ON CHARACTERIZATION OF THE SPECTRUM ASSOCIATED WITH A MATRIX DIFFERENTIAL OPERATOR

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ABSTRACT

In this paper authors prove the following three theorems on the characterization of the spectrum associated with the matrix differential operator

$$L = \begin{pmatrix} -d^2/dx^2 + p(x) & r(x) \\ r(x) & -d^2/dx^2 + q(x) \end{pmatrix}.$$

Theorem (I). The point spectrum is identical with the set of real values fof λ *or which the equation* $(L - \lambda I)\psi = 0$ *has non-trivial solution of* L^2

Theorem (II). If λ' *is in the point spectrum, the equation*

$$(L - \lambda I)\phi = -f,$$

where f *is a given column vector such that* $(f, f) \in L$, *has no solution in* L^2 *unless* f *is orthogonal to* E_k $(k = 1, 2)$ *at* λ'.

Theorem (III). The spectrum is the complement of the set of real values of λ *for which the equation*

$$(L - \lambda I)\phi = -f,$$

where f *has continuous first order derivative and* $(f(x), f(x)) \in L$, *has a solution* ϕ *such that* $(\phi, \phi) \in L$.

These theorems are generalisations of the corresponding theorems of Titchmarsh for a linear differential operator discussed in [5].

Keywords: Spectrum, eigenvector, boundary condition L^2-solution, saltus, Green's matrix, mermorphic, orthogonal.

1. Introduction

The object of this paper is to prove three theorems on the characterization of the spectrum associated with the matrix differential operator

$$L = \begin{pmatrix} -d^2/dx^2 + p(x) & r(x) \\ r(x) & -d^2/dx^2 + q(x) \end{pmatrix} \tag{1.1}$$

studied by Tiwari [1].

The homogeneous and corresponding non-homogeneous differential equations considered are

$$(L - \lambda I)\,\phi = 0, \tag{1.2}$$

and

$$(L - \lambda I)\,\phi = -f, \tag{1.3}$$

respectively, where $\phi = \phi(x) = [(u(x),\ v(x)]$ is a two component column vector, λ is a variable parameter real or complex, $p(x)$, $q(x)$ and $r(x)$ are all real valued and continuous function of x throughout the interval $[O,\ b)$ where $b \to \infty$ and $f = f(x) = \{f_1,\ f_2\}$ is a real valued two component column vector function of x. The boundary conditions considered are

$$\left.\begin{aligned} a_{j1}u(O) + a_{j2}u'(O) + a_{j3}v(O) + a_{j4}v'(O) &= 0 \\ b_{j1}u(b) + b_{j2}u'(b) + b_{j3}v(b) + b_{j4}v'(b) &= 0 \end{aligned}\right\} \tag{1.4}$$

$j = 1,\ 2$, accents denoting differentiation with respect to x and

$$\left.\begin{aligned} a_{11}a_{22} - a_{12}a_{21} + a_{13}a_{24} - a_{14}a_{23} &= 0 \\ b_{11}b_{22} - b_{12}b_{21} + b_{13}b_{24} - b_{14}b_{23} &= 0 \end{aligned}\right\} \tag{1.5}$$

The relations (1.5) are necessary for self-adjointness of the problem.

2. Notations and Some Results

(*a*) $M_k(x,\ y)$ denotes the kth column of the matrix $M(x,\ y)$,

(*b*) $(y,\ z)$ represents the product $y^T z$, where y and z are two column vectors,

(*c*) $\langle y,z\rangle_{0,a}$ stands for $\int_0^a (y,\ z)\,dr$ and $\|y\|_{0,a}$ for $\langle y,\ y\rangle_{0,a} \equiv \langle y, \bar{y}\rangle_{0,a}$

if y is complex.

We mention the following results obtained by Tiwari [1] which are required for our subsequent analysis:

The Green's matrix for the boundary value problem is denoted by $G = G(x, y, \lambda) = (G_{rs}(x, y, \lambda))$, $r, s, = 1, 2$, which is symmetric in the sense that $G_{ij}(x, y, \lambda) = G_{ji}(y, x, \lambda)$, where λ is not an eigenvalue and it has the usual properties (*cf.* Chakrabarty [2], Courant and Hilbert [3] and Neumark [4])

Let

$$\Phi = \Phi(x, \lambda, f) = (\Phi_1, \Phi_2) = \int_0^\infty G^T(y, x, \lambda) f(y)\, dy, \tag{2.1}$$

where λ is not an eigenvalue and $(f(x), f(x)) \in L$, then Φ satisfies the system of equations (1.3) and the boundary conditions (1.4). Also, if $\lambda = \mu + i\nu$,

$$\| \Phi(x, \lambda, f) \|_{0,\infty} \leqslant \nu^{-2} \| f \|_{0,\infty} . \tag{2.2}$$

If

$$H(x, y, \mu) = (H_{rs}(x, y, \mu)) = Lt_{\nu \to 0} \int_0^\mu Im\, G(x, y, \sigma + i\nu)\, d\sigma, \quad (\mu > 0)$$

$$= - Lt._{\nu \to 0} \int_\mu^0 Im.\ G(x, y, \sigma + i\nu)\, d\sigma, \quad (\mu < 0)$$

$$= 0, \quad \mu = 0$$

and if $H(x, y, \mu)$ is discontinuous at μ, where its saltus is denoted by $\pi E(x, y) = \pi(E_{rs}(x, y))$, then $E_k(x, y)$, $k = 1, 2$ satisfy (1.2) for any y and $(E_k, E_k) \in L$.

If the Green's matrix G is unique, then the following two results hold:
(*I*) If $g(x, \lambda_1)$ and $h(x, \lambda_2)$ are two solutions of (1.2) or (1.3) for distinct eigenvalues λ_1 and λ_2, then

$$\langle h, Lg \rangle_{0,\infty} = \langle g, Lh \rangle_{0,\infty} \tag{2.4}$$

$$(II)\ (\lambda - \lambda') \int_0^\infty G(x, y, \lambda)\ G(x, u, \lambda')\, dx = G(u, y, \lambda') - G(y, u, \lambda') \tag{2.5}$$

for any non-real λ and λ'.

3. Spectral Theorems

Following Titchmarsh [5], [6] we define spectrum as follows:

If the Green's matrix $G(x, y, \lambda)$ is meromorphic, each of its poles is an eigen value, the spectrum in this case is the set of eigen values and is called a discrete spectrum.

The point spectrum is defined to be the set of real λ for which $H(x, y, \lambda+0) - H(x, y, \lambda-0)$ is not an identically null matrix, when Green's matrix is meromorphic, the spectrum is the same as the point spectrum.

Theorem (I). The point spectrum is identical with the set of real values of λ for which the equation

$$(L - \lambda I)\psi = 0 \tag{3.1}$$

has a non-trivial solution of L^2.

Proof. It follows from the definition of the point spectrum that

$$E(x, y) = (E_{rs}(x, y)) = \frac{1}{\pi}\{H_{rs}(x, y, \lambda+0) - H_{rs}(x, y, \lambda-0)\} \tag{3.2}$$

is not an identically null matrix.

Since $E_k(x, y)$ satisfies (3.1) for any y and $(E_k, E_k) \in L$, $k = 1, 2$; the first part of the theorem follows.

Conversely, let $\psi(x)$ be a non-trivial solution of (3.1) for a particular value λ' of λ and let $(\psi(x), \psi(x)) \in L$. Then it follows from (2.1) that $(L - \lambda I)\Phi(x, \lambda, \psi) = -\psi$.

Let

$$f(x) = \Phi(x, \lambda, \psi) - \frac{\psi(x)}{\lambda' - \lambda}$$

then

$$(L - \lambda I) f = -\psi + \psi = 0.$$

From the uniqueness of the solution of (3.1), it follows that f is an identically null column vector. Hence

$$\Phi(x\ \lambda, \psi) = \frac{\psi(x)}{\lambda' - \lambda}.$$

Putting $\lambda = \mu + i\nu$, we obtain

$$\int_0^\infty Im\, G^T(y, x, \mu + i\nu)\, \psi(y)\ dy = \frac{-\nu\,\psi(x)}{(\mu - \lambda')^2 + \nu^2}.$$

Hence

$$\int_0^\infty \Big[\int_{\lambda'-\epsilon}^{\lambda'+\epsilon} \{Im\, G(x, y, \mu + i\nu)\, d\mu\}\, \psi(y)\Big]\, dy$$

$$= -\nu\psi(x) \int_{\lambda'-\epsilon}^{\lambda'+\epsilon} \frac{d\mu}{(\mu - \lambda')^2 + \nu^2}$$

that is

$$\int_0^\infty [H(x, y, \lambda' + \epsilon) - H(x, y, \lambda' - \epsilon)]\, \psi(y)\, dy = -\pi\psi(x)$$

on taking the limit as $\nu \to 0$ and using (2.3).

Therefore,

$$\int_0^\infty E(x, y)\, \psi(y)\, dy = -\psi(x).$$

It follows therefore that $E(x, y)$ is not an identically null matrix and so λ' is in the point spectrum.

THEOREM (II). If λ' is in the point spectrum, the equation

$$(L - \lambda I)\, \Phi = -f,$$

where f is a given column vector such that $(f, f) \in L$, has no solution in L^2 unless f is orthogonal to $E_k\ (k = 1, 2)$ at λ'.

Proof. It is sufficient to show that if $E_k\ (k = 1, 2)$ satisfies $(L - \lambda I)\, \psi = 0$ at λ' and $(E_k, E_k) \in L$, then E_k is orthogonal to f.

Now, using (2.4), we have

$$\langle \Phi, \lambda' E_k \rangle_{0,\infty} = \langle \Phi, LE_k \rangle_{0,\infty} = \langle E_k, L\Phi \rangle_{0,\infty} = \langle E_k, \lambda' \Phi - f \rangle_{0,\infty}$$

$$\therefore\ \langle E_k, f \rangle_{0,\infty} = 0.$$

THEOREM (III). The spectrum is the complement of the set of real values of λ for which the equation

$$(L - \lambda I)\, \phi = -f, \tag{3.3}$$

where f has continuous first order derivative and $(f(x), f(x)) \in Le$, has a solution ϕ such that $(\phi, \phi) \in L$.

Proof. If λ' is not in the spectrum, then $H(x, y, \lambda)$ is constant in some internal (α, β) containing λ' and hence $G(x, y, \lambda)$ is regular for $\alpha < Re\,\lambda < \beta$. Let f and g be column vectors such that $(f, f), (g, g) \in L$. Let

$$\Phi(y, \lambda, b) = \int_0^b G^T(y, x, \lambda) f(x)\, dx$$

and

$$F(\lambda) = \langle \Phi(y, \lambda, b), g(y) \rangle_{0,b}. \tag{3.4}$$

Then

$$|F(\lambda)|^2 \leqslant \| \Phi(y, \lambda, b) \|_{0,b} \| g(y) \|_{0,b}$$

$$\leqslant \nu^{-2} \| f(y) \|_{0,b} \| g(y) \|_{0,b} \tag{3.5}$$

where $\lambda = \mu + i\nu$. Hence

$$|F(\lambda)|^2 < K \| f \|_{0,b} \| g \|_{0,b}, \tag{3.6}$$

where K is independent of f, g and b, $\alpha + \delta < Re\,\lambda < \beta - \delta$, $-\delta < \nu < \delta$. Taking $g(y) = \Phi(y, \lambda, b)$ for any given λ, we obtain from (3.4) and (3.6)

$$\| \Phi(y, \lambda, b) \|_{0,b} < K \| f(y) \|_{0,b}\ .$$

Hence if $a < b$, $\| \Phi(y, \lambda, b) \|_{0,a} < K \| f(y) \|_{0,b}$. (3.7)

Let us put $f = \{0, 0\}$ in $[O, c)$ where $c < b$, then

$$\| \Phi(y, \lambda, b) - \Phi(y, \lambda, c) \|_{0,a} < K \| f(y) \|_{0,b-c}.$$

Taking the limit as $b \to \infty$, $\Phi(y, \lambda, b)$ converges in mean for y in $[o, a)$ say to $\Phi(y, \lambda)$ uniformly for λ in the above region and in particular at λ' If y is in $[o, b]$ and λ is not real

$$(L - \lambda I)\, \Phi(y, \lambda, b) = -f(y).$$

By arguments similar to those of Titchmarsh [5] and Tiwari [1] we obtain on making $\lambda \to \lambda'$ and then $b \to \infty$ through a suitable sequence

$$(L - \lambda' I)\, \Phi(y, \lambda) = -f(y).$$

Also, by making $b \to \infty$ through a suitable sequence and then $a \to \infty$, it follows from (3.7) that

$$(\Phi(x, \lambda, f), \Phi(x, \lambda, f)) \in L.$$

This proves the first part of the theorem.

Conversely, we suppose that λ' is a real number such that for every $f(x)$ which is continuous and has continuous derivatives, the equation

$$(L - \lambda' I) = \phi - f \tag{3.8}$$

has at least one solution of L^2.

If the equation (3.8) has more than one solution for any f, then the difference of two such solutions satisfies the equation

$$(L - \lambda' I)\phi = 0.$$

From this it follows that λ' is in the point spectrum and it would follow that every f of the above class was orthogonal to E_k, $(k = 1, 2)$, which is impossible.

The possibility of more than one ϕ corresponding to any f is thus ruled out. Adapting the analysis analogous to that of Titchmarsh [5] it follows quite easily that

$$\| \phi \|_{0,\infty} / \| f \|_{0,\infty}$$

is bounded. Let

$$\| \phi \|_{0,\infty} = M^2(\lambda') \| f \|_{0,\infty}.$$

Since,

$$\Phi(x, \lambda) = \int_0^\infty G^T(y, x, \lambda) f(y)\, dy = \int_0^\infty G(x, y, \lambda) f(y)\, dy \tag{3.9}$$

satisfies $(L - \lambda I)\Phi = -f$,

it follows that

$$(L - \lambda' I)\Phi = (L - \lambda I + \lambda I - \lambda' I)\Phi = -f + (\lambda - \lambda')\Phi.$$

Therefore,

$$\| \Phi \|_{0,\infty} \leqslant M^2(\lambda') \| -f + (\lambda - \lambda')\Phi \|_{0,\infty}$$

$$\leqslant \frac{2M^2(\lambda')}{1 - 2|\lambda - \lambda'|^2 M^2(\lambda')} \| f \|_{0,\infty},$$

where

$$|\lambda - \lambda'| < [\sqrt{2}\, M(\lambda')]^{-1}.$$

For such values of λ the transformation $f \to \Phi$ is, therefore, bounded. Let $M_1(\lambda)$ be its bound.

Putting $f(x) = G_k(x, u, i)$, $k = 1, 2$ in (3.9), denoting the corresponding Φ by $\Phi^{(k)}$ and using (2.5), we obtain

$$\Phi^{(k)}(x, \lambda) = \int_0^\infty G(x, y, \lambda)\, G_k(y, u, i)\, dy$$

$$= \frac{1}{(\lambda - i)} [G_k(x, u, \lambda) - G_k(x, u, i)].$$

Hence

$$\| G_k(x, u, \lambda) - G_k(x, u, i) \|_{0,\infty} \leq M_1^2(\lambda)\, | \lambda - i|^2\, \| G_k(x, u, i)\|_{0,\infty}.$$

Therefore $\| G(x, u, \lambda) \|_{0,\infty}$ is bounded if λ is in some neighbourhood of λ' and consequently

$$\int_0^\infty | G_{rs}(x, y, \lambda) |^2\, dx; \quad r, s = 1, 2$$

is bounded in that neighbourhood. Hence, if $\lambda = \mu + i\nu$,

$$2\, Im\, G(x, u, \lambda) = G(x, u, \lambda) - G(x, u, \bar{\lambda})$$

$$= (\lambda - \bar{\lambda}) \int_0^\infty G(x, y, \lambda)\, G(y, u, \bar{\lambda})\, dy$$

$$= 0(| \nu |),$$

as $\nu \to 0$, uniformly in some neighbourhood of λ'. Hence $H(x, u, \lambda)$ is constant in some neighbourhood of λ' and so λ' does not belong to the spectrum.

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