ON THE NATURE OF THE SPECTRUM FOR A PAIR OF SECOND-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT

In connection with the differential system

$$(L - \lambda)\, U = 0 \qquad (0 \leqslant x < \infty) \tag{A}$$

where

$$L = \begin{pmatrix} -\dfrac{d^2}{dx^2} + p(x) & r(x) \\ r(x) & -\dfrac{d^2}{dx^2} + q(x) \end{pmatrix}, \quad U \equiv \{u, v\} \equiv \begin{pmatrix} u \\ v \end{pmatrix}$$

with a prescribed set of boundary conditions at $x = 0$, *the nature of the spectrum (continuous and discrete) is studied. The method used is Titchmarsh's complex variable method initiated in his 'Eigenfunction expansions'.*

Keywords: Spectrum, Boundary-condition vector, Kronecker delta, Entire function, Bilinear concomitant. Differential equations; Eigenfunctions.

1. INTRODUCTION

We consider the differential system (A) *viz.*

$$LU = \lambda U, \quad \lambda \text{ eigenvalue parameter } (0 \leqslant x < \infty), \tag{1.1}$$

where (i) $p(x)$, $q(x)$ are real valued and $p''(x)$, $q''(x) < \infty$ (ii) $r(x)$ is real valued and continuous in $0 \leqslant x < \infty$ (iii) $p(x)$, $q(x)$ and/or $r(x)$ tend to minus infinity as x tends to infinity. As usual accent denotes differentiation with respect to x.

The boundary condition at $x = 0$ is defined as in [6] by

$$a_{j1}\, u(0) + a_{j2}\, u'(0) + a_{j3}\, v(0) + a_{j4}\, v'(0) = 0 \; (j = 1, 2) \tag{1.2}$$

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Let $\phi_j(x, \lambda) \equiv \phi_j(0/x, \lambda) = \{u_j(0/x, \lambda), v_j(0/x, \lambda)\}$, $(j = 1, 2)$ be the boundary condition vectors at $x = 0$ (*see* [6]).

Let $\theta_k = \{x_k, y_k\}$ $(k = 1, 2)$ be two other vectors satisfying (1.1) and the relations

$$[\phi_j, \theta_k] = \delta_{jk}, \quad [\theta_1, \theta_2] = 0 \quad (j, k = 1, 2),$$

δ_{jk} being the Kronecker delta and [..] is the bilinear concomitant of the two vectors (*see* Chakrabarty [2]). Then the pair of L^2-solutions of the system (1.1) is given by

$$\psi_r(x, \lambda) = \theta_r(x, \lambda) + \sum_{s=1}^{2} m_{rs}(\lambda)\, \phi_s(x, \lambda) \quad (r = 1, 2)$$

(compare Chakrabarty [3]).

If $m(\lambda) = (m_{rs}(\lambda))$ be a meromorphic matrix function of λ we define the spectrum to be discrete. On the other hand an interval throughout which

$$\rho_{rs}(\lambda) = \lim_{\nu \to 0} \int_0^{\lambda} I_m\, m_{rs}(\mu + i\nu)\, d\mu$$

are continuous belongs to the continuous spectrum. (*see* Bhagat [1]).

When $p(x)$, $q(x)$, $r(x) \in L$, Bhagat [1] establishes a theorem on the continuous nature of the spectrum associated with the system (1.1). The present author in his paper [6] establishes some conditions for the disreteness of the spectrum of the system (1.1). The present paper is concerned with two theorems on the nature of the spectrum of (1.1) under different sets of conditions.

2. Some Notations and Abbreviations

We make use of the following notation:

$$(y, z) = y_1(t)\, z_1(t) + y_2(t)\, z_2(t)$$

for two vectors $y = \{y_1(t), y_2(t)\}$, $z = \{z_1(t), z_2(t)\}$ (*see* Chakrabarty [2] and Naimark [5]).

In what follows we write

$$Z(t) = (\lambda - p(t))(\lambda - q(t))$$

$$M(t) = -\tfrac{1}{4} Z(t)^{-1/4} [p''(\lambda - p)^{-1} + q''(\lambda - q)^{-1} + \tfrac{5}{4} p'^2 (\lambda - p)^{-2} + \tfrac{5}{4} q'^2 (\lambda - q)^{-2} + \tfrac{1}{2} p' q' Z(t)^{-1}]$$

$$\xi_1(t) \equiv \xi_1(p, q, t) = Z(t)^{1/2} + M(t)\, Z(t)^{-1/4}$$
$$- (\lambda - p(t))^{1/2}/(\lambda - q(t))^{1/2}$$

$$\eta_1(t) = r(t)\, Z(t)^{-1/2}, \quad \zeta_1(t) = \xi_1(q, p, t)$$

$$N_1(t) = \begin{pmatrix} \xi_1(t) & \eta_1(t) \\ \eta_1(t) & \zeta_1(t) \end{pmatrix}$$

$$N_2(t) = \{Z(t)^{-1}, \; Z(t)^{-1}\}, \; S(x) = \{\cos \xi(x), \; \sin \xi(x)\}.$$

3. A Basic Transformation

We consider the system (1.1) which is equivalent to the equations

$$\left.\begin{aligned} \frac{d^2u}{dx^2} + (\lambda - p(x))\, u &= r(x)\, v \\ \frac{d^2v}{dx^2} + (\lambda - q(x))\, v &= r(x)\, u \end{aligned}\right\} \tag{3.1}$$

By means of the transformation

$$\left.\begin{aligned} \xi(x) &= \int_0^x (\lambda - p(t))^{1/2} (\lambda - q(t))^{1/2}\, dt \\ \{\eta(x), \zeta(x)\} &= (\lambda - p(x))^{1/4} (\lambda - q(x))^{1/4} \{u(x), \; v(x)\} \end{aligned}\right\} \tag{3.2}$$

the system (3.1) is transformed to

$$\left.\begin{aligned} \frac{d^2\eta}{d\xi^2} + [K(x, \lambda) + 1/(\lambda - q(x))]\, \eta &= R(x)\, \zeta \\ \frac{d^2\zeta}{d\xi^2} + [K(x, \lambda) + 1/(\lambda - p(x))]\, \zeta &= R(x)\, \eta \end{aligned}\right\} \tag{3.3}$$

where

$$K(x, \lambda) = 1/4\, [p''(\lambda - p)^{-2} (\lambda - q)^{-1} + q''(\lambda - q)^{-2} (\lambda - p)^{-1}]$$
$$+ 5/16\, [p'^2 (\lambda - p)^{-3} (\lambda - q)^{-1} + q'^2 (x - q)^{-3} (\lambda - p)^{-1}]$$
$$+ 1/8\; p'q' (\lambda - p)^{-2} (\lambda - q)^{-2}$$

$$R(x) \equiv R(x, \lambda) = r(x)\, Z(x)^{-1}.$$

In the above λ may be real or complex. If λ is complex, we take $0 < \arg \lambda < \pi$ so that $\mathrm{Im}\ \lambda > 0$ and $0 < \arg (\lambda - p)^m, \arg (\lambda - q)^n < M\pi$ for each m, n, where $M = \max (m, n)$. Then $\mathrm{Im}\ \xi(x) > 0$. (*see* Titchmarsh [8]).

If λ be real and p or $q > \lambda$, we take arg $(\lambda - p)^{1/2}$ or arg $(\lambda - q)^{1/2}$ as equal to $\pi/2$ as the case may be.

$$\text{Let } P(x) = Z(x)^{1/4} H(x) \tag{3.4}$$

where

$$H(x) = \{H_1(x), H_2(x)\}$$

with

$$H_1(x) = \frac{d}{dx}\left[Z(x)^{-1/2}\frac{d\eta}{dx}\right] - Z(x)^{-1/4}\frac{d^2u}{dx^2}$$

$$H_2(x) = \frac{d}{dx}\left[Z(x)^{-1/2}\frac{d\zeta}{dx}\right] - Z(x)^{-1/4}\frac{d^2v}{dx^2}$$

Then following the method used in [6], we obtain

$$\eta(x) = \eta(0)\cos\xi(x) + \eta'(0)\,Z(0)^{-1/2}\sin\xi(x) + \int_0^x \sin(\xi(x) - \xi(t))\,(l_1(t), \Omega(t))\,dt \tag{3.5}$$

and

$$\zeta(x) = \zeta(0)\cos\xi(x) + \zeta'(0)\,Z(0)^{-1/2}\sin\xi(x) + \int_0^x \sin(\xi(x) - \xi(t))\,(l_2(t), \Omega(t))\,dt \tag{3.6}$$

where

$$l_1(t) = \{\xi_1(t), \eta_1(t)\},\ l_2(t) = \{\eta_1(t), \zeta_1(t)\} \text{ and}$$
$$\Omega(t) = \{\eta(t), \zeta(t)\}.$$

Thus

$$\Omega(x) = N(0)\,S(x) + \int_0^x \sin(\xi(x) - \xi(t))\,N_1(t)\,\Omega(t)\,dt \tag{3.7}$$

with

$$N(o) = \begin{pmatrix} \eta(0) & \eta'(0) & Z(0)^{-1/2} \\ \zeta(0) & \zeta'(0) & Z(0)^{-1/2} \end{pmatrix}$$

4. A LEMMA

In what follows we use $|B|$ to represent the matrix whose elements are the moduli of the elements of the corresponding matrix B. We then have the following lemma.

LEMMA I. Let (1) $p(x)$, $q(x) < -Q(x)$ where $Q(x) \geqslant \delta > 0$ or $p(x)$, $q(x)$, $r(x) < -Q(x)$ where $Q(x) \geqslant \delta > 0$ with $r(x) = O(p(x)\,q(x))$, as $x \to \infty$.

(2) $Q(x)^{-1} \in L\,[0, \infty)$

(3) $p'(x) = O\,[\,|\,p(x)\,|^c]$, $q'(x) = O\,[\,|\,q(x)\,|^c]$
$(0 < c < 5/2)$

(4) $p''(x)$, $q''(x)$ are ultimately of one sign

(5) $p'(x)$, $q'(x) < 0$.

Then

$$\int_0^\infty |\,N_1(t)\,|\,|\,N_2(t)\,|\,dt$$

is uniformly convergent with respect to λ (real or complex) in any region for which $|\,\lambda - p(x)\,|$, $|\,\lambda - q(x)\,| \geqslant \delta > 0$ for $0 \leqslant x < \infty$.

The lemma follows in the same way as that indicated in Paladhi [6].

5. SOME ASYMPTOTIC RELATIONS

Using the substitution

$$\Omega_2(x) = Z(x)\,\Omega(x)\exp(i\xi(x))$$

in (3.7) and then applying Conte and Sangren's Lemma [4], it follows that

$$|\,\eta(x), |\,\zeta(x)\,| = O\,[\,|\exp(K_1 p(x)\,q(x))\,|\,|\exp(-i\,\xi(x))\,|\,] \tag{5.1}$$

where K_1 is a positive constant.

The system (3.1) has the solution

$$U = \{u(x), v(x)\} = Z(x)^{-1/4}\,[N(0)\,S(x) + \int_0^x \sin(\xi(x) - \xi(t))\,N_1(t)\,\Omega(t)\,dt] \tag{5.2}$$

Now

$$u(x, \lambda) = Z(x)^{3/4}\exp(K_1\,p(x)\,q(x)) \times [\cos\xi(x)\,u_1(x, \lambda)/\{Z(x)\exp(K_1\,p(x)\,q(x))\}] + \sin\xi(x)\,v_1(x, \lambda)/\{Z(x)\exp(K_1\,p(x)\,q(x))\}]$$

where

$$\left.\begin{aligned}\mu_1(x,\lambda) &= \eta(0) - \int_0^x (\Omega(t), l_1(t)) \sin\xi(t)\,dt \\ \nu_1(x,\lambda) &= \eta'(0)\,Z(0)^{-1/2} + \int_0^x (\Omega(t), l_1(t)) \cos\xi(t)\,dt\end{aligned}\right\} \tag{5·3}$$

with a similar expression for $v(x, \lambda)$ with $\mu_2(x, \lambda)$, $\nu_2(x, \lambda)$ defined in the same manner as μ_1, ν_1 with l_1 replaced by l_2, $\eta(0)$ by $\zeta(0)$ and $\eta'(0)$ by $\zeta'(0)$.

Several cases are now considered.

(i) Let λ be real and positive.

We have

$$\begin{aligned}&\mu_1(x,\lambda)/Z(x)\exp(K_1\,p(x)\,q(x)) \\ &\quad = \eta(0)/\{Z(x)\exp(K_1\,p(x)\,q(x))\} \\ &\qquad - \int_0^x (\Omega(t), l_1(t)) \sin\xi(t)/(Z(x)\exp(K_1\,p(x)\,q(x))\,dt\end{aligned} \tag{5.31}$$

Now,

$$\begin{aligned}&\left|\int_0^x \sin\xi(t)\,(\Omega(t), l_1(t))/Z(x)\exp(K_1\,p(x)\,q(x))\,dt\right| \\ &\quad \leqslant \int_0^x (|l_1(t)|, |N_2(t)|)\,dt, \quad \text{by } (5.1).\end{aligned}$$

As $x \to \infty$ the integral on the right is convergent (uniformly with respect to λ) by the Lemma I. Hence the integral on the left is convergent as $x \to \infty$.

Therefore, the left-hand side of (5.31) $\to \mu_1(\lambda)$, $< \infty$, say.

Similarly,

$$\nu_1(x,\lambda)/Z(x)\exp(K_1\,p(x)\,q(x)) \to \nu_1(\lambda), < \infty, \text{ say.}$$

Again,

$$\begin{aligned}\mu_2(\lambda) &= \lim_{x\to\infty} \mu_2(x,\lambda)/Z(x)\exp(K_1\,p(x)\,q(x)) \\ &= \lim_{x\to\infty} \left[-\int_0^x \sin\xi(t)\,(l_2(t), \Omega(t))\,dt\right]/Z(x)\exp(K_1\,p(x)\,q(x))\end{aligned} \tag{5.32}$$

and

$$\nu_2(\lambda) = \lim_{x\to\infty} \nu_2(x, \lambda)/Z(x) \exp(K_1 p(x) q(x))$$

$$= \lim_{x\to\infty} [\int_0^x \cos \xi(t) (l_2(t), \Omega(t))\, dt]/Z(x) \exp(K_1 p(x) q(x)], \quad (5.33)$$

$\mu_2(\lambda)$, $\nu_2(\lambda)$ being finite limits as before.

Thus

$$u(x, \lambda) \sim (\lambda - p(x))^{-1/4} (\lambda - q(x))^{-1/4} [\mu_1(\lambda) \cos \xi(x) + \nu_1(\lambda) \sin \xi(x)] Z(x) \exp(K_1 p(x) q(x)) \quad (5.4)$$

Similarly,

$$v(x, \lambda) \sim (\lambda - p(x))^{-1/4} (\lambda - q(x))^{-1/4} [\mu_2(\lambda) \cos \xi(x) + \nu_2(\lambda) \sin \xi(x)] Z(x) \exp(K_1 p(x) q(x)) \quad (5.4\,a)$$

Again,

$$\eta'(x) = \frac{d}{dx} [Z(x)^{1/4} u(x, \lambda)]$$

$$= Z(x)^{1/4} u'(x, \lambda) - (1/4) [(\lambda - p(x))^{-3/4} (\lambda - q(x))^{1/4} p'(x) + (\lambda - q(x))^{-3/4} (\lambda - p(x))^{1/4} q'(x)] u(x, \lambda) \quad (5.5)$$

Differentiating (3.5),

$$\eta'(x) = Z(x)^{1/2} [-\eta(0) \sin \xi(x) + \eta'(0) Z(0)^{-1/2} \cos \xi(x) + \int_0^x \cos(\xi(x) - \xi(t)) (l_1(t), \Omega(t))\, dt \quad (5.6)$$

Therefore from (5.5) and (5.6),

$$u'(x, \lambda) \sim (\lambda - p(x))^{1/4} (\lambda - q(x))^{1/4} [\nu_1(\lambda) \cos \xi(x) - \mu_1(\lambda) \sin \xi(x)] Z(x) \exp(K_1 p(x) q(x)) \quad (5.7)$$

as $x \to \infty$.

Similarly,

$$v'(x, \lambda) \sim (\lambda - p(x))^{1/4} (\lambda - q(x)))^{1/4} [\nu_2(\lambda) \cos \xi(x) - \mu_2(\lambda) \sin \xi(x)] Z(x) \exp(K_1 p(x) q(x)) \quad (5.8)$$

as $x \to \infty$,

Let $\{u_j, v_j\}$ and $\{x_j, y_j\}$, $j=1, 2$, be the solutions of (1.1) and let A_1, B_1 be associated with u_1; A_2, B_2 with v_1; A_3, B_3 with u_2; A_4, B_4 with v_2; A_5, B_5 with x_1, A_6, B_6 with y_1; A_7, B_7 with x_2, A_8, B_8 with y_2, in the same way as μ_i, ν_i are associated with the solution $\{u, v\}$ of (1.1) in (5.4) and (5.4a). Then

$$\begin{pmatrix} u_j\ (x, \lambda) \\ u'_j\ (x, \lambda) \end{pmatrix} \sim Z(x) \exp\left(K_1\, p(x)\, q(x)\right) C(x)\, (AB)_i\, S(x) \tag{5.9}$$

$[(j, i) = (1, 1), (2, 3)]$

where

$$C(x) = \begin{pmatrix} Z(x)^{-1/4} & 0 \\ 0 & Z(x)^{1/4} \end{pmatrix}, \qquad (AB)_i = \begin{pmatrix} A_i(\lambda) & B_i(\lambda) \\ B_i(\lambda) & -A_i(\lambda) \end{pmatrix}$$

with similar expressions for $\{x_k(x, \lambda),\ x'_k(x, \lambda)\}$

$[(k, i) = (1, 5), (2, 7)]$, for $v_j(x, \lambda)$, $v'_j(x, \lambda)$ $[(j, i) = (1, 2), (2, 4)]$

and for $\{y_k(x, \lambda), y'_k(x, \lambda)\}$ $[(k, i) = (1, 6), (2, 8)]$.

Then substituting for $[\phi_1\ \theta_1]$, $[\phi_2\ \theta_2]$ in terms of A_j, B_j from (5.9) we can assume that

$$\begin{aligned} &A_1(\lambda), B_1(\lambda), A_2(\lambda), B_2(\lambda) \neq 0 \text{ simultaneously;} \\ &A_3(\lambda), B_3(\lambda), A_4(\lambda), B_4(\lambda) \neq 0 \text{ simultaneously} \\ &A_i(\lambda) = \lim_{x\to\infty} A_i(x, \lambda)/Z(x) \exp\left(K_1\, p(x)\, q(x)\right) \\ &B_i(\lambda) = \lim_{x\to\infty} B_i(x, \lambda)/Z(x) \exp\left(K_1\, p(x)\, q(x)\right) \\ &\qquad (i = 1, 2, \ldots 8) \end{aligned} \tag{5.10}$$

(since $[\phi_j\ \theta_j] = 1$, $j = 1, 2$).

A change of argument is necessary if $\lambda < 0$. In this case we choose X so that $\lambda - p(x)$, $\lambda - q(x) > 0$ for $x \geqslant X$, and the interval $[0, \infty)$ is replaced by $[X, \infty)$.

(ii) Let λ be complex : $\lambda = \alpha + i\beta\ (\beta > 0)$

Let x_0 be so chosen that $\alpha - p(t)$, $\alpha - q(t) > \beta$ for $x > x_0$.

From (3.2),

$$\xi(x) = \left[\int_0^{x_0} + \int_{x_0}^{x}\right] (\alpha - p(t) + i\beta)^{1/2} (\alpha - q(t) + i\beta)^{1/2}\, dt$$

$$= \xi_0 + \int_{x_0}^{x} (\alpha - p(t))^{1/2} (\alpha - q(t))^{1/2} \, dt$$

$$+ \tfrac{1}{2} i\beta \int_{x_0}^{x} [(\alpha - p(t))^{1/2}/(\alpha - q(t))^{1/2}$$

$$+ (\alpha - q(t))^{1/2}/(\alpha - p(t))^{1/2}] \, dt$$

$$+ O\,[\beta^2 \int_{x_0}^{x} | p(t) \, q(t) |^{1/2} (| p(t) |^{-2} + | q(t) |^{-2}) \, dt]$$

$$(\xi_0 = \text{Constant}).$$

Hence

$$\operatorname{Im} \xi(x) \sim \tfrac{1}{2} \beta \int_{x_0}^{x} [(\alpha - p(t))^{1/2}/(\alpha - q(t))^{1/2}$$

$$+ (\alpha - q(t))^{1/2}/(\alpha - p(t))^{1/2}] \, dt.$$

Therefore, if

$$\int^{\infty} (p(t) + q(t))/(p(t) \, q(t))^{1/2} \, dt \tag{5.11}$$

is divergent, it follows that $| \exp(- i \xi(x)) |$ is large for large x.

6. Spectral Theorem (Continuous Case)

Using (3.7) and proceeding as in [6], it follows that

$$\Omega(x) \sim i/2 \, Z(x) \exp[- i\xi(x) + K_1 p(x) \, q(x)] \, \bar{R} \tag{6.1}$$

where $\bar{R} \equiv \{R_1, R_2\}$

$$= \lim_{x \to \infty} \int_0^{x} \exp(i \, \xi(t)) \, Z(x)^{-1} \exp(- K_1 p(x) \, q(x)) \, N_3(t) \, dt < \infty \tag{6.2}$$

with $N_3(t) = \{(l_1(t), \Omega(t)), (l_2(t), \Omega(t))\}$.

Let

$$\left. \begin{aligned} X_k(x) &= Z(x)^{1/4} \, \theta_k(x, \lambda) \\ Y_k(x) &= Z(x)^{1/4} \, \phi_k(x, \lambda) \end{aligned} \right\} \quad (k = 1, 2) \tag{6.3}$$

where $X_k = \{X_{k1}, X_{k2}\}$, $Y_k = \{Y_{k1}, Y_{k2}\}$, say.

Proceeding as before we have for a fixed λ, as $x \to \infty$

$$X_k(x) \sim (i/2) \, Z(x) \exp[- i\xi(x) + K_1 p(x) \, q(x)] \, T_k(\lambda)$$

$$Y_k(x) \sim (i/2) \, Z(x) \exp[- i\xi(x) + K_1 p(x) \, q(x)] \, S_k(\lambda)$$

$$(k = 1, 2) \tag{6.4}$$

where $T_k(\lambda) = \{R_{1k}(\lambda), R_{2k}(\lambda)\}$, $S_k(\lambda) = \{S_{1k}(\lambda), S_{2k}(\lambda)\}$ and R_{ik}, S_{ik} $(i, k = 1, 2)$ are independent of x.

It follows from (6.4), (5.32), (5.33) and (5.9) that

$$\begin{aligned} R_{lk} &= B_j(\lambda) - i A_j(\lambda) \; (l = 1;\ k = 1, 2,\ j = 5, 7. \\ &\qquad \text{Also } \; l = 2;\ k = 1, 2,\ j = 6, 8) \\ S_{lk} &= B_j(\lambda) - i A_j(\lambda) \; (l = 1;\ k = 1, 2,\ j = 1, 3. \\ &\qquad \text{Also } \; l = 2;\ k = 1, 2,\ j = 2, 4). \end{aligned} \tag{6.5}$$

Now considering the solutions

$$\psi_k = \theta_k(x, \lambda) + \sum_{r=1}^{2} m_{kr}(\lambda)\, \phi_r(x, \lambda) \qquad (k = 1, 2)$$

and proceeding as in [6] we have

$$m_{rs}(\lambda) = N_{rs}(\lambda)/D(\lambda) \qquad (r, s = 1, 2) \tag{6.6}$$

where

$$\begin{aligned} N_{rs}(\lambda) &= R_{2r} S_{12} - R_{1r} S_{22}, \quad \text{if } s = 1,\ r = 1, 2 \\ &= R_{1r} S_{21} - R_{2r} S_{11}, \quad \text{if } s = 2,\ r = 1, 2 \end{aligned}$$

and

$$D(\lambda) = S_{11} S_{22} - S_{12} S_{21}.$$

Thus

$$\begin{aligned} m_{11}(\lambda) &= \frac{(B_6(\lambda) - iA_6(\lambda))\,(B_3(\lambda) - iA_3(\lambda)) - (B_5(\lambda) - iA_5(\lambda))\,(B_4(\lambda) - i A_4(\lambda))}{(B_1(\lambda) - iA_1(\lambda))\,(B_4(\lambda) - iA_4(\lambda)) - (B_3(\lambda) - iA_3(\lambda))\,(B_2(\lambda) - i A_2(\lambda))} \\ &= (Re\, N_{11}(\lambda) + i\, \text{Im}\, N_{11}(\lambda))\,[(B_1 B_4 - A_1 A_4) - (B_3 B_2 - A_3 A_2) \\ &\qquad + i\,(A_1 B_4 + B_1 A_4 - A_3 B_2 - A_2 B_3)]/w(\lambda), \end{aligned}$$

the numerator and denominator being continuous functions of λ; where

$$\begin{aligned} w(\lambda) &= [(B_1 B_4 - A_1 A_4) - (B_3 B_2 - A_3 A_2)]^2 \\ &\qquad + [(A_1 B_4 + B_1 A_4) - (A_3 B_2 + A_2 B_3)]^2 \end{aligned}$$

Therefore

$$\lim_{\beta \to 0} \text{Im}\,[m_{11}(\lambda)] = M_{11}(\alpha)/w(\alpha), \text{ say.}$$

Now by the Schwartz inequality, we have

$$[(B_1 B_4 - A_1 A_4) - (B_3 B_2 - A_3 A_2)]^{-2}$$

$$\geqslant [B_1^2 + A_1^2 + B_2^2 + A_2^2]^{-1} \, [B_3^2 + A_3^2 + B_4^2 + A_4^2]^{-1}, \qquad (6.7)$$

A_i, B_i $(i = 1, 2, 3, 4)$ being real.

Since A_j, B_j $(j = 1, 2$ or $j = 3, 4)$ cannot vanish simultaneously at a point on the λ axis, it follows from (6.7) that the denominator in the expression for $\lim_{\beta \to 0} \operatorname{Im} [m_{11}(\lambda)]$ is not zero at any point on the λ axis. Similar result holds for other $\lim_{\beta \to 0} \operatorname{Im} [m_{ij}(\lambda)]$

Hence the spectrum of the system (1.1), (1.2) is continuous over the whole range $(-\infty, \infty)$.

We thus obtain the following theorem.

THEOREM 1. *If all the conditions of the Lemma 1 are satisfied and if* $\int^{\infty} [(p/q)^{1/2} + (q/p)^{1/2}] \, dt$ *be divergent, then the spectrum of the system* (1.1), (1.2) *is continuous over the whole* λ*-axis* $(-\infty, \infty)$.

7. Spectral Theorem [Discrete Case]

In what follows we assume that all the conditions of Theorem 1 are satisfied except that

$$\int^{\infty} (p(t) + q(t))/(p(t)\, q(t))^{1/2} \, dt \qquad (7.1)$$

is now convergent.

We have,

$$\xi(x, \lambda) - \xi(x, 0) = \int_0^x [\lambda^2 - \lambda(p(t) + q(t))] \, g(t) \, dt$$

where

$$g(t) = [(\lambda - p(t))^{1/2} (\lambda - q(t))^{1/2} + (p(t)\, q(t))^{1/2}]^{-1},$$

Then

$$\xi(x, \lambda) - \xi(x, 0) \to \int_0^{\infty} [\lambda^2 - \lambda(p(t) + q(t)] \, g(t) \, dt$$

$$= \chi(\lambda) < \infty, \text{ as } x \to \infty, \qquad (7.2)$$

For, the integrand in (7·2)

$$= O\,[(p(t)+q(t))/(p(t)\,q(t))^{1/2}] + O\,[(p(t)\,q(t))^{-1/2}]$$

and by condition (1) of lemma I

$$(p(t)\,q(t))^{-1/2} < Q(t)^{-1}$$

The finiteness of $X(\lambda)$ therefore follows from (7.1) and condition (of Lemma I. Thus Im $\xi(x)$ is bounded and so are $\cos\xi(x)$ and $\sin\xi($ (λ real or complex).

We then have from (3·5), (3·6) and (3·2)

$$\begin{aligned} u(x,\lambda) &= C_1 Z(x)\exp(K_1 p(x)\,q(x))\,[\mu_1(\lambda)\cos\xi(x) \\ &\qquad + \nu_1(\lambda)\sin\xi(x) + o(1)] \\ v(x,\lambda) &= C_1 Z(x)\exp(K_1 p(x)\,q(x))\,[\mu_2(\lambda)\cos\xi(x) \\ &\qquad + \nu_2(\lambda)\sin\xi(x) + o(1)] \\ u'(x,\lambda) &= C_2 Z(x)\exp(K_1 p(x)\,q(x))\,[\nu_1(\lambda)\cos\xi(x) \\ &\qquad - \mu_1(\lambda)\sin\xi(x) + o(1)] \\ v'(x,\lambda) &= C_2 Z(x)\exp(K_1 p(x)\,q(x))\,[\nu_2(\lambda)\cos\xi(x) \\ &\qquad - \mu_2(\lambda)\sin\xi(x) + o(1)] \end{aligned} \tag{7.3}$$

for large x and for all values of λ real or complex, where $C_1 = Z(x)^{-1/4}$, $C_2 = Z(x)^{1/4}$.

Since $(\lambda - p(t))^{1/4}$, $(\lambda - q(t))^{1/4}$ are analytic functions of λ regular except on the negative real axis, with similar arguments for $\xi(t,\lambda)$, $|N_1(t)|\;|N_2(t)|$, therefore the integrals in the expressions for $\mu_i(\lambda)$, $\nu_i(\lambda)$ $(i = 1, 2)$ converge uniformly with respect to λ in any finite region. Hence $\mu_i(\lambda)$, $\nu_i(\lambda)$ and therefore $A_j(\lambda)$, $B_j(\lambda)$ $(j = 1, 2 \ldots.)$ are analytic functions of λ regular except possibly on the negative real axis. Similar arguments hold for the interval (X, ∞) which replaces $[0, \infty)$, X being sufficiently large. We therefore have,

$$\mu_1(\lambda) = \lim_{x\to\infty} \int_X^x - \frac{(U(t), l_1(t))\,\sin\xi(t)\,Z(t)^{1/4}}{Z(x)\exp(K_1\,p(x)\,q(x))}\,dt$$

$$\nu_1(\lambda) = \lim_{x\to\infty} \int_X^x \frac{(U(t), l_1(t))\,\cos\xi(t)\,Z(t)^{1/4}}{Z(x)\exp(K_1\,p_1(x)\,q(x))}\,dt$$

with similar expressions for $\mu_2(\lambda)$, $\nu_2(\lambda)$ and for $A_j(\lambda)$, $B_j(\lambda)$ $(j = 1, 2, \ldots)$. Hence $A_j(\lambda)$, $B_j(\lambda)$ are regular except possibly on the real axis between $-\infty$ and max $[p(X), q(X)]$. In fact, $A_j(\lambda)$, $B_j(\lambda)$ $(j = 1, 2, \ldots)$ are entire functions of λ in the interval $(-\infty, \infty)$.

We consider the solution

$$\psi_k = \theta_k(x, \lambda) + \sum_{s=1}^{2} l_{ks}(\lambda)\, \phi_s(x, \lambda) \quad (k = 1, 2) \tag{7·4}$$

such that

$$\begin{aligned}\psi_1(x, \lambda) &\equiv \psi_1(b/x, \lambda)\\ &= ([\phi_2\phi_4]\, \phi_3(b/x, \lambda) - [\phi_2\phi_3]\, \phi_4(b/x, \lambda))/D_1(\lambda)\\ \psi_2(x, \lambda) &\equiv \psi_2(b/x, \lambda)\\ &= ([\phi_1\phi_3]\, \phi_4(b/x, \lambda) - [\phi_1\phi_4]\, \phi_3(b/x, \lambda))/D_1(\lambda)\end{aligned} \tag{7. 5}$$

where ϕ_3, ϕ_4 are the boundary-condition vectors at $x = b$ $(b > 0)$ and

$$\begin{aligned}D_1(\lambda) &\equiv D_1(b, \lambda)\\ &= [\phi_1\phi_3](b, \lambda)\, [\phi_2\phi_4](b, \lambda) - [\phi_1\phi_4](b, \lambda)\, [\phi_2\phi_3](b, \lambda)\end{aligned}$$

and θ_k are defined as before.

Then,

$$l_{rs}(b, \lambda) \equiv l_{rs}(\lambda) = [\psi_r(b/x, \lambda)\, \theta_s(0/x, \lambda)] \quad (r, s = 1, 2),$$

$l_{rs}(b, \lambda)$ being dependent on b, λ.

We therefore have,

$$\begin{pmatrix} u_j(x, \lambda) \\ u'_j(x, \lambda) \end{pmatrix} = Z(x)\, C(x) \exp(K_1\, p(x)\, q(x))\, [(AB)_i\, S(x) + 0(1)], \tag{7.6}$$

by (7.3) $((j, i) = (1, 1), (2, 3), (3, 9), (4, 10))$, with similar results for $x_k(x, \lambda)$, $x'_k(x, \lambda)$ $((k, i) = (1, 5), (2, 7))$ and for $y_k(x, \lambda)$, $y'_k(x, \lambda)$ $((k, i) = (1, 6), (2, 8))$ and for $v_j(x, \lambda)$, $v'_j(x, \lambda)$ $((j, i) = (1, 2), (2, 4), (3, 11), (4, 12))$.

Putting $A_{13}(\lambda) = r_1 \cos \gamma_1$, $B_{13}(\lambda) = r_1 \sin \gamma_1$ and $-\gamma_1 = \tilde{C}_1 - \xi(b, 0)$ in the expression $A_{13}(\lambda) \cos \xi(b) + B_{13}(\lambda) \sin \xi(b)$ ($\tilde{C}_1$ is a constant) where

$$A_{13}(\lambda) = A_1 + B_9 + A_2 + B_{11} - A_9 - B_1 - A_{11} - B_2,$$

$$B_{13}(\lambda) = A_1 + B_2 + A_2 - A_9 - B_9 - A_{11} - B_{11},$$

we have,

$$A_{13}(\lambda)\cos\xi(b)+B_{13}(\lambda)\sin\xi(b)$$
$$=r_1\cos[\xi(b,\lambda)-\xi(b,0)+\tilde{C}_1]$$
$$\to r_1\cos[x(\tau)+\tilde{C}_1]<\infty,\ \text{as}\ b\to\infty.$$

Hence substituting for u_j, u'_j, etc., in $[\phi_1\phi_3](b,\lambda)$ and simplifying, it follows that for large b

$$[\phi_1\phi_3](b,\lambda)=Z(b)^2\exp^2[K_1\,p(b)\,q(b)]\ [S_{13}(\lambda)+o(1)] \tag{7.7}$$

where $s_{13}(\lambda)=(A_1B_9-A_9B_1)+(A_2B_{11}-A_{11}B_2)$.

Proceeding as before, for large b, if $h(b)=Z(b)^2\exp^2(K_1p(b)q(b))$

$$\left.\begin{aligned}
&[\phi_2\phi_4](b,\lambda)=h(b)[s_{24}(\lambda)+o(1)]\\
&[\phi_1\phi_4](b,\lambda)=h(b)[s_{14}(\lambda)+o(1)]\\
&[\phi_2\phi_3](b,\lambda)=h(b)[s_{23}(\lambda)+o(1)]\\
&[\phi_3\theta_1](b,\lambda)=h(b)[s_{31}(\lambda)+o(1)]\\
&[\phi_4\theta_1](b,\lambda)=h(b)[s_{41}(\lambda)+o(1)]
\end{aligned}\right\} \tag{7.8}$$

where

$$s_{24}(\lambda)=(A_3B_{10}-A_{10}B_3)+(A_4B_{12}-A_{12}B_4)$$
$$s_{14}(\lambda)=(A_1B_{10}-A_{10}B_1)+(A_2B_{12}-A_{12}B_2)$$
$$s_{23}(\lambda)=(A_3B_9-A_9B_3)+(A_4B_{11}-A_{11}B_4)$$
$$s_{31}(\lambda)=(A_9B_5-A_5B_9)+(A_{11}B_6-A_6B_{11})$$
$$s_{41}(\lambda)=(A_{10}B_5-A_5B_{10})+(A_{12}B_6-A_6B_{12}).$$

Hence substituting from (7.7), (7.8) we have

$$l_{11}(b,\lambda)=N_{11}(b,\lambda)/D_{11}(b,\lambda),$$

where

$$D_{11}(b,\lambda)=\{[s_{13}(\lambda)+o(1)][s_{24}(\lambda)+o(1)]$$
$$-[s_{14}(\lambda)+o(1)][s_{23}(\lambda)+o(1)]\}\ [Z(b)\exp(K_1p(b)\,q(b))]^4$$

for large b, with a similar expression for $N_{11}(b,\lambda)$.

Letting $b\to\infty$ in $l_{11}(b,\lambda)$,

$$l_{11}(b,\lambda)\to m_{11}(\lambda)=N_{11}(\lambda)/D_{11}(\lambda),\ \text{say},$$

$N_{11}(\lambda)$, $D_{11}(\lambda)$ being analytic functions of λ, with similar expressions for other $m_{ij}(\lambda)$.

Hence $m_{ij}(\lambda)$ are meromorphic functions of λ. Therefore the spectrum of the system (1, 1), (1, 2) is discrete. Hence we obtain the following theorem.

THEOREM 2. *If all the conditions of lemma I are satisfied and if* $\int^{\infty} [(p/q)^{1/2} + (q/p)^{1/2}]\, dt$ *be convergent then the spectrum of the system* (1.1), (1.2) *is discrete over the interval* $(-\infty, \infty)$.

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