, we obtain

$$\int_{u_1}^{u_2} - \operatorname{Im} m_{rs}(\lambda)\, d\mu = \int_{-U}^{U} d\rho_{rs}(u) \int_{u_1}^{u_2} \nu d\mu | \{(\mu - u)^2 + \nu^2\} + 0(\nu).$$

where $U > u_2$ and $-U < u_1$ (*cf.* Titchmarsh[3], p. 137).

Making $\nu \to 0$ and using (3.11) the right hand side tends to

$$\pi [\rho_{rs}(u_2) - \rho_{rs}(u_{21})] = \pi \int_{u_2}^{u_2} d\rho_{rs}(u),$$

where u_1 and u_2 are the points of continuity of $\rho_{rs}(u)$.

Now, it follows from the definitions of functions $K_{rs}(u)$ and $\rho_{rs}(u)$ that

$$K(u) = \pi\rho(u) \tag{7.4}$$

Further

$$\chi_r(x, \lambda) = \sum_{s=1}^{2} \int_0^{\lambda} \phi_s(0 \mid x, u)\, dK_{rs}(u) \qquad (\lambda \text{ real})$$
$$= \pi \sum_{s=1}^{2} \int_0^{\lambda} \phi_s(0 \mid x, u)\, d\rho_{rs}(u) = \pi W_r(x, \lambda). \tag{7.5}$$

8. Singular Surfaces

Following Everitt[7] and Bhagat[8] we get the generalization of Weyl's circle obtained by Titchmarsh[3] for our boundary value problem. We only mention the relevant results required for the purpose of our tranform theory and omit the details. Let us define

$$S_r(b, \lambda, b_{jk}) = S_r(b) = -i[\psi_r(b, x, \lambda), \bar{\psi}_r(b, x, \lambda)]_{x=b} = 0 \tag{8.1}$$

$r = 1, 2$. For fixed b and $\lambda = \mu + i\nu$ $(\nu \neq 0)$, as b_{jk} vary, the points (l_{r1}, l_{r2}) describe a surface in the two-dimensional complex space, whose equation is expressed as

$$S_r(b) = 0 \quad (r = 1, 2).$$

We call these surfaces the singular surfaces of our problem. These surfaces are 'central surfaces' which tend to a limit surface $S_r(\infty) = 0$ as $b \to \infty$. The surface $S_r(\infty) = 0$ is also a central surface and $l_{rs}(b, \lambda) \to m_{rs}(\lambda)$ as $b \to \infty$ through a suitable sequence; the point
$\big(m_{r1}(\lambda), m_{r2}(\lambda)\big) \in S_r(\infty) = 0.$

Let $\big(M_{r1}(b), M_{r2}(b)\big)$ $(r = 1, 2)$ denote the centre of the singular surface $S_r(b) = 0$ in the two-dimensional complex space and let (Z_{r1}, Z_{r2}) be any point on this surface, then the range of the values of Z_{rs} is completely determined by

$$| Z_{rs} - M_{rs}^{(b)} |^2 \leqslant \frac{\| \phi_{3-r}(0 \mid x, \lambda) \| \, \| \phi_{3-s}(0 \mid x, \lambda) \|}{4\nu^2 [\, \| \phi_1(0 \mid x, \lambda) \| \, \| \phi_2(0 \mid x, \lambda) \| - | \langle \phi_1(0 | x, \lambda), \bar{\phi}_2(0 | x, \lambda) \rangle |^2]^2}, \tag{8.2}$$

where

$$[1 - | \langle \phi_1, \bar{\phi}_2 \rangle |^2 / \| \phi_1 \| \, \| \phi_2 \|] > 0. \tag{8.3}$$

for all $b > 0$.

9. The Reverse Transform

We define the following two classes of vectors:

(i) The class of vectors

$$f(x) = \{f_1(x), f_2(x)\} \in L^2 \quad \text{if} \quad \| f \|_{0,\infty} <\rangle \infty \tag{9.1}$$

(ii) The class of vectors

$$F(u) = \{F_1(u), F_2(u)\} \in \mathcal{L}^2 \quad \text{if} \quad \| F, d\rho \| < < \infty. \tag{9.2}$$

THEOREM (9.1). If $F(u) \in \mathcal{L}^2$. Then it has a 'reverse transform' $f(x) \in L^2$.

Proof: Let us define $f_a(x)$ by (4.8). Let $g(x) = \{g_1(x), g_2(x)\} \in L^2[0, X]$ and $g(x) = \{0, 0\}$ for $x > X$, and let $G(u)$ be its transform.

Then the conditions leading to (4.10) are satisfied, and hence, if $0 \leqslant a < b$, we obtain

$$[\langle (f_a(x) - f_b(x)), g(x)\rangle_{0, X}]^2 \leqslant [\| F, d\rho \|_{-b, -a} + \| F, d\rho \|_{a, b}] \, \| g(x) \|_{0, X}.$$

Putting $g(x) = f_a(x) - f_b(x)$ in $(0, X)$ and then making $X \to \infty$, we get

$$\| f_a(x) - f_b(x) \|_{0, \infty} \leqslant \| F, d\rho \|_{-b, -a} + \| F, d\rho \|_{a, b}. \tag{9.3}$$

Hence the sequence of vectors $f_a(x)$ converges in mean over $[0, \infty)$, say, to $f(x)$. Putting $a = 0$ and making $b \to \infty$ in (9.3), it follows that

$$\| f(x) \|_{0, \infty} \leqslant \| F, d\rho \|. \tag{9.4}$$

$f(x)$ is the reverse transform of $F(u)$.

Thus, starting from a vector $f(x)$ of L^2 with transform $F(u)$, it follows that $F(u)$ has the reverse transform $h(x)$ such that $f(x)$ and $h(x)$ are the limits in mean of the sequence of vectors $f_a(x)$ defined by (4.8). Hence

$h(x) = f(x)$ almost everywhere.

Lemma (9.1)

$$\lim_{b \to \infty} \| \psi_r(b, x, \lambda) - \psi_r(x, \lambda)) \| = 0 \tag{9.5}$$

$(\mathrm{Im}(\lambda) \neq 0)$ as $b \to \infty$ through a suitable sequence.

Proof: For simplicity we evaluate the limit when $r = 1$.
We have

$$\| \psi_1(b, x, \lambda) - \psi_1(x, \lambda) \| \leqslant | l_{11} - m_{11} |^2 \| \phi_1 \| + 2 \, | l_{11} - m_{11} | \, | l_{12} - m_{12} | \, | \langle \phi_1, \bar{\phi}_2 \rangle | + | l_{12} - m_{12} |^2 \| \phi_2 \|. \tag{9.6}$$

If ϕ_1 and $\phi_2 \in L^2[0, \infty)$ then the right hand side tends to zero as $b \to \infty$ through a suitable sequence, for $l_{rs}(b, \lambda) \to m_{rs}(\lambda)$ and the lemma follows. When ϕ_1 and ϕ_2 both do not belong to $L^2[0, \infty)$, using (8.2) in (9.6), we obtain

$$\| \psi_1 (b, x, \lambda) - \psi_1 (x, \lambda) \|$$

$$\leqslant \frac{2 \{\| \phi_2 \|\}^2 \| \phi_1 \| + 2 \| \phi_2 \| \{\| \phi_2 \| \| \phi_1 \|\}^2 | \langle \phi_1, \bar{\phi}_2 \rangle |}{4\nu^2 [\| \phi_1 \| \| \phi_2 \| - | \langle \psi_1, \bar{\phi}_2 \rangle |^2]^2}$$

$$\leqslant \frac{1}{\nu^2 \| \phi_1 \| [1 - | \langle \psi_1, \bar{\phi}_2 \rangle |^2 / \| \phi_1 \| \| \phi_2 \|]^2}$$

which tends to zero as $b \to \infty$ if $\phi_1 \notin L^2 [0, \infty)$, since (8.3) holds for all values of $b > 0$. Similarly

$$\| \psi_2 (b, x, \lambda) - \psi_2 (x, \lambda) \| \to 0 \quad \text{as} \quad b \to \infty \text{ if } \phi_2 \notin L^2 [0, \infty).$$

Lemma (9.2)

(i) $\big(W_r (x, u), W_r (x, u)\big) \in L [0, \infty)$ in x.

(ii) $\sum_{s=1}^{2} \langle W_r (x, u_2) - W_r (x, u_1), W_s (x, v_2) - W_s (x, v_1) \rangle$

$$\left.\begin{aligned} &= \sum_{s=1}^{2} \int_{w_1}^{w_2} d\rho_{rs} (u), \quad (w_1 < w_2) \\ &= 0, \qquad\qquad\qquad (w_1 \geqslant w_2), \end{aligned}\right\} \tag{9.7}$$

where $w_1 = \max (u_1, v_1)$, $w_2 = \min (u_2, v_2)$ are the points of continuity of $\rho_{rs} (u)$.

Proof : Let

$$W_r (b ; x, u) = \sum_{s=1}^{2} \int_0^u \phi_s (0 \mid x, t) \, d\rho_{rs} (b ; t).$$

Then

$$W_1 (b ; x, u) = \sum_{0 \leqslant \lambda_{nb} \leqslant u}{}' \big(\phi_1 (0 \mid x, \lambda_{nb}) R_{11} (b ; n) + \phi_2 (0 \mid x, \lambda_{nb}) \times R_{12} (b ; n)\big), \tag{9.8}$$

where the dash denotes that the terms with $\lambda_{nb} = 0$ or u are halved. Two cases arise according as $D (b ; \lambda)$ has a simple or a double zero at $\lambda = \lambda_{nb}$.

CASE I. Let $D (b ; \lambda)$ have a double zero at $\lambda = \lambda_{nb}$. Then from (9.8) and (2.11)

$$W_1(b;x,u)$$

$$= \sum_{0\leqslant\lambda_{nb}\leqslant u}{}' R^{\frac{1}{2}}_{11}(b;n)\,\psi_n^{(1)}(b;x)$$

$$= \sum_{0\leqslant\lambda_{nb}\leqslant u}{}' R^{\frac{1}{2}}_{11}(b;n)\big(A_n(A_n^2+B_n^2)^{-\frac{1}{2}}\psi_n^{(1)}(b;x)$$

$$+B_n(A_n^2+B_n^2)^{-\frac{1}{2}}\psi_n^{(2)}(b;x)\big)$$

$$= \sum_{0\leqslant\lambda_{nb}\leqslant u}{}' R^{\frac{1}{2}}_{11}(b;n)\,\psi_n(b;x), \tag{9.9}$$

where

$$A_n = \langle\psi_n^{(1)}(b;x),\quad \psi_1(b;x,\lambda)\rangle = R^{\frac{1}{2}}_{11}(b;n)/(\lambda-\lambda_{nb})$$

$$B_n = \langle\psi_n^{(2)}(b;x),\quad \psi_1(b;x,\lambda)\rangle = 0.$$

CASE—II. Let $D(b;\lambda)$ have a simple zero at $\lambda=\lambda_{nb}$. Then from (9.8), (2.9) and (2.10)

$$W_1(b;x,u) = \sum_{0\leqslant\lambda_{nb}\leqslant u}{}' R^{\frac{1}{2}}_{11}(b;n)\,\psi_n(b;x)$$

which is of the same form as (9.9). Hence if $u>0$

$$\|W_1(b;x,u)\| = \sum_{0\leqslant\lambda_{nb}\leqslant u}{}'' R_{11}(b;n)\leqslant\rho_{11}(b;u), \tag{9.10}$$

where double dash denotes a factor $\frac{1}{4}$ at the ends. Therefore, if $c<b$

$$\|W_1(b;x,u)\|_{0,c}\leqslant\rho_{11}(b;u)\leqslant K(u),$$

where K is independent of b and c. Making first $b\to\infty$ and then $c\to\infty$ we obtain

$$\|W_1(x,u)\|_{0,\infty}\leqslant K(u) \tag{9.11}$$

and similarly if $u<0$.

Again

$$W_2(b;x,u) = \sum_{0\leqslant\lambda_{nb}\leqslant u}{}' \big(\phi_1(0\mid x,\lambda_{nb})\,R_{21}(b;n)+\phi_2(0\mid x,\lambda_{nb})\,R_{22}(b;n)\big)$$

$$= \sum_{0\leqslant\lambda_{nb}\leqslant u}{}' R^{\frac{1}{2}}_{22}(b;n)\,\psi_n(b;x)$$

by (2.9) and (2.10) if λ_{nb} is a simple zero of $D(b;\lambda)$.

If λ_{nb} be a double zero of $D(b; \lambda)$, we have

$$W_2(b; x, u) = \sum_{0 \leqslant \lambda_{nb} \leqslant u}' R^{\frac{1}{2}}_{22}(b; n)\left(A_n \left(A_n^2 + B_n^2\right)^{-\frac{1}{2}} \psi_n^{(1)}(b; x) + B_n \left(A_n^2 + B_n^2\right)^{-\frac{1}{2}} \psi_n^{(2)}(b; x)\right)$$

$$= \sum_{0 \leqslant \lambda_{nb} \leqslant u}' R^{\frac{1}{2}}_{22}(b; n)\, \psi_n(b; x),$$

where

$$A_n = \langle \psi_n^{(1)}(b; x), \psi_2(b; x, \lambda)\rangle = R_{21}(b, n) | R^{\frac{1}{2}}_{11}(b; n)(\lambda - \lambda_{nb}),$$

$$B_n = \langle \psi_n^{(2)}(b; x), \psi_2(b; x, \lambda)\rangle = -\frac{\{R_{11}(b; n) R_{22}(b; n) - R_{12}^2(b; n)\}^{\frac{1}{2}}}{R^{\frac{1}{2}}_{11}(b; n)(\lambda - \lambda_{nb})}.$$

The analysis now proceeds as in the case of $W_1(b; x, u)$ and first part of the lemma follows. Let $\mathrm{Im}(\lambda) > 0$. Then

$$\langle\left(W_1(b; x, u_2) - W_1(b; x, u_1)\right), \quad \psi_r(b; x, \lambda)\rangle$$

$$= \sum_{u_1 \leqslant \lambda_{nb} \leqslant u_2}' [R_{11}(b; n)\langle \phi_1(0 \mid x, \lambda_{nb}), \psi_r(b; x, \lambda)\rangle + R_{12}(b; n)\langle \phi_2(0 \mid x, \lambda_{nb}), \psi_r(b; x, \lambda)\rangle]$$

$$= \sum_{u_1 \leqslant \lambda_{nb} \leqslant u_2}' R_{1r}(b; n)/(\lambda - \lambda_{nb}) = \int_{u_1}^{u_2} d\rho_{1r}(b; t) | (\lambda - t)$$

by arguments similar to those leading to (7.2). Hence

$$\sum_{r=1}^{2} [\langle\left(W_1(b; x, u_2) - W_1(b; x, u_1)\right), \quad \psi_r(b; x, \lambda)\rangle]$$

$$= \sum_{r=1}^{2} \int_{u_1}^{u_2} d\rho_{1r}(b; t) | (\lambda - t)\,. \tag{9.12}$$

From (9.5), (9.10) and the Schwarz inequality for vectors, we obtain

$$\lim_{b \to \infty} \langle W_1(b; x, u), \quad \left(\psi_r(b; x, \lambda) - \psi_r(x, \lambda)\right)\rangle = 0.$$

Also, since $W_1(b; x, u) \in L^2[0, \infty)$ for some b-sequence and

$$W_1(b; x, u) \to W_1(x, u) \in L^2[0, \infty),$$

$$\lim_{b \to \infty} \langle\left(W_1(b; x, u) - W_1(x, u)\right), \quad \psi_r(x, \lambda)\rangle = 0.$$

Hence

$$\sum_{r=1}^{2} [\langle \big(W_1(x, u_2) - W_1(x, u_1)\big), \quad \psi_r(x, \lambda)\rangle_{0,\infty}] = \sum_{r=1}^{2} \int_{u_1}^{u_2} d\rho_{1r}(t)/(\lambda - t).$$

Therefore

$$\sum_{r=1}^{2} [\langle \big(W_1(x, u_2) - W_1(x, u_1)\big), \int_{v_1}^{v_2} - \operatorname{Im} \psi_r(x, \mu + i\nu)\, d\mu\rangle_{0,\infty}]$$

$$= \sum_{r=1}^{2} \int_{u_1}^{u_2} d\rho_{1r}(t) \int_{v_1}^{v_2} \nu d\mu / \{(\mu - t)^2 + \nu^2\}.$$

Making $\nu \to 0$, using the relations (6.7), (7.5) on the left hand side and he relation (3.11) on the right hand side, we obtain

$$\sum_{r=1}^{2} [\langle \big(W_1(x, u_2) - W_1(x, u_1)\big), \quad \big(W_r(x, v_2) - W_r(x, v_1)\big)\rangle_{0,\infty}]$$

$$\left.\begin{aligned} &= \sum_{r=1}^{2} \int_{w_2}^{w_1} d\rho_{1r}(t) && (w_1 < w_2) \\ &= 0 && (w_1 \geqslant w_2) \end{aligned}\right\} .$$

'or the justification of the limiting process under the sign of integration, ve note that

$$\int_0^{\sigma} - \operatorname{Im} \psi_r(x, \mu + i\delta)\, d\mu = \chi_r(x, \sigma + i\delta) \in L^2[0, \infty)$$

or $\delta = \delta_1, \delta_2, \delta_3 \ldots$ and as $\delta \to 0$, $\chi_r(x, \sigma + i\delta) \to \chi_r(x, \sigma) \in L^2[0, \infty$ imilar arguments apply when we start with $W_2(b; x, u)$ and (ii) follows.

We now start for the reverse transform by considering two column ectors $F(u)$ and $G(u)$ defined as follows:

$F(u) = \{M_1, M_2\}$ in $u_1 \leqslant u \leqslant u_2$;

$G(u) = \{N_1, N_2\}$ in $v_1 \leqslant v \leqslant v_2$

$F(u) = \{0, 0\} = G(u)$ otherwise,

vhere M_1, M_2, N_1 and N_2 are constants.
'he reverse transforms of $F(u)$ and $G(u)$ respectively are then given by

$$f(x) = \sum_{r=1}^{2} \int_{-\infty}^{\infty} \phi_r(0 \mid x, u)\big(F(u), d\rho_r(u)\big)$$

$$= \sum_{r=1}^{2} \sum_{s=1}^{2} M_r \int_{u_1}^{u_2} \phi_s(0 \mid x, u)\, d\rho_{rs}(u)$$

$$= \sum_{r=1}^{2} M_{,r} \big(W_r(x, u_2) - W_r(x, u_1)\big)$$

and

$$g(x) = \sum_{r=1}^{2} N_r\big(W_r(x, v_2) - W_r(x, v_1)\big).$$

Hence

$$\langle f, g\rangle_{0,\infty} = \langle \sum_{r=1}^{2} M_r\big(W_r(x, u_2) - W_r(x, u_1)\big), \sum_{s=1}^{2} N_s\big(W_s(x, v_2) - W_s(x, v_1)\big)\rangle_{0,\infty}$$

$$\left.\begin{aligned} &= \sum_{r=1}^{2}\sum_{s=1}^{2} M_r N_s \int_{w_1}^{w_2} d\rho_{rs}(t) \qquad (w_1 < w_2) \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad (w_1 \geqslant w_2) \end{aligned}\right\} \tag{9.13}$$

by (9.7), where $w_1 = \max(u_1, v_1)$, $w_2 = \min(u_2, v_2)$ are the points of continuity of $\rho_{rs}(t)$ $(r, s = 1, 2)$. Also

$$\langle F, G, d\rho\rangle = \left.\begin{aligned} &\sum_{r=1}^{2}\sum_{s=1}^{2} M_r N_s \int_{w_1}^{w_2} d\rho_{rs}(t) \qquad (w_1 < w_2) \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad (w_1 \geqslant w_2) \end{aligned}\right\}. \tag{9.14}$$

It follows from (9.13) and (9.14) that the Parseval formula

$$\langle F, G, d\rho\rangle = \langle f, g\rangle_{0,\infty}$$

holds in this case.

Thus, defining a step-vector as one each of whose components is a step function, we obtain, by addition of vectors, such as $F(u)$ and $G(u)$ above, the Parseval formula when $F(u)$ and $G(u)$ are any step-vectors with two components having their steps at the points of continuity of $(\rho_{rs}(u))$, and $F(u) = (0, 0) = G(u)$ outside finite intervals. Now, let $F(u)$ be any vector of $\mathcal{L}^2$. Then we can define a sequence of step-vectors $F^{(n)}(u)$, each of the previous type, such that

$$\| F - F^{(n)}, d\rho \| \to 0.$$

Let $f^{(n)}(x)$ be the reverse transform of $F^{(n)}(u)$. Then $\big(f^{(m)}(x) - f^{(n)}(x)$ is the reverse transform of $\big(F^{(m)}(u) - F^{(n)}(u)\big)$ |, and

$$\| f^{(m)} - f^{(n)} \|_{0,\infty} \doteq \| F^{(m)} - F^{(n)}, d\rho \| \to 0$$

as m and n tend to infinity independently of each other.

Hence $f^{(n)}(x)$ converge in mean to $f(x)$, say. Then $f(x)$ is the reverse transform of $F(u)$, and

$$\| F, d\rho \| = \| f \|_{0,\,\infty} \tag{9.15}$$

which may be termed 'reverse Parseval formula'.

It follows from the arguments used in § 4 that the reverse transform defined in the above manner is equal almost everywhere to that defined in § 4.

THEOREM (9.2). If $F(u)$ is a given two component column vector of $\mathcal{L}^2$, $f(x)$ is its reverse transform, and $H(u)$ is the transform of $f(x)$, then $H(u)$ is equivalent to $F(u)$ in the sense that

$$\| F - H, d\rho \| = 0. \tag{9.16}$$

Proof: Let

$$F_{ra}(u) = \langle \phi_r (0 \mid x, u), f(x) \rangle_{0,\,a}.$$

Then the reverse transform of $F_a(u) = \{F_{1a}(u), F_{2a}(u)\}$ is $f(x)$ in $[0, a]$ and $\{0, 0\}$ in $[a, \infty)$. Therefore the reverse transform of $\big(F(u) - F_a(u)\big)$ is $\{0, 0\}$ in $[0, a]$ and $f(x)$ in $[a, \infty)$.

Hence, by the reverse Parseval formula (9.15)

$$\| F - F_a, d\rho \| = \| f \|_{a,\,\infty}.$$

Therefore $F_a(u)$ converges in mean with respect to $\rho(u)$ to $F(u)$. Further, by the arguments of § 4, $F_a(u)$ converges in mean with respect to $\rho(u)$ to $H(u)$.

Hence (9.16) follows.

Combining the relevant results of § 4 and § 9, we obtain the following:

THEOREM (9.3). A necessary and sufficient condition that $f(x) \in L^2$ is that $F(u) \in \mathcal{L}^2$.

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REFERENCES

1. Chakrabarty, N. K. — Some problems in eigenfunction expansions (I). *Quart. J. of Math.* (*Oxford*), 1965, **16** (2), 135–150.

2. Kodaira, K. — On ordinary differential equations of any even order and the corresponding eigenfunction expansion. *Amer. J. Math.*, 1950, **72**, 502–544.

3. Titchmarsh, E. C. — *Eigenfunction Expansions Associated with Second-order Differential Equations*, Part I, 2nd ed., Clarendon Press, Oxford 1962.

4. Chakrabarty, N. K. — Some problems in eigenfunction expansions (III), *Quart. J. of Math.* (*Oxford*), 1968, **19** (2), 213–224.

5. Hardy, G. H., Littlewood, J. L. and Polya, G. — *Inequalities*, Cambridge University Press, 1952.

6. Levinson, N. — The expansion theorem for singular self-adjoint differential operator. *Ann. Math.*, 1954, **59** (2), 300–315.

7. Everitt, W. N. — Fourth order singular differential equations, *Math. Ann.*, 1963, **149**, 320–340.

8. Bhagat, B. — *A Thesis for the Degree of Doctor of Philosophy*, (unpublished), Patna University, 1966.

9. Tiwari, S. — *On Eigenfunction Expansions Associated with Differential Equations*, Thesis (unpublished), University of Calcutta, 1971.