On the convergence of eigenfunction expansions associated with a vector-matrix differential equation

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Received on July 8, 1977 ; Revised on November 24, 1977

Abstract

We discuss the convergence under Fourier conditions of the eigenfunction expansions associated with the system

$$L\phi = \begin{pmatrix} -\frac{d^2}{dx^2} + p(x) & q(x) \\ q(x) & -\frac{d^2}{dx^2} + r(x) \end{pmatrix} \phi = \lambda \phi$$

together with the boundary conditions

$$A_1 \phi(0) + A_2 \phi'(0) = 0$$

where A_1 and A_2 are 2×2 symmetric matrices with real constant elements and $\phi(x)$ is the column vector with components $u(x)$ and $v(x)$.

Key words: Limit-2 case, boundary condition vector, self-adjointness, Green's matrix, eigenvalue, eigenvector.

1. Introduction

The eigenfunction theory to be discussed is that associated with the differential system

$$\left.\begin{aligned} -u''(x) + p(x)u(x) + q(x)v(x) = \lambda u(x) \\ -v''(x) + q(x)u(x) + r(x)v(x) = \lambda v(x) \end{aligned}\right\} (0 \le x < \infty) \tag{1}$$

together with the boundary conditions

$$a_{j1}u(0) + a_{j2}v(0) + a_{j3}u'(0) + a_{j4}v'(0) = 0, \quad (j = 1, 2) \tag{2}$$

where

(i) $p(x)$, $q(x)$ and $r(x)$ are real-valued, continuous and Lebesgue integrable over the interval $[0, \infty)$,

(ii) $-u''(x) + p(x)u(x) + q(x)v(x)$ and $-v''(x) + q(x)u(x) + r(x)v(x)$ belong to $L^2[0, \infty)$,

(iii) a_{jk} $(j = 1, 2;\ k = 1, 2, 3, 4)$ are real valued constants,

(iv) the set $\{a_{1k}\}$ is linearly independent of the set $\{a_{2k}\}$,

(v) $a_{12}\, a_{24} + a_{11}\, a_{23} - a_{14}\, a_{22} - a_{13}\, a_{21} = 0,$ (3)

(vi) $a_{13}\, a_{22} + a_{14}\, a_{21} - a_{11}\, a_{24} - a_{12}\, a_{23} = 1.$ (4)

The system

$$\left.\begin{array}{l} u''(x) + \lambda u(x) = 0 \\ v''(x) + \lambda v(x) = 0 \end{array}\right\} \qquad (0 \le x < \infty) \tag{5}$$

whose solutions $(u, v)^T = \{u, v\}$ satisfy the boundary conditions (2) is called the "Fourier System" corresponding to the general system (1). The elements of this problem are distinguished from those of the original problem (1)–(2) by a superscript F.

Our object in the present paper is to show that the eigenfunction expansions of a vector function $f(x) = \{f_1(x),\ f_2(x)\}$ associated with the system (1) behaves as regards convergence in the same way as the corresponding eigenfunction expansions of $f(x)$ associated with Fourier system (5).

The minimum and maximum number of linearly independent Lebesgue square-integrable solutions of the problem are 2 and 4 respectively. We assume that the problem possesses two and only two linearly independent L^2 solutions, *i.e.*, our problem is in the limit–2 case.

2. Notations and abbreviations

The eigenfunction theory associated with a pair of second-order differential system has been developed among others by Chakravarty.[1] His notations will be adopted here.

(i) For any two vectors

$$Y(x) = \{y_1(x),\ y_2(x)\} \quad \text{and} \quad Z(x) = \{z_1(x),\ z_2(x)\};$$

$$(Y, Z) = y_1(x)\, z_1(x) + y_2(x)\, z_2(x), \quad \langle Y, Z\rangle_{0,x} = \int_0^x (Y, Z)\, dt,$$

$$\| Y \|_{0,x} = \langle Y, Y\rangle_{0,x} \quad \text{and} \quad \| Y \| = \| Y \|_{0,\infty}, \quad \langle Y, Z\rangle = \langle Y, Z\rangle_{0,\infty}.$$

(ii) The boundary condition vectors $\phi_r(x, \lambda) = \phi_r(0/x, \lambda) = \{u_r(0/x, \lambda),\ v_r(0/x, \lambda)\}$ at $x = 0$ are the solutions of (1) satisfying

$$\left.\begin{array}{ll} u_r(0/0, \lambda) = -a_{r3}, & u_r'(0/0, \lambda) = a_{r1} \\ v_r(0/0, \lambda) = -a_{r4}, & v_r'(0/0, \lambda) = a_{r2} \end{array}\right\} \quad (r = 1, 2) \tag{6}$$

The functions $\theta_r(0/x, \lambda) = \theta_r(x, \lambda) = \{x_r(0/x, \lambda),\ y_r(0/x, \lambda)\}$ solutions of (1) which take real constant values independent of λ at $x = 0$ are defined by the relations

$$[\phi_r\, \theta_s] = \delta_{rs} \quad \text{and} \quad [\theta_r\, \theta_s] = 0, \quad (r, s = 1, 2) \tag{7}$$

where $[YZ]$ stands for the bilinear concomitant

$$y_1 z_1' - y_1' z_1 + y_2 z_2' - y_2' z_2$$

for any two solutions $Y = \{y_1, y_2\}$, $Z = \{z_1, z_2\}$ of (1). It is well known that $[YZ]$ is independent of x. The relations (7) are satisfied if we take

$$\left.\begin{aligned} x_k(0/0, \lambda) &= (-1)^{k-1} a_{l1}, & x_k'(0/0, \lambda) &= (-1)^k a_{l2} \\ y_k(0/0, \lambda) &= (-1)^{k-1} a_{l3}, & y_k'(0/0, \lambda) &= (-1)^k a_{l1} \end{aligned}\right\} \tag{8}$$

[when $k = 1$, $l = 2$ and when $k = 2$, $l = 1$].

As usual to consider the problem (1)–(2) in the interval $[0, \infty)$, we first consider it in the interval $[0, b]$, (to be referred to as the b-case) and then make b tend to infinity. The two boundary conditions at $x = b$ for the b-case are expressed in terms of the boundary condition vectors

$$\chi_r(b/x, \lambda) = \chi_r(x, \lambda) = \{u_{r+2}(b/x, \lambda), v_{r+2}(b/x, \lambda)\}, \ (r = 1, 2)$$

in the form

$$[U(x, \lambda)\, \chi_r(b/x, \lambda)] = 0 \qquad (r = 1, 2) \tag{9}$$

where $\chi_r(b/b, \lambda) = \{-b_{r3}, -b_{r4}\}$, $\chi_r'(b/b, \lambda) = \{b_{r1}, b_{r2}\}$, the real valued constants, b_{ij} $(i = 1, 2;\ j = 1, 2, 3, 4)$ satisfying

$$b_{12} b_{24} + b_{11} b_{23} - b_{14} b_{22} - b_{13} b_{21} = 0.$$

Moreover the self-adjointness condition for the problem is

$$[\phi_1(x, \lambda)\, \phi_2(x, \lambda)] = [\chi_1(x, \lambda)\, \chi_2(x, \lambda)] = 0. \tag{10}$$

In the singular case the boundary conditions (9) are replaced by the L^2 conditions.

Now as in Chakravarty[1] it follows that the Green's matrix $G(b, x, \xi, \lambda)$ for the boundary value problem (1)–(2) with (9)–(10) is given by

$$\left.\begin{aligned} G(b, x, \xi, \lambda) &= \begin{pmatrix} G_{11}(b, x, \xi, \lambda) & G_{21}(b, x, \xi, \lambda) \\ G_{12}(b, x, \xi, \lambda) & G_{22}(b, x, \xi, \lambda) \end{pmatrix} \\ &= \begin{pmatrix} u_1(x, \lambda) & u_2(x, \lambda) \\ v_1(x, \lambda) & v_2(x, \lambda) \end{pmatrix} \begin{pmatrix} \psi_{11}(b, \xi, \lambda) & \psi_{12}(b, \xi, \lambda) \\ \psi_{21}(b, \xi, \lambda) & \psi_{22}(b, \xi, \lambda) \end{pmatrix} (0 \le x < \xi) \\ &= \begin{pmatrix} \psi_{11}(b, x, \lambda) & \psi_{21}(b, x, \lambda) \\ \psi_{12}(b, x, \lambda) & \psi_{22}(b, x, \lambda) \end{pmatrix} \begin{pmatrix} u_1(\xi, \lambda) & v_1(\xi, \lambda) \\ u_2(\xi, \lambda) & v_2(\xi, \lambda) \end{pmatrix} (\xi \le x < b) \end{aligned}\right\} \tag{11}$$

where

$$\begin{aligned} \psi_1(b, x, \lambda) &= \{\psi_{11}(b, x, \lambda), \psi_{12}(b, x, \lambda)\} \\ &= \frac{[\phi_2 \chi_2]\, \chi_1(b/x, \lambda) - [\phi_2 \chi_1]\, \chi_2(b/x, \lambda)}{W(b, \lambda)} \end{aligned} \tag{12}$$

$$\psi_2(b, x, \lambda) = \{\psi_{21}(b, x, \lambda), \psi_{22}(b, x, \lambda)\}$$
$$= \frac{[\phi_1\chi_1]\,\chi_2(b/x, \lambda) - [\phi_1\,\chi_2]\,\chi_1(b/x, \lambda)}{W(b, \lambda)} \tag{13}$$

$W(b, \lambda)$ being the Wornskian

$$[\phi_1\,\chi_2]\,[\phi_2\chi_1] - [\phi_2\,\chi_2]\,[\phi_1\chi_1]. \tag{14}$$

Also as in Chakravarty[2]

$$\psi_r(b, x, \lambda) = \theta_r(x, \lambda) + \sum_{s=1}^{2} l_{rs}(\lambda)\,\phi_s(x, \lambda), \quad (r = 1, 2) \tag{15}$$

where $l_{rs}(\lambda) = [\psi_r(b, x, \lambda)\,\theta_s(x, \lambda)]$, $(r, s = 1, 2)$, $l_{rs}(\lambda) = l_{sr}(\lambda)$ for all b and λ and when b tends to infinity,

$$\psi_r(x, \lambda) = \theta_r(x, \lambda) + \sum_{s=1}^{2} m_{rs}(\lambda)\,\phi_s(x, \lambda) \tag{16}$$

$$m_{rs}(\lambda) = m_{sr}(\lambda) = \lim_{b\to\infty} l_{rs}(\lambda).$$

From Green's formula

$$(\lambda - \lambda')\langle\, Y(x, \lambda), Z(x, \lambda')\,\rangle_{o,\,b} = [Y(x, \lambda)\,Z(x, \lambda')]_{x=0}^{x=b}$$

it then follows easily that

$$\langle\psi_r(b, x, \lambda), \quad \psi_s(b, x, \lambda')\rangle_{0,\,b} = \frac{l_{rs}(\lambda) - l_{rs}(\lambda')}{\lambda' - \lambda} \tag{17}$$

whence taking $\lambda = \bar{\lambda}$, $(\lambda = \mu + i\nu)$ we have

$$\|\,\psi_r(b, x, \lambda)\,\|_{0.\,b} = -\frac{im\, l_{rr}(\lambda)}{\nu} \tag{18}$$

and

$$\langle\psi_1(b, x, \lambda), \psi_2(b, x, \bar{\lambda})\rangle_{0,\,b} = -\frac{im\, l_{12}(\lambda)}{\nu}. \tag{19}$$

3. The vector $U_r(b, x, \lambda_{n,\,b})$

It follows as in Chakravarty[2] and Everitt[5] that for each fixed b, the only singularities of $l_{rs}(\lambda)$ are simple poles on the real axis. Let $\lambda_{n,\,b}$ be a simple pole of $l_{rs}(\lambda)$ with residue $R_{rs}(b, n)$. Since $l_{rs}(\lambda) = l_{sr}(\lambda)$ it follows that $R_{rs}(b, n) = R_{sr}(b, n)$.

Now as v tends to zero, $iv\psi_r(b, x, \lambda_{n,b} + iv)$ $(r = 1, 2)$ (each belongs to $L^2[0, b]$) converge in mean square to

$$\sum_{s=1}^{2} R_{rs}(b, n)\,\phi_s(x, \lambda_{n,b}) = U_r(b, x, \lambda_{n,b}) = \{U_{r1}(b, x, \lambda_{n,b}),\; U_{r2}(b, x, \lambda_{n,b})\} \quad (r = 1, 2). \tag{20}$$

The proof follows in the same way as Chaudhuri.[3]

Put $\lambda' = \lambda_{n,b} + iv$ in (17), multiply by iv and then make v tend to zero, so as to obtain

$$\langle\psi_r(b, x, \lambda)\, U_s(b, x, \lambda_{n,b})\rangle_{0,b} = \frac{R_{rs}(b, n)}{\lambda - \lambda_{n,b}}, \qquad (\lambda \neq \lambda_{n,b}). \tag{21}$$

Next putting $\lambda = \lambda_{m,b} + iv$ in (21), we obtain on making v tend to zero

$$\langle U_r(b, x, \lambda_{m,b}),\; U_s(b, x, \lambda_{n,b})\rangle_{0,b} = \delta_{m,n} R_{rs}(b, n) \tag{22}$$

where $\delta_{m,n}$ is the Kronecker delta.

4. Preliminary results

It is well known from Chakravarty[1] that the eigenvalues $\lambda_{n,b}$ of the boundary value problem in the finite interval $[0, b]$ are either simple zeros or double zeros of $W(b, \lambda)$ and corresponding to a simple zero there is only one eigenvector $U(b, x, \lambda_{n,b})$ and corresponding to a double zero there are two eigenvectors $U^{(r)}(b, x, \lambda_{n,b})$, $(r = 1, 2)$ which are orthogonal to each other.

It is easy to prove that if $\lambda_{n,b}$ is a double zero of $W(b, \lambda)$

$$U^{(1)}(b, x, \lambda_{n,b}) = R_{11}^{-1/2}(b, n)\, U(b, x, \lambda_{n,b})$$

$$U^{(2)}(b, x, \lambda_{n,b}) = \frac{R_{11}(b, n)\, U_2(b, x, \lambda_{n,b}) - R_{12}(b, n)\, U(b, x, \lambda_{n,b})}{R_{11}^{1/2}(b, n)\,[R_{11}(b, n)\, R_{22}(b, n) - R_{12}^2(b, n)]^{1/2}} \tag{23}$$

whereas if $\lambda_{n,b}$ is a simple zero of $W(b, \lambda)$

$$U(b, x, \lambda_{n,b}) = R_{11}^{-1/2}(b, n)\, U_1(b, x, \lambda_{n,b}) = R_{22}^{-1/2}(b, n)\, U_2(b, x, \lambda_{n,b}). \tag{24}$$

For, let $\lambda_{n,b}$ be a double zero of $W(b, \lambda)$. If $R_{12}(b, n) = R_{21}(b, n) = 0$, then clearly the two normalised orthogonal eigenvectors are given by $R_{rr}^{-1/2}(b, n)\, U_r(b, x, \lambda_{n,b})$, $(r=1, 2)$ But if $R_{12}(b, n) = R_{21}(b, n) \neq 0$, the two normalised eigenvectors can be represented as

$$U^{(1)}(b, x, \lambda_{n,b}) = R_{11}^{-1/2}(b, n)\, U_1(b, x, \lambda_{n,b})$$

$$U^{(2)}(b, x, \lambda_{n,b}) = A_1 U_1(b, x, \lambda_{n,b}) + A_2 U_2(b, x, \lambda_{n,b})$$

where the constants A_1, A_2 are to be determined from the relations

$$\langle U^{(1)}(b, x, \lambda_{n,b}), U^{(2)}(b, x, \lambda_{n,b})\rangle_{0,b} = 0 \quad \text{and} \quad \| U^{(2)}(b, x, \lambda_{n,b}) \|_{0,b} = 1.$$

It follows easily that

$$A_1 = -R_{12} R_{11}^{-1/2} (R_{11} R_{22} - R_{12}^2)^{-1/2}, \qquad A_2 = R_{11}^{1/2} (R_{11} R_{22} - R_{12}^2)^{-1/2}$$

and (23) follows.

On the other hand if $\lambda_{n,b}$ is a simple zero of $W(b, \lambda)$, let $U(b, x, \lambda_{n,b})$ be the eigenvector corresponding to the eigenvalue $\lambda_{n,b}$, then

$$R_{rr}^{-1/2}(b, n)\, U_r(b, x, \lambda_{n,b}). \quad (r = 1, 2)$$

are eigenvectors. We show that

$$R_{11}^{-1/2}(b, n)\, U_1(b, x, \lambda_{n,b}) = -R_{22}^{-1/2}(b, n)\, U_2(b, x, \lambda_{n,b})$$

If $[\phi_2\chi_2] \neq 0$

$$[\phi_2\chi_2](\lambda_{n,b})\, \chi_1(x, \lambda_{n,b}) - [\phi_2\chi_1](\lambda_{n,b})\, \chi_2(x, \lambda_{n,b})$$
$$= k\{[\phi_2\chi_2](\lambda_{n,b})\, \phi_1(x, \lambda_{n,b}) - [\phi_2\chi_1](\lambda_{n,b})\, \phi_2(x, \lambda_{n,b})\} \tag{25}$$

[Compare Chakravarty[1]]

where k is a finite constant not equal to zero. Now replacing λ in (12) by $\lambda_{n,b} + iv$, multiplying both sides by iv, on making v tend to zero, it follows, on using (25) that

$$iv\, \psi_1(b, x, \lambda_{n,b} + iv) \to \frac{1}{W'(b, \lambda_{n,b})} \{[\phi_2\chi_2](\lambda_{n,b})\, \chi_1(x, \lambda_{n,b})$$
$$- [\phi_2\chi_1](\lambda_{n,b})\, \chi_2(x, \lambda_{n,b})\}$$
$$= \frac{k}{W'(b, \lambda_{n,b})} \{[\phi_2\chi_2](\lambda_{n,b})\, \phi_1(x, \lambda_{n,b})$$
$$- [\phi_1\chi_2](\lambda_{n,b})\, \phi_2(x, \lambda_{n,b})\} \tag{26}$$

The accent denotes differentiation with respect to λ. Again we have from (15) as v tends to zero

$$iv\, \psi_1(b. x. \lambda_{n,b} + iv) \to R_{11}(b, n)\, \phi_1(x, \lambda_{n,b}) + R_{12}(b, n)\, \phi_2(x, \lambda_{n,b}). \tag{27}$$

Comparing the coefficients of ϕ_1 and ϕ_2 in (26) and (27) we get

$$R_{11}(b, n) = \frac{k\,[\phi_2\chi_2](\lambda_{n,b})}{W'(b, \lambda_{n,b})}$$

and

$$R_{12}(b, n) = \frac{-k\,[\phi_1\chi_2](\lambda_{n,b})}{W'(b, \lambda_{n,b})} \tag{28}$$

imilarly from (13), it follows that as ν tends to zero

$$i\nu\psi_2(b, x, \lambda_{n,b} + i\nu) \to \frac{1}{W'(b, \lambda_{n,b})} \{[\phi_1\chi_1](\lambda_{n,b})\,\chi_2(x, \lambda_{n,b}) - [\phi_1\chi_2](\lambda_{n,b})\,\chi_1(x, \lambda_{n,b})\}$$

$$= \frac{1}{W'(b, \lambda_{n,b})} \frac{[\phi_1\chi_2](\lambda_{n,b})}{[\phi_2\chi_2](\lambda_{n,b})} \{[\phi_2\chi_1](\lambda_{n,b})\,\chi_2(x, \lambda_{n,b}) - [\phi_2\chi_2](\lambda_{n,b})\,\chi_1(x, \lambda_{n,b})\}.$$

Since

$$[\phi_1\chi_1](\lambda_{n,b})\,[\phi_2\chi_2](\lambda_{n,b}) - [\phi_1\chi_2](\lambda_{n,b})\,[\phi_2\chi_1](\lambda_{n,b}) = 0]$$

$$= \frac{-k}{W'(b, \lambda_{n,b})} \frac{[\phi_1\chi_2](\lambda_{n,b})}{[\phi_2\chi_2](\lambda_{n,b})} \{[\phi_2\chi_2](\lambda_{n,b})\,\phi_1(x, \lambda_{n,b}) - [\phi_1\chi_2](\lambda_{n,b})\,\phi_2(x, \lambda_{n,b})\}. \tag{29}$$

Also from (15) as ν tends to zero

$$i\nu\,\psi_2(b, x, \lambda_{n,b} + i\nu) \to R_{21}(b, n)\,\phi_1(x, \lambda_{n,b}) + R_{22}(b, n)\,\phi_2(x, \lambda_{n,b}). \tag{30}$$

Comparing the coefficients of ϕ_1 and ϕ_2 in (29) and (30) we obtain

$$R_{21}(b, n) = \frac{-k\,[\phi_1\chi_2](\lambda_{n,b})}{W'(b, \lambda_{n,b})}$$

and

$$R_{22}(b, n) = \frac{k\,\{[\phi_1\chi_2](\lambda_{n,b})\}^2}{W'(b, \lambda_{n,b})\,[\phi_2\chi_2](\lambda_{n,b})}. \tag{31}$$

From (28) and (31) it follows that

$$R_{11}(b, n)\,R_{22}(b, n) = R_{12}^2(b, n). \tag{32}$$

Now multiplying both sides of (20) by $R_{rr}^{-1/2}(b, n)$ and making use of the result (32) we get

$$R_{11}^{-1/2}\,U_1(b, x, \lambda_{n,b}) = R_{11}^{1/2}(b, n)\,\phi_1(x, \lambda_{n,b}) - R_{22}^{1/2}(b, n)\,\phi_2(x, \lambda_{n,b})$$

$$R_{22}^{-1/2}\,U_2(b, x, \lambda_{n,b}) = -R_{11}^{1/2}(b, n)\,\phi_1(x, \lambda_{n,b}) + R_{22}^{1/2}(b, n)\,\phi_2(x, \lambda_{n,b}).$$

Hence

$$R_{11}^{-1/2}\,U_1(b, x, \lambda_{n,b}) = -R_{22}^{-1/2}\,U_2(b, x, \lambda_{n,b}).$$

5. Asymptotic formulae

Let $\phi_r^F(x, \lambda) = \{u_r^F(0/x, \lambda),\ v_r^F(0/x, \lambda)\}$, $(r = 1, 2)$ be the boundary condition vectors for the system (5)–(2) satisfying the initial conditions

$$\left.\begin{aligned} u_r^F(0/0, \lambda) &= -a_{r3}, & u_r'^F(0/0, \lambda) &= a_{r1} \\ v_r^F(0/0, \lambda) &= -a_{r4}, & v_r'^F(0/0, \lambda) &= a_{r2} \end{aligned}\right\} \quad (r = 1, 2). \tag{33}$$

By considering the most general solution of the system (5) and the relations (33) we can easily deduce that

$$\left.\begin{aligned} u_r^F(x, \lambda) &= \frac{1}{2}\left(-a_{r3} + \frac{a_{r1}}{i\mu}\right)e^{i\mu x} - \frac{1}{2}\left(a_{r3} + \frac{a_{r1}}{i\mu}\right)e^{-i\mu x} \\ v_r^F(x, \lambda) &= \frac{1}{2}\left(-a_{r4} + \frac{a_{r2}}{i\mu}\right)e^{i\mu x} - \frac{1}{2}\left(a_{r4} + \frac{a_{r2}}{i\mu}\right)e^{-i\mu x} \end{aligned}\right\} \quad (r = 1, 2) \tag{34}$$

$\lambda = \mu^2$ where $\mu = \sigma + it$, $t > 0$.

Similarly for the vectors $\theta_r^F(x, \lambda) = \{x_r^F(0/x, \lambda),\ y_r^F(0/x, \lambda)\}$ $(r = 1, 2)$ which take the initial conditions

$$\left.\begin{aligned} x_r^F(0, \lambda) &= (-1)^{r-1} a_{s4}, & x_r'^F(0, \lambda) &= (-1)^r a_{s2} \\ y_r^F(0, \lambda) &= (-1)^{r-1} a_{s3}, & y_r'^F(0, \lambda) &= (-1)^r a_{s1} \end{aligned}\right\} \tag{35}$$

[when $r = 1$, $s = 2$ and when $r = 2$, $s = 1$]

it follows as before that

$$\left.\begin{aligned} x_r^F(x, \lambda) &= \frac{(-1)^{r-1}}{2}\left[\left(a_{s4} - \frac{a_{s2}}{i\mu}\right)e^{i\mu x} + \left(a_{s4} + \frac{a_{s2}}{i\mu}\right)e^{-i\mu x}\right] \\ y_r^F(x, \lambda) &= \frac{(-1)^{r-1}}{2}\left[\left(a_{s3} - \frac{a_{s1}}{i\mu}\right)e^{i\mu x} + \left(a_{s3} + \frac{a_{s1}}{i\mu}\right)e^{-i\mu x}\right] \end{aligned}\right\} \tag{36}$$

[when $r = 1$, $s = 2$ and when $r = 2$, $s = 1$].

If

$$\psi_1^F(x, \lambda) = \lim_{b \to \infty} \psi_1^F(b, x, \lambda)$$

and

$$m_{rs}^F(\lambda) = \lim_{b \to \infty} l_{rs}^F(\lambda)$$

we have

$$\psi_1^F(x, \lambda) = \theta_1^F(x, \lambda) + m_{11}^F(\lambda)\,\phi_1^F(x, \lambda) + m_{12}^F(\lambda)\,\phi_2^F(x, \lambda).$$

Hence

$$\psi^F_{11}(x,\lambda) = \frac{1}{2}\Big[\Big\{\Big(a_{24} - \frac{a_{22}}{i\mu}\Big)e^{i\mu x} + \Big(a_{24} + \frac{a_{22}}{i\mu}\Big)e^{-i\mu x}\Big\}$$
$$+ m^F_{11}(\lambda)\Big\{\Big(-a_{13} + \frac{a_{11}}{i\mu}\Big)e^{i\mu x} - \Big(a_{13} + \frac{a_{11}}{i\mu}\Big)e^{-i\mu x}\Big\}$$
$$+ m^F_{12}(\lambda)\Big\{\Big(-a_{23} + \frac{a_{21}}{i\mu}\Big)e^{i\mu x} - \Big(a_{23} + \frac{a_{21}}{i\mu}\Big)e^{-i\mu x}\Big\}\Big] \tag{37}$$

with a similar expression for $\psi^F_{12}(x,\lambda)$.

Since $\psi^F_1(x,\lambda)$ belongs to $L^2[0,\infty)$ and $im\,\mu > 0$ it follows that the coefficients of $e^{-i\mu x}$ should vanish. Thus

$$\frac{1}{2}\Big(a_{24} + \frac{a_{22}}{i\mu}\Big) - \frac{1}{2}\Big(a_{13} + \frac{a_{11}}{i\mu}\Big)m^F_{11}(\lambda) - \frac{1}{2}\Big(a_{23} + \frac{a_{21}}{i\mu}\Big)m^F_{12}(\lambda) = 0$$

$$\frac{1}{2}\Big(a_{23} + \frac{a_{21}}{i\mu}\Big) - \frac{1}{2}\Big(a_{14} + \frac{a_{12}}{i\mu}\Big)m^F_{11}(\lambda) - \frac{1}{2}\Big(a_{24} + \frac{a_{22}}{i\mu}\Big)m^F_{12}(\lambda) = 0$$

leading to

$$m^F_{11}(\lambda) = -(\mu^2 A_1 + 2i\mu A_2 + A_3)M_1^{-1}$$

and

$$m^F_{12}(\lambda) = -(\mu^2 E_1 + i\mu E_2 + E_3)M_1^{-1} \tag{38}$$

where,

$$A_1 = a^2_{23} - a^2_{24},\quad A_2 = a_{22}a_{24} - a_{21}a_{23},\quad A_3 = a^2_{22} - a^2_{21}$$
$$E_1 = a_{14}a_{24} - a_{13}a_{23},\quad E_2 = a_{13}a_{21} + a_{11}a_{23} - a_{12}a_{24} - a_{14}a_{22}$$
$$E_3 = a_{11}a_{21} - a_{12}a_{22},\quad B_1 = a_{13}a_{24} - a_{14}a_{23},$$
$$B_2 = a_{13}a_{22} + a_{11}a_{23} - a_{12}a_{24} - a_{14}a_{21},$$
$$B_3 = a_{12}a_{21} - a_{11}a_{22} \text{ and } M_1 = B_1\mu^2 - i\mu B_2 + B_3. \tag{39}$$

Similarly since

$$\psi^F_2(x,\lambda) = \theta^F_2(x,\lambda) + m^F_{21}(\lambda)\phi^F_1(x,\lambda) + m^F_{22}(\lambda)\phi^F_2(x,\lambda)$$

belongs to $L^2[0,\infty)$, it follows that

$$m^F_{22}(\lambda) = -(\mu^2 C_1 + 2i\mu C_2 + C_3)M_1^{-1},$$

where

$$C_1 = a^2_{13} - a^2_{14},\quad C_2 = a_{12}a_{14} - a_{11}a_{13},\quad C_3 = a^2_{12} - a^2_{11} \tag{40}$$

also

$$m_{12}^F(\lambda) = m_{21}^F(\lambda).$$

Substituting the values of $m_{11}^F(\lambda), m_{12}^F(\lambda)$ in (37), it follows on slight reduction that

$$\psi_{11}^F(x,\lambda) = \frac{e^{i\mu x}}{2}\Big[\Big(a_{24} - \frac{a_{22}}{i\mu}\Big) + \Big(a_{13} - \frac{a_{11}}{i\mu}\Big)(\mu^2 A_1 + 2i\mu A_2 + A_3)$$

$$+ \Big(a_{23} - \frac{a_{21}}{i\mu}\Big)(\mu^2 E_1 + i\mu E_2 + E_3)\, M_1^{-1}\Big]$$

$$= \frac{e^{i\mu x}}{2} M_1^{-} \Big[\Big(a_{24} - \frac{a_{22}}{i\mu}\Big)(B_1 \mu^2 - i\mu B_2 + B_3) + \Big(a_{13} - \frac{a_{11}}{i\mu}\Big)$$

$$\times (\mu^2 A_1 + 2i\mu A_2 + A_3) + \Big(a_{23} - \frac{a_{21}}{i\mu}\Big)(\mu^2 E_1 + i\mu E_2 + E_3)\Big] \tag{41}$$

$$= o\Big(\frac{e^{-|t|\,|x|}}{|\mu|}\Big), \quad \text{provided} \quad a_{11}A_1 + a_{21}E_1 + a_{22}B_1 \neq 0. \tag{42}$$

It follows in a similar manner that

$$\psi_{12}^F(x,\lambda),\ \psi_{21}^F(x,\lambda),\ \psi_{22}^F(x,\lambda) \ \text{ are each } o\Big(\frac{e^{-|t|\,|x|}}{|\mu|^l}\Big) \tag{42a}$$

where $l = 1$ if $a_{12}A_1 + a_{21}B_1 + a_{22}E_1, a_{21}C_1 + a_{11}E_1 - a_{12}B_1$ and $a_{22}C_1 - a_{11}B_1 + a_{12}E_1$ are all non-vanishing and $l > 1$ when all of them vanish. Therefore from (11) using (34) and the triangle inequality, we have for $x < \xi$

$$G_{11}^F(x,\xi,\lambda) = u_1^F(x,\lambda)\,\psi_{11}^F(\xi,\lambda) + u_2^F(x,\lambda)\,\psi_{21}^F(\xi,\lambda) \tag{43}$$

$$= \frac{1}{2}\Big[\Big\{\Big(-a_{13} + \frac{a_{11}}{i\mu}\Big)e^{i\mu x} - \Big(a_{13} + \frac{a_{11}}{i\mu}\Big)e^{-i\mu x}\Big\}$$

$$+ \Big\{\Big(-a_{23} + \frac{a_{21}}{i\mu}\Big)e^{i\mu x} - \Big(a_{23} + \frac{a_{21}}{i\mu}\Big)e^{-i\mu x}\Big\}\Big]$$

$$\times\, o\Big(\frac{e^{-|t|\,|\xi|}}{|\mu|}\Big) = o\Big(\frac{e^{-|t|\,|x-\xi|}}{|\mu|}\Big)$$

with similar expressions for the other $G_{ij}^F(x,\xi,\lambda)$, $(i,j = 1,2)$ and hence

$$G_{ij}^F(x,\xi,\lambda) = o\Big(\frac{e^{-t|x-\xi|}}{|\mu|}\Big), \quad (i,j = 1,2). \tag{44}$$

Lemma 1: Let $p(x)$, $q(x)$ and $r(x)$ all belong to $L[0,\infty)$, then

$$G_{ij}(x,\xi,\lambda) = G_{ij}^F(x,\xi,\lambda) + o\Big(\frac{e^{-t|x-\xi|}}{|\mu|}\Big), \quad (i,j = 1,2) \tag{45}$$

where $G_{ij}(x, \xi, \lambda)$ and $G_{ij}^F(x, \xi, \lambda)$ are the elements of the Green's matrices $G(x, \xi, \lambda)$ of the system (1)–(2) and $G^F(x, \xi, \lambda)$ of the system (5)–(2) respectively.

PROOF: We consider the differential system

$$(L - \lambda)\, U(x) = P(x)\, G_l^F(x, \xi, \lambda) \tag{46}$$

where

$$U(x) = \{u(x), v(x)\}, \qquad G_l^F(x, \xi, \lambda) = \{G_{l1}^F(x, \xi, \lambda), G_{l2}^F(x, \xi, \lambda)\},\ l = 1, 2$$

and

$$P(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & r(x) \end{pmatrix}.$$

Since both $G'_{ij}(x, \xi, \lambda)$ and $G'^F_{ij}(x, \xi, \lambda)$ considered as functions of x, have singularities at the point $x = \xi$ of the same order with the same saltus, $\{G_l(x, \xi, \lambda) - G_l^F(x, \xi, \lambda)\}$ is continuous and satisfies the system (46), (2) and

$$G_l(x, \xi, \lambda) = G_l^F(x, \xi, \lambda) + \int_0^\infty G(x, y, \lambda)\, P(y)\, G_l^F(y, \xi, \lambda)\, dy, \tag{47}$$

the integral on the right being convergent by (ii) of § 1.

To solve this integral equation we define a sequence of vector functions $\{G_l^{(n)}(x, \xi, \lambda)\}$ by the relations

$$G_l^{(0)}(x, \xi, \lambda) = G_l^F(x, \xi, \lambda)$$

$$G_l^{(n)}(x, \xi, \lambda) = G_l^{(0)}(x, \xi, \lambda) + \int_0^\infty G^{(n-1)}(x, y, \lambda)\, P(y)\, G_l^F(y, \xi, \lambda)\, dy.$$

From (44) it follows that

$$| G_{l1}^{(1)}(x, \lambda) - G_{l1}^{(0)}(x, \xi, \lambda) |$$

$$\leq \frac{K^2}{|\mu|^2} \int_0^\infty e^{-t(|x-y|+|y-\xi|)} \{|p(y)| + 2\,|q(y)| + |r(y)|\}\, dy,$$

with similar expression for $| G_{l2}^{(1)}(x, \xi, \lambda) - G_{l2}^{(0)}(x, \xi, \lambda) |$, K being an absolute constant.

Again since $e^{-t(|x-y|+|y-\xi|)} \leq e^{-t|x-\xi|}$ by the triangle inequality, we get

$$| G_{l1}^{(1)}(x, \xi, \lambda) - G_{l1}^{(0)}(x, \xi, \lambda) | \leq \frac{K^2 e^{-t|x-\xi|}}{|\mu|^2} \int_0^\infty \{|p(y)| + 2\,|q(y)| + |r(y)|\}\, dy$$

$$= \frac{AKe^{-t|x-\xi|}}{|\mu|^2}, \quad \text{(say)} \tag{48}$$

where, $A = K \int_0^\infty \{|p(y)| + 2|q(y)| + |r(y)|\} dy < \infty$; $p(x), q(x), r(x)$ being integrable in $[0, \infty)$: with similar results for $| G_{i2}^{(1)}(x, \xi, \lambda) - G_{i2}^{(0)}(x, \xi, \lambda) |$.

Using (47) and (48)

$$| G_{i1}^{(2)}(x, \xi, \lambda) - G_{i1}^{(1)}(x, \xi, \lambda) |$$

$$\leqslant \frac{AK^2}{|\mu|^3} \int_0^\infty e^{-t(|x-y|+|y-\xi|)} \{|p(y)| + 2|q(y)| + |r(y)|\} dy$$

$$= \frac{A^2 K e^{-t|x-\xi|}}{|\mu|^3}$$

with similar results for

$$| G_{i2}^{(2)}(x, \xi, \lambda) - G_{i2}^{(1)}(x, \xi, \lambda) |.$$

Now let

$$| G_{ij}^{(n)}(x, \xi, \lambda) - G_{ij}^{(n-1)}(x, \xi, \lambda) | \leqslant \frac{A^n K e^{-t|x-\xi|}}{|\mu|^{n+1}} \tag{49}$$

for some fixed positive integer n.

Then from

$$| G_{ij}^{(n+1)}(x, \xi, \lambda) - G_{ij}^{(n)}(x, \xi, \lambda) | \leqslant \int_0^\infty \{ | G_{1j}^{(n)}(x, y, \lambda) - G_{1j}^{(n-1)}(x, y, \lambda) | | F_{i1}(y) |$$

$$+ | G_{2j}^{(n)}(x, y, \lambda) - G_{2j}^{(n-1)}(x, y, \lambda) | | F_{i2}(y) | \} dy$$

where

$$F_{i1}(x) = p(x) G_{i1}^F(x, y, \lambda) + q(x) G_{i2}^F(x, y, \lambda)$$

and

$$F_{i2}(x) = q(x) G_{i1}^F(x, y, \lambda) + r(x) G_{i2}^F(x, y, \lambda)$$

we get

$$G_{ij}^{(n+1)}(x, \xi, \lambda) - G_{ij}^{(n)}(x, \xi, \lambda) | \leqslant \frac{A^{n+1} K e^{-t|x-\xi|}}{|\mu|^{n+2}}.$$

Thus (49) holds by induction, for all integral values of n. The uniform convergence of the sequence $\{G_i^{(n)}(x, \xi, \lambda)\}$ to the limit $G_i(x, \xi, \lambda)$, as n tends to infinity follows easily. The functions $G_{ij}(x, \xi, \lambda)$, $(i, j = 1, 2)$ satisfy all the properties of the Green's matrix for the system (1), (2) and are therefore the elements of the Green's matrix; the integral equation (47) therefore possesses a solution.

Also,

$$| G_{ij}(x, \xi, \lambda) | = \lim_{n\to\infty} | G_{ij}^{(n)}(x, \xi, \lambda) |$$

$$= \lim_{n\to\infty} | G_{ij}^{(0)}(x, \xi, \lambda) + \sum_{r=1}^{n} \{G_{ij}^{(r)}(x, \xi, \lambda) - G_{ij}^{(r-1)}(x, \xi, \lambda)\} |$$

$$\leqslant \lim_{n\to\infty} \left[\frac{KAe^{-t|x-\xi|}}{|\mu|} + \sum_{r=1}^{n} \frac{A^r Ke^{-t|x-\xi|}}{|\mu|^{r+1}}\right]$$

Hence,

$$G_{ij}(x, \xi, \lambda) \leqslant \frac{AKe^{-t|x-\xi|}}{|\mu|}. \tag{50}$$

It then follows from (47) that

$$G_1(x, \xi, \lambda) = G_1^F(x, \xi, \lambda) + o\left\{\int_0^\infty \frac{e^{-t|x-\xi|}}{|\mu|^2} (|p(y)| + 2|q(y)| + |r(y)|)\, e^{-t|y-\xi|}\, dy\right\}$$

$$= G_1^F(x, \xi, \lambda) + o\left(\frac{e^{-t|x-\xi|}}{|\mu|^2}\right). \tag{51}$$

Lemma 2 : For any fixed complex λ and λ'

$$[\psi_r(x, \lambda),\ \psi_s(x, \lambda')] \to 0 \quad \text{as} \quad x \to \infty. \qquad (r, s = 1, 2). \tag{52}$$

PROOF : We consider the integral equation

$$\psi(x, \lambda) = \psi^F(x, \lambda) + \int_0^\infty G^F(x, y, \lambda)\ P(y)\ \psi(y, \lambda)\, dy \tag{53}$$

where $\psi^F(x, \lambda)$ is the L^2 solution of the Fourier system. To solve the integral equation (53) by iteration we define the sequence of vectors $\{\psi^{(n)}(x, \lambda)\}$ as follows:

$$\psi^{(0)}(x, \lambda) = \psi^F(x, \lambda)$$

$$\psi^{(n)}(x, \lambda) = \psi^{(0)}(x, \lambda) + \int_0^\infty G^F(x, y, \lambda)\ P(y)\ \psi^{(n-1)}(y, \lambda)\, dy.$$

Now

$$\psi^{(1)} - \psi^{(0)} = o\left\{\int_0^x P(y) \frac{e^{-t(x-y)}}{|\mu|} \frac{e^{-ty}}{|\mu|} dy\right\} + o\left\{\int_x^\infty |P(y)| \frac{e^{-t(y-x)}}{|\mu|} \frac{e^{-ty}}{|\mu|} dy\right\}$$

$$= o\left(\frac{e^{-ts}}{|\mu|^2}\right) + o\left(\frac{e^{-ts}}{|\mu|^2} \int_s^{\infty} |P(y)|\, dy\right)$$

$$= o\left(\frac{e^{-ts}}{|\mu|^2}\right), \quad \text{since } p(x), q(x), r(x) \text{ are } L[0, \infty).$$

Put

$$\psi^{(n)} - \psi^{(n-1)} = o\left(\frac{e^{-ts}}{|\mu|^{n+1}}\right)$$

Then

$$\psi^{(n+1)} - \psi^{(n)} = o\left\{\int_0^s |P(y)| \frac{e^{-t(s-y)}}{|\mu|} \cdot \frac{e^{-ty}}{|\mu|^{n+1}}\, dy\right\}$$

$$+ o\left\{\int_s^{\infty} |P(y)| \frac{e^{-t(y-s)}}{|\mu|} \cdot \frac{e^{-ty}}{|\mu|^{n+1}}\, dy\right\} = o\left(\frac{e^{-ts}}{|\mu|^{n+2}}\right)$$

Now comparing with the geometric series $\Sigma \; |\mu|^{-(n+1)}$ we conclude that

$$\sum_{n=1}^{\infty} \{\psi^{(n)}(x, \lambda) - \psi^{(n-1)}(x, \lambda)\}$$

converges for $|\mu| > 1$. Hence arguing as before and making $n \to \infty$ we obtain

$$\psi(x, \lambda) = \lim_{n \to \infty} [\psi^{(0)}(x, \lambda) + \sum_{r=1}^{n} \{\psi^{(r)}(x, \lambda) - \psi^{(r-1)}(x, \lambda)\}]$$

$$= o\left(\frac{e^{-ts}}{|\mu|}\right)$$

We have from (34), (41) and (43)

$$\frac{\partial}{\partial x} G^F_{ij}(x, \xi, \lambda) = o(e^{-t|s-\xi|}), \quad \frac{d}{dx} \psi^F(x, \lambda) = o(e^{-ts})$$

By virtue of these relations the integral

$$\int_0^{\infty} \frac{\partial}{\partial x} G^F(x, y, \lambda) P(y) \psi^F(y, \lambda)\, dy$$

is uniformly convergent with respect to x and hence differentiating (53) with respect to x we obtain

$$\frac{d}{dx} \psi(x, \lambda) = \frac{d}{dx} \psi^F(x, \lambda) + \int_0^{\infty} \frac{\partial}{\partial x} G^F(x, y, \lambda) P(y) \psi^F(y, \lambda)\, dy$$

from which it follows that

$$\frac{d}{dx}\psi(x, \lambda) = o(e^{-tx})$$

Now for $\lambda' = (\mu')^2$, $\mu' = \sigma' + it'$, $t' > 0$, we obtain from the definition of bilinear concomitant

$$[\psi_r(x, \lambda)\,\psi_s(x, \lambda')] = o\left(\frac{e^{-(t+t')x}}{|\mu|}\right) + o\left(\frac{e^{-(t+t')x}}{|\mu'|}\right)$$

[see Titchmarsh,[7] Pt. I, p. 26 and Chaudhuri,[3] p. 263].

The result follows by making x tend to infinity.

6. The matrix $k_{rs}(\lambda)$

Following Everitt,[5] we have

$$m_{11}(\lambda)\, m_{22}(\lambda) - m_{12}^2(\lambda) \neq 0 \quad im\lambda \neq 0. \tag{54}$$

Hence $(m_{rs}(\lambda))$, $(r, s = 1, 2)$ is a non-singular matrix. Each $m_{rs}(\lambda)$, $(r, s = 1, 2)$ has singularities on the real axis and that $m_{rs}(\lambda)$ are analytic functions of λ regular in either of the half planes $im\,\lambda > 0$ or $im\,\lambda < 0$.

Lemma 3 : The functions

$$k_{rs}(\lambda) = \lim_{\delta\to 0} \int_0^{\lambda} - im\, m_{rs}(\gamma + i\delta)\, d\gamma, \quad (r, s = 1, 2) \tag{55}$$

exist for all real λ; each $k_{rs}(\lambda)$ is a function of bounded variation and

$$k_{rs}(\lambda) = \tfrac{1}{2}\{k_{rs}(\lambda + 0) + k_{rs}(\lambda - 0)\} \tag{56}$$

and

$$\lim_{\delta\to 0} \int_0^{\lambda} - im\, \psi_r(x, \gamma + i\delta)\, d\gamma = \sum_{s=1}^{2} \int_0^{\lambda} \phi_s(x, \gamma)\, dk_{rs}(\gamma) \tag{57}$$

[see refs. 8 and 3].

Lemma 4 : Let $\chi_r(x, \lambda) = \sum_{s=1}^{2} \int_0^{\lambda} \phi_s(x, \gamma)\, dk_{rs}(\gamma)$, $(r = 1, 2)$; λ real then $\chi_r(x, \lambda)$ belong to $L^2[0, \infty)$.

PROOF: If $\lambda_{n,b}$ is a simple zero of $W(b; \lambda)$, we obtain from (21), (24) and (32),

$$\langle \psi_r(b, x, \lambda), U(b, x, \lambda_{n,b})\rangle_{o,b} = \frac{R_{rr}^{1/2}(b, n)}{\lambda - \lambda_{n,b}}, \quad (r = 1, 2). \tag{58}$$

If $\lambda_{n,b}$ be a double zero of $W(b, \lambda)$

$$\langle U^{(1)}(b, x, \lambda_{n,b}), \psi_r(b, x, \lambda)\rangle_{o,b} = \frac{R_{r1}(b, n) R_{11}^{-1/2}(b, n)}{\lambda - \lambda_{n,b}} = A_{nr}, \text{ say}$$

$$\langle U^{(2)}(b, x, \lambda_{n,b}), \psi_r(b, x, \lambda)\rangle_{o,b}$$

$$= \frac{R_{12}(b, n) R_{r1}(b, n) - R_{11}(b, n) R_{r2}(b, n)}{R_{11}^{1/2}(b, n) \{R_{11}(b, n) R_{22}(b, n) - R_{12}^2(b, n)\}^{1/2} (\lambda - \lambda_{n,b})} = B_{nr}, \text{ say.}$$

Clearly,

$$(A_{nr}^2 + B_{nr}^2)^{1/2} = \frac{R_{rr}^{1/2}(b, n)}{\lambda - \lambda_{n,b}}$$

Thus in any case

$$\frac{R_{rr}^{1/2}(b, n)}{\lambda - \lambda_{n,b}}$$

is the Fourier coefficient of $\psi_r(b, x, \lambda)$. Following Titchmarsh[7] [Pt. I, p. 54] we have from (58)

$$\left\langle U(b, x, \lambda_{n,b}), \int_0^\lambda im\, \psi_r(b, x, \gamma + i\delta)\, d\gamma \right\rangle_{o,b} = o\left(\frac{R_{rr}^{1/2}(b, n)}{1 + \lambda_{n,b}^2}\right), \quad \lambda \text{ finite.}$$

Hence by Parseval's theorem[2] § 7

$$\left\| \int_0^\lambda im\, \psi_r(b, x, \gamma + i\delta)\, d\gamma \right\|_{o,b} = o\left(\sum_{n=-\infty}^{\infty} \frac{R_{rr}(b, n)}{1 + \lambda_{n,b}^2} \right)$$

Making b tend to infinity through a suitable sequence, we obtain

$$\left\| \int_0^\lambda im\, \psi_r(b, x, \gamma + i\delta)\, d\gamma \right\| = o(1).$$

Finally making $\delta \to 0$ and using (57) we obtain

$$\left\| \sum_{s=1}^{2} \int_0^\lambda \phi_s(x, \gamma)\, dk_{rs}(\gamma) \right\| = o(1)$$

so that $\chi_r(x, \lambda), (r = 1, 2)$ belong to $L^2[0, \infty)$

7. Expansion theorem

Lemma 5 : Let $f(x) = \{f_1(x), f_2(x)\}$ belong to $L^2[0, \infty)$ and

$$F_r(\gamma) = \langle \chi_r(y, \gamma), f(y)\rangle \qquad (r = 1, 2) \tag{59}$$

γ real, then for any fixed x

$$\lim_{\delta\to 0} \; im \int_{-R+i\delta}^{R+i\delta} \Phi(x,\lambda)\,d\lambda = -\sum_{r=1}^{2} \int_{-R}^{R} \phi_r(x,\gamma)\,dF_r(\gamma), \quad (\lambda = \gamma + i\delta) \tag{60}$$

where $F_r(\gamma)$ is a function of bounded variation.

PROOF: Since $\chi_r(x, \gamma)$ and $f(x)$ both belong to $L^2[0, \infty)$, $F_r(\gamma)$ exists, now as in Chakravarty[2] and from (11)

$$\Phi(x,\lambda) = \int_0^{\infty} G^T(x,y,\lambda) f(y)\,dy$$

$$= \sum_{r=1}^{2} [\psi_r(x,\lambda)\langle\phi_r(y,\lambda), f(y)\rangle_{0,x} + \phi_r(x,\lambda)\langle\psi_r(y,\lambda), f(y)\rangle_{x,\infty}].$$

Therefore

$$im\,[\int_{-R+i\delta}^{R+i\delta} \Phi(x,\lambda)\,d\lambda]$$

$$= im\,[\int_{-R+i\delta}^{R+i\delta} d\lambda \sum_{r=1}^{2} \phi_r(x,\lambda)\langle\psi_r(y,\lambda), f(y)\rangle]$$

$$+ im\,[\int_{-R+i\delta}^{R+i\delta} d\lambda \sum_{r=1}^{2} \int_0^x \{(\psi_r(x,\lambda), \phi_r(y,\lambda))$$

$$- (\phi_r(x,\lambda), \psi_r(y,\lambda))\}\, f(y)\,dy\,]$$

$= I_1 + I_2$, say.

Now

$$= im \sum_{r=1}^{2} [(\sum_{s=1}^{2} m_{rs}(\lambda)\,\phi_s(x,\lambda) + \theta_r(x,\lambda), \phi_r(y,\lambda))$$

$$- (\phi_r(x,\lambda), \sum_{s=1}^{2} m_{\cdot s}(\lambda)\,\phi_s(y,\lambda) + \theta_r(y,\lambda))]$$

$$= im \sum_{r=1}^{2} [(\theta_r(x,\lambda), \phi_r(y,\lambda)) - (\phi_r(x,\lambda), \theta_r(y,\lambda))]$$

$= o(\delta)$, as $\delta \to 0$, for x, y in fixed intervals.

It therefore follows that $I_2 = o(\delta)$, as $\delta \to 0$.

Again,

$$I_1 = im \int_{-R+i\delta}^{R+i\delta} \sum_{r=1}^{2} \langle\psi_r(y,\lambda), f(y)\rangle\,\phi_r(x,\lambda)\,d\lambda$$

$$= \int_{-R}^{R} \sum_{r=1}^{2} \langle im\,\psi_r(y,\gamma+i\delta), f(y)\rangle\,\mathrm{Re}\,\phi_r(x,\gamma+i\delta)\,d\gamma$$

$$+ \int_{-R}^{R} \sum_{r=1}^{2} \langle \mathrm{Re}\, \psi_r (y, \gamma + i\delta), f(y) \rangle \, im\, \phi_r (x, \gamma + i\delta)\, d\gamma$$

$$= I_{11} + I_{12}, \text{ say.}$$

Then

$$I_{12} = o(\delta) \int_{-R}^{R} d\gamma \int_{0}^{\infty} |(\psi_r (y, \gamma + i\delta), f(y))|\, dy$$

$$= o(\delta) [\int_{-R}^{R} \{ \int_{0}^{\infty} |(\psi_r (y, \gamma + i\delta), f(y))|\, dy\}^2 d\gamma]^{1/2}.$$

On applying the Schwartz inequality for vectors and noting that fact $f(x) \in L^2 [0, \infty)$

$$I_{12} = o(\delta) [\int_{-R}^{R} \| \psi_r (y, \gamma + i\delta) \| \, d\gamma]^{1/2}$$

$$= o(\delta^{1/2}), \text{ by (18).}$$

Similarly $I_{11} = o(\delta^{1/2})$.

Since $\theta_r (x, \gamma)$ and $\phi_r\ (x, \gamma)$, $(r = 1, 2)$ are real for real γ, $im\, \theta_r (x, \gamma + i\delta)$ and $\phi_r (x, \gamma + i\delta)$, $(r = 1, 2)$ are $o(\delta)$, uniformly with respect to γ over a finite interval.

So that

$$\int_{-R+i\delta}^{R+i\delta} \Phi(x, \lambda)\, d\lambda = I_{11} + o(\delta^{1/2}) \text{ as } \delta \to 0. \tag{61}$$

We also have

$$\int_{0}^{\eta} d\gamma \int_{0}^{\infty} (-im\, \psi_r (y, \gamma + i\delta), f(y))\, dy$$

$$= \int_{0}^{\infty} (\int_{0}^{\eta} -im\, \psi_r (y, \gamma + i\delta)\, d\gamma,\ f(y))\, dy$$

$$\to \langle \chi_r (y, \eta), f(y) \rangle = F_r (\eta), \text{ as } \delta \to 0. \tag{62}$$

The change in the order of integration being permissible, since $\psi_r (y, \gamma + i\delta)$, $(r = 1, 2)$ are continuous in y and γ and $|\langle -im\, \psi_r (y, \gamma + i\delta), f(y) \rangle| < \infty$, by the Schwartz inequality for vectors.

For the validity of the limiting process under the sign of integration, we note that [See ref. 7, Pt I, Lemma 24, 27]

$$\int_{0}^{\eta} -im\, \psi_r (y, \gamma + i\delta)\, d\gamma = \chi_r (y, \eta + i\delta) \in L^2 [0, \infty),$$

$$\delta = \delta_1, \delta_2, \delta_3, \ldots \text{ and as } \delta \to 0$$

$$\chi_r (y, \eta + i\delta) \to \chi_r (y, \eta) \in L^2 [0, \infty).$$

Integrating by parts we obtain from (61) using (62)

$$\lim_{\delta\to 0} im \left(-\frac{1}{\pi} \int_{-R+i\delta}^{R+i\delta} \Phi(x,\lambda)\, d\lambda \right)$$

$$= \lim_{\delta\to 0} \frac{1}{\pi} \sum_{r=1}^{2} \Bigg[\Big\{ \phi_r(x,\gamma) \int_0^{\gamma} d\gamma' \int_0^{\infty} (im\psi_r(y,\gamma'+i\delta),\ f(y)) \Big\}_{\gamma=-R}^{\gamma=R}$$

$$- \int_{-R}^{R} \frac{\partial}{\partial\gamma} \phi_r(x,\gamma)\, d\gamma \int_0^{\gamma} d\gamma' \int_0^{\infty} (-im\,\psi_r(y,\gamma+i\delta),\ \bar{f}(y))\, dy \Bigg]$$

On integration by parts the integral on the right we obtain

$$\lim_{\delta\to 0} im \left(-\frac{1}{\pi} \int_{-R+i\delta}^{R+i\delta} \Phi(x,\lambda)\, d\vec{\lambda} \right) = \frac{1}{\pi} \sum_{r=1}^{2} \int_{-R}^{R} \phi_r'(x,\gamma)\, dF_r(\gamma).$$

Finally, if $F_r(\gamma)$, $(r = 1, 2)$ is of bounded variation, the required result (60) follows.

8. The convergence theorem

THEOREM: If all the conditions given in § 1 are satisfied and $f(x)$ is both $L[0,\infty)$ and $L^2[0,\infty)$, then the expansion of $f(x) = \{f_1(x), f_2(x)\}$ corresponding to the system (1), (2) converges under the same conditions as the corresponding expansion of $f(x)$ when the differential system is replaced by (5).

PROOF: Let C a closed semi circular contour in the upper half of the λ-plane be with base the line joining the points $-R+i\delta$, $R+i\delta$ $(\delta > 0)$. As $\Phi(x,\lambda)$ is analytic inside and on this closed contour, applying Cauchy's theorem

$$\int_C \Phi(x,\lambda)\, d\lambda + \int_{-R+i\delta}^{R+i\delta} \Phi(x,\lambda)\, d\lambda = 0$$

and hence by (60)

$$\lim_{\substack{R\to\infty\\ \delta\to 0}} im \int_C \Phi(x,\lambda)\, d\lambda = - \lim_{\substack{R\to\infty\\ \delta\to 0}} im \int_{-R+i\delta}^{R+i\delta} \Phi(x,\lambda)\, d\lambda$$

$$= \sum_{r=1}^{2} \int_{-\infty}^{\infty} \phi_r(x,\lambda)\, dF_r(\lambda). \tag{63}$$

Similarly,

$$\lim_{\substack{R\to\infty\\ \delta\to 0}} im \int_C \Phi^F(x,\lambda)\, d\lambda = - \lim_{\substack{R\to\infty\\ \delta\to 0}} im \int_{-R+i\delta}^{R+i\delta} \Phi^F(x,\lambda)\, d\lambda$$

$$= \sum_{r=1}^{2} \int_{-\infty}^{\infty} \phi_r^F(x,\lambda)\, dF_r^F(\lambda) \tag{64}$$

The extreme right-hand side of (63) and (64) give rise to the expansion of $f(x)$ corresponding respectively to (1), (2) and to (5), (2). To establish the theorem we have therefore to prove that

$$\lim_{\substack{R\to\infty\\ \delta\to 0}} im \int_C \Phi(x,\lambda)\,d\lambda = \lim_{\substack{R\to\infty\\ \delta\to 0}} im \int_C \Phi^F(x,\lambda)\,d\lambda \tag{65}$$

Multiplying the transpose equation of (51) by $f^T(y)$ and integrating with respect to y over the interval $[0,\infty)$, it follows that

$$\Phi(x,\lambda) = \Phi^F(x,\lambda) + o\left(\int_0^\infty \frac{e^{-t|x-y|}}{|\lambda|}\,|f(y)|\,dy\right).$$

Finally integrating, on the part of the upper semi-circle of centre $i\delta$ $(\delta > 0)$ and radius R, of the contour C, we get

$$\int_C \Phi(x,\lambda)\,d\lambda = \int_C \Phi^F(x,\lambda)\,d\lambda + 0\left(\int_C |d\lambda| \int_0^\infty \frac{e^{-t|x-y|}}{|\lambda|}\,|f(y)|\,dy\right) \tag{66}$$

Now

$$\int_C \frac{|d\lambda|}{|\lambda|} \int_0^\infty e^{-t|x-y|}\,|f(y)|\,dy$$

$$= \int_C \frac{|d\lambda|}{|\lambda|} \left\{\int_0^{x-\zeta} \pm \int_{x-\zeta}^{x} + \int_x^{x+\zeta} + \int_{x+\zeta}^{\infty}\right\} e^{-t|x-y|}\,|f(y)|\,dy$$

$= I_1 + I_2 + I_3 + I_4$, say, where $\zeta > 0$.

Since $f(x)$ belongs to $L[0,\infty)$, we can choose ζ so that $\int_{x-\zeta}^{x} |f(y)|\,dy < \epsilon$ and $\int_x^{x+\zeta} |f(y)|\,dy < \varepsilon$, where $\varepsilon > 0$ is small but arbitrary. Then

$$I_2 = \int_C \frac{|d\lambda|}{|\lambda|} \int_{x-\zeta}^{x} e^{-t|x-y|}\,|f(y)|\,dy < \int_C \frac{|d\lambda|}{|\lambda|} \int_{x-\zeta}^{x} |f(y)|\,dy. = o(1)$$

Similarly $I_3 = o(1)$.

We put $\lambda = i\delta + \mathrm{Re}^{i\theta}$.

Then

$$I_1 = \int_C \frac{|d\lambda|}{|\lambda|} \int_0^{x-\zeta} e^{-t|x-y|}\,|f(y)|\,dy$$

$$< \int_C \frac{|d\lambda|}{|\lambda|} e^{-t\zeta} \int_0^{x-\zeta} |f(y)|\, dy < K \int_C \frac{|d\lambda|}{|\lambda|} e^{-t\zeta}$$

[where $\int_0^{x-\zeta} |f(y)|\, dy < K$]

$$= o\left(\int_0^{\pi} e^{-R^{1/2} |\sin 1/2\, \theta| \zeta} d\theta \right).$$

Proceeding as in Titchmarsh[6] p 104, it follows that

$$o\left(\int_0^{\pi} e^{-R^{1/2} |\sin 1/2\, \theta| \zeta} d\theta \right)$$

can be made arbitrarily small by making R tend to infinity. Similar conclusions follow for I_4.

Hence

$$o\left(\int_C |d\lambda| \int_0 \frac{e^{-t|x-y|}}{|\lambda|} |f(y)|\, dy \right) = o(1) \text{ as } R \to \infty.$$

The theorem therefore follows from (66).

9. Acknowledgements

The author wishes to express his thanks to Prof. N. K. Chakravarty, Department of Pure Mathematics, Calcutta University, for his kind guidance and encouragement with development of this work. He is also indebted to Dr. (Mrs.) Jyoti Das (nee Chaudhuri) for her several helpful suggestions during the preparation of this paper.

The financial support for the investigation was provided, in part, by the Education Directorate, Government of West Bengal.

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