Surface waves in a nonuniformly corrugated cylindrical metallic structure

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Abstract

The characteristic equation of a cosinusoidally spacing modulated corrugated cylindrical metallic structure excited in E_0-wave has been formulated by using non-homogeneous mixed boundary condition. Power carried by surface waves inside and outside the modulated medium which is treated as an artificial dielectric and power lost and hence attenuation constant have been determined. Expression for the relative amplitudes of Floquet harmonics with respect to that of the fundamental has been derived by finding the Fourier gap coefficient and using WKBJ approximation. Surface impedance of the modulated structure has also been determined.

Key words: Surface wave modulated structure, Floquet harmonics.

1. Introduction

In a recent paper,[2] the authors have introduced the concept of equivalent dielectric constant for inhomogeneous surface wave structures, *viz.*, nonuniformly corrugated cylindrical metallic structures with spacings between the thin ($t \ll \lambda_0$) discs being modulated cosinusoidally. Hence, transforming the surface wave structure to an equivalent structure consisting of a thin metallic rod coated with a dielectric whose dielectric constant is modulated cosinusoidally in the direction of propagation, the problem has been formulated in the form of Hill's equation which has been solved[1] to yield the phase constant $\beta = (2/\pi)$ arc sin $(\sqrt{\Delta(0)}$ sin $\sqrt{\theta_0}/2)$. The present paper is concerned with the derivation of the characteristic equation, the solution of which yields the radial propagation constant k_1 outside the modulated dielectric medium. The parameter θ_0 involved in β contains k_1. The radial propagation constant k_2 inside the modulated dielectric medium is related to k_1 and is a function of the modulation index $\delta\,(\ll 1)$ and unmodulated dielectric constant ϵ° which is a function of the spacing(s) between discs of radius b and the radius a of the inner supporting rod of the uniformly corrugated structure. The theory of power carried by surface waves outside the modulated dielectric medium relative to the total power has been developed. The Fourier gap coefficient C_n involved in the power flow expression is determined and using WKBJ approximation, the relative amplitudes of both backward and forward space harmonics with respect to that of the fundamental have been studied. The paper concludes with the derivation of expressions for the attenuation constant by the power loss method and surface impedance of the modulated structure. The present paper deals only with the theoretical aspects of the problem. Numerical calculations to illustrate the behaviour of surface waves and experimental verification of the theory will be reported elsewhere.

2. Wave equation and its solution

The wave equation governing the propagation of E-wave in a modulated dielectric medium with spatially varying dielectric constant $\varepsilon(z)$ in the direction (z) of propagation is given by[1]

$$\nabla \times \nabla \times \nabla \times [\Psi(\rho, \phi, z)\vec{i}_z] - k_0^2 \epsilon(z) \nabla \times [\Psi(\rho, \phi, z)\vec{i}_z] - \frac{\nabla \epsilon(z)}{\epsilon(z)} \times \nabla \times \nabla \times [\Psi(\rho, \phi, z)\vec{i}_z] = 0 \tag{1}$$

In order that the axial cylindrical surface wave may be supported by a cylindrical surface the relation between the radial (k) and axial (γ) propagation constants may be written as

$$k^2 = -[k_0^2 \epsilon(z) + \gamma^2], \tag{2}$$

where $\gamma = i\beta$, if the structure is assumed lossless.

Since for E_0-wave, $\partial/\partial\phi = 0$ the wave equation (1) is transformed to

$$\frac{\partial^2 \psi_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi_z}{\partial \rho} - k^2 \psi_z + \frac{\partial^2 \psi_z}{\partial z^2} - \frac{1}{\epsilon(z)}\frac{\partial \epsilon(z)}{\partial z}\frac{\partial \psi_z}{\partial z} - \gamma^2 \psi_z = 0 \tag{3}$$

which yields the solution for Ψ in terms of the modified Bessel functions as follows:

$$\Psi(\rho, \phi, z) = [A I_0(k\rho) + B K_0(k\rho)]\, \Phi^{1,2}(z) \tag{4}$$

where $\Phi^{1,2}(z)$ satisfies the following equation:

$$\left[\frac{d^2}{dz^2} - \frac{d\epsilon(z)}{dz}\frac{1}{\epsilon(z)}\frac{d}{dz} + \{k_0^2 \epsilon(z) + k^2\}\right]\Phi^{1,2}(z) = 0 \tag{5}$$

Introducing[2],

$$\epsilon(z) = \epsilon^0\left(1 - \delta \cos\frac{2\pi z}{L}\right)$$

and the new variable $\xi = \pi z/L$, eq. (5) is transformed to

$$\left[\frac{d^2}{d\xi^2} + \theta_0 + 2\sum_{\theta=1}^{\infty} \theta_n \cos 2n\xi\right] W^{1,2}(\xi) = 0 \tag{6}$$

where

$$W^{1,2}(\xi) = \exp(\pm i\beta\xi) \sum_{n=-\infty}^{\infty} C_n(\beta) \exp(\pm 2\, in\, \xi) \tag{6 a}$$

which accounts for the Floquet harmonics. Hence

$$\Psi(\rho, \phi, z) = \left(1 - \delta \cos \frac{2\pi z}{L}\right)^{1/2} [AI_0(k\rho) + BK_0(k\rho)].$$

$$\times \exp\left(\pm i\beta \frac{\pi z}{L}\right) \sum_{n=-\infty}^{\infty} C_n(\beta) \exp\left(\pm \frac{2in\pi z}{L}\right)$$

$$= \chi [AI_0(k\rho) + BK_0(k\rho)] \tag{7}$$

In eq. (6) $\theta_0 = f(k, \delta)$, whereas $\theta_n = 1, 2, \ldots (n \neq 0)$ are $f(\delta)$ only (1).

3. Field components

Using the following relations between Ψ and the field quantities

$$H_\phi = -\frac{\partial \Psi_z}{\partial \rho}$$

$$E_z = \frac{1}{\omega \epsilon_0 \epsilon(z)} \left[\frac{\partial H_\phi}{\partial \rho} + \frac{1}{\rho} H_\phi\right]$$

$$E_\rho = \frac{i}{\omega \epsilon_0 \epsilon(z)} \frac{\partial H_\phi}{\partial z} \tag{8}$$

the field components in the modulated medium ($a \leqslant \rho \leqslant b$) where b denotes the radius of the discs and a is the radius of the inner conductor supporting the corrugated medium are given by

Med 2 : $a \leqslant \rho \leqslant b$

$$H_{\phi 2} = -A_2\chi \left[I_1(k_2\rho) + \frac{I_0(k_2a)}{K_0(k_2a)} K_1(k_2\rho)\right]$$

$$E_{z2} = -\frac{A_2\chi}{\omega \epsilon_0 \epsilon(z)} \left[I_0(k_2\rho) - \frac{I_0(k_2a)}{K_0(k_2a)} K_0(k_2\rho)\right]$$

$$E_{\rho 2} = -A_2\chi' \left[I_1(k_2\rho) + \frac{I_0(k_2a)}{K_0(k_2a)} K_1(k_2\rho)\right] \tag{9}$$

since, at $\rho = a$, $E_{z2} = 0$ and

$$I_1'(k_2\rho) = -\frac{I_1(k_2\rho)}{k_2\rho} + \frac{I_0(i_2\rho)}{k_2} \tag{9 a}$$

Since in medium outside the modulated structure

$$\Psi = \chi A_1 H_0^{(1)}(k_1\rho)$$

so that the condition at infinity is satisfied, the field components are

Med 1 : $b \leqslant \rho < \infty$

$$H_{\phi 1} = -i\chi \frac{2A_1}{\pi} K_1(k_1\rho)$$

$$E_{z1} = i\chi \frac{2A_1}{\pi\omega\epsilon_0\epsilon(z)} K_0(k_1\rho)$$

$$E_{\rho 1} = -i\chi' \frac{2A_1}{\pi} K_1(k_1\rho) \tag{10}$$

where the following transformations have been used:

$$H_0^{(1)\prime}(ik_1\rho) = \frac{2}{\pi k_1} K_1(k\rho)$$

$$K_0'(k_1\rho) = -\frac{K_1(k_1\rho)}{k_1}$$

$$H_0^{(1)\prime}(ik_1\rho) = -\frac{2}{\pi} K_0'(k_1\rho)$$

$$K_1'(k_1\rho) = -\frac{K_1(k_1\rho)}{k_1\rho} - \frac{K_0(k_1\rho)}{k_1} \tag{10a}$$

and

$$\chi = \left(1 - \delta\cos\frac{2\pi z}{L}\right)^{1/2} \sum_{n=-\infty}^{\infty} C_n(\beta)\exp(\pm i\theta)$$

$$\chi' = \frac{i}{\omega\epsilon_0\epsilon(z)}\left[\frac{1}{2}\left(1 - \delta\cos\frac{2\pi z}{L}\right)^{-1/2}\frac{2\pi\delta}{L}\sin\frac{2\pi z}{L}\right.$$

$$\sum_{n=-\infty}^{\infty} C_n(\beta)\exp(\pm i\theta)$$

$$+\left(1 - \delta\cos\frac{2\pi z}{L}\right)^{1/2}\left(z\frac{d\beta}{dz} + \beta\right)\left(\pm\frac{i\pi}{L}\right)\sum_{n=-\infty}^{\infty} C_n(\beta)\exp(\pm i\theta)$$

$$+\left(1 - \delta\cos\frac{2\pi z}{L}\right)^{1/2}\sum_{n=-\infty}^{\infty}\frac{dC_n(\beta)}{dz}\exp(\pm i\theta)$$

$$\left.+\left(1 - \delta\cos\frac{2\pi z}{L}\right)^{1/2}\sum_{n=-\infty}^{\infty} C_n(\beta)\left(\pm\frac{2in\pi}{L}\right)\exp(\pm i\theta)\right] \tag{10b}$$

and

$$\theta = \frac{2n\pi z}{L} + \frac{\pi\beta z}{L}.$$

4. Boundary condition

The inhomogeneous mixed boundary conditions

$$a_1 E_{z1} + b_1 \frac{\partial E_{z1}}{\partial\rho} = \kappa_{1z}$$

$$a_2 E_{z2} + b_2 \frac{\partial E_{z2}}{\partial\rho} = \kappa_{2z}$$

$$a_1 H_{\phi 1} + b_1 \frac{\partial H_{\phi 1}}{\partial\rho} = \kappa_{1\phi}$$

$$a_2 H_{\phi 2} + b_2 \frac{\partial H_{\phi 2}}{\partial\rho} = \kappa_{2\phi} \tag{11}$$

can be reduced to the following mixed homogeneous boundary conditions:

$$a_1 \frac{\partial E_{z1}}{\partial\rho} + b_1 \frac{\partial^2 E_{z1}}{\partial\rho^2} = 0$$

$$a_2 \frac{\partial E_{z2}}{\partial\rho} + b_2 \frac{\partial^2 E_{z2}}{\partial\rho^2} = 0$$

$$a_1 \frac{\partial H_{\phi 1}}{\partial\rho} + b_1 \frac{\partial^2 H_{\phi 1}}{\partial\rho^2} = 0$$

$$a_2 \frac{\partial H_{\phi 2}}{\partial\rho} + b_2 \frac{\partial^2 H_{\phi 2}}{\partial\rho^2} = 0 \tag{12}$$

Hence by matching appropriate boundary conditions at $\rho = b$ and replacing b_1 and b_2 in terms of a_1 and a_2 respectively, the following condition is obtained:

$$\frac{a_1}{a_2} = \frac{\left[E_{z2} - \dfrac{(\partial E_{z2}/\partial\rho)^2}{\partial^2 E_{z2}/\partial\rho^2} \right]_{\rho=b}}{\left[E_{z1} - \dfrac{(\partial E_{z1}/\partial\rho)^2}{\partial^2 E_{z1}/\partial\rho^2} \right]_{\rho=b}} = \frac{\left[H_{\phi 2} - \dfrac{(\partial H_{\phi 2}/\partial\rho)^2}{\partial^2 H_{\phi 2}/\partial\rho^2} \right]_{\rho=b}}{\left[H_{\phi 1} - \dfrac{(\partial H_{\phi 1}/\partial\rho)^2}{\partial^2 H_{\phi 1}/\partial\rho^2} \right]_{\rho=b}} \tag{13}$$

5. Characteristic equation

By using appropriate field components in eq. (13) the following characteristic equation is obtained:

$$
-\frac{\left[K_1(k_1b) - \dfrac{\left\{\dfrac{K_1(k_1b)}{b} + K_0(k_1b)\right\}^2}{\left\{2\dfrac{K_1(k_1b)}{b^2} + \dfrac{K_0(k_1 b)}{b} + K_1(k_1b)\right\}}\right]}{\left[K_0(k_1b) - \dfrac{K_1^2(k_1b)}{\dfrac{K_1(k_1b)}{b} + K_0(k_1 b)}\right]}
$$

$$
= \frac{\times\left[\left\{I_1(k_2b) + \dfrac{I_0(k_2a)}{K_0(k_2a)}K_1(k_2b)\right\} + \dfrac{\left\{\dfrac{I_1(k_2b)}{b} - I_0(k_2b) + \dfrac{I_0(k_2a)}{K_0(k_2a)}\left(\dfrac{K_1(k_2b)}{b} + K_0(k_2b)\right)\right\}^2}{\left\{\dfrac{I_0(k_2b)}{b} - I_1(k_2b) - \dfrac{I_0(k_2a)}{K_0(k_2a)}\left(\dfrac{2K_1(k_2b)}{b^2} + \dfrac{K_0(k_2b)}{b} + K_1(k_2b)\right)\right\}}\right]}{\left[-\dfrac{I_0(k_2a)}{K_0(k_2a)}K_0(k_2b) + I_0(k_2b) + \dfrac{\left\{I_1(k_2b) + \dfrac{I_0(k_2a)}{K_0(k_2a)}K_1(k_2b)\right\}^2}{\left\{\dfrac{I_1(k_2b)}{b} - I_0(k_2b) + \dfrac{I_0(k_2a)}{K_0(k_2a)}\left(\dfrac{K_1(k_2b)}{b} - K_0(k_2b)\right)\right\}}\right]} \tag{14}
$$

the solution of which yields k_1 and hence k_2 can be obtained by using the relation

$$
k_2^2 = k_1^2 + k_0^2[1 - \epsilon(z)] \tag{15}
$$

which is obtained from eq. (2)

6. Fourier gap coefficient $C_n(\beta)$

Consider a uniformly corrugated structure (Fig. 1). Each cell of the corrugated medium may be considered as a short-circuited (at $\rho = a$) radial transmission line. The field components in the medium 2 ($a < \rho < b$) are given by

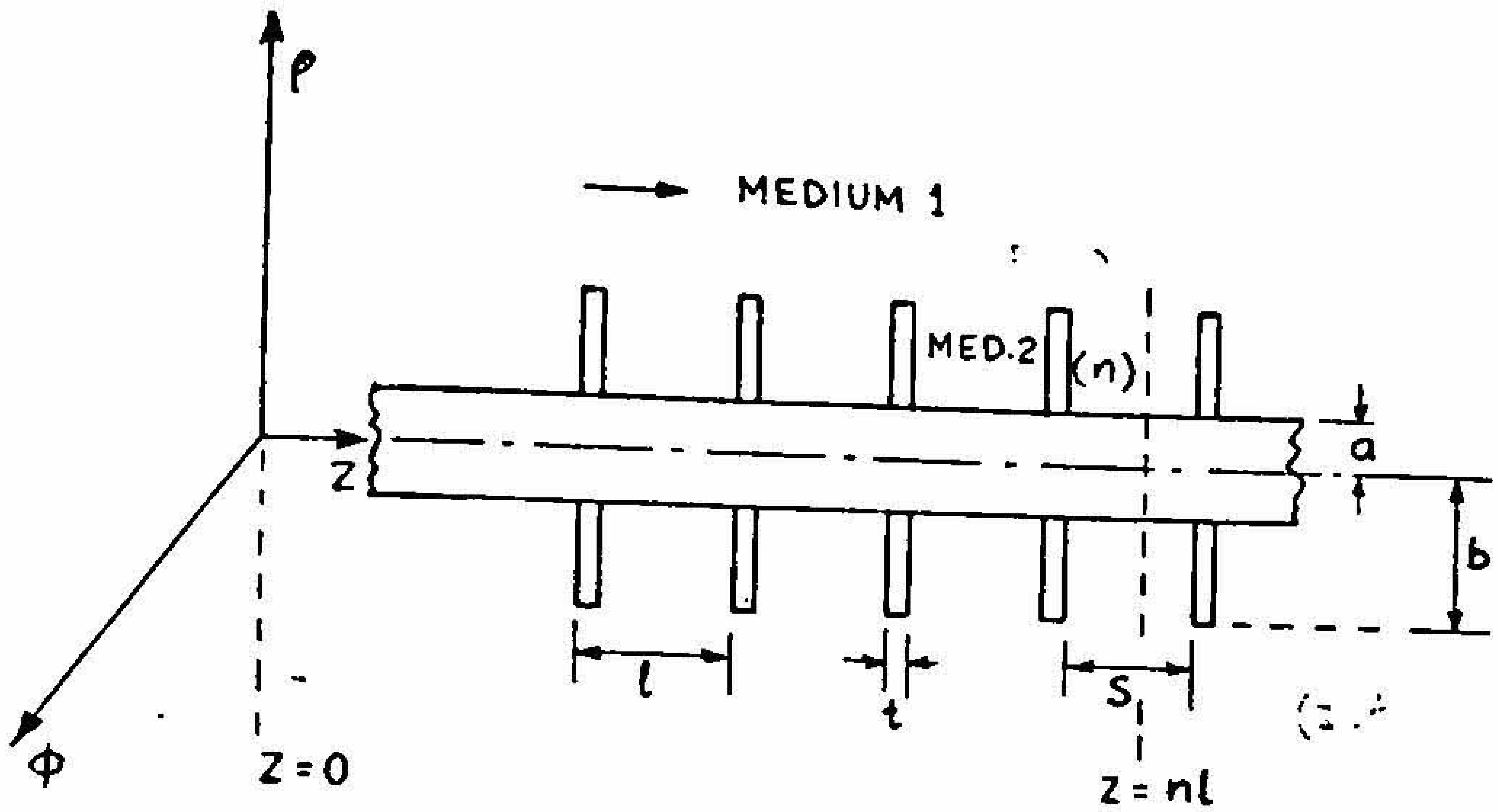

FIG. 1. Uniformly corrugated metal rod excited in E_{01}-mode.

$$
\begin{aligned}
E_{z2}^{um} &= -\mu_0 \frac{i\omega}{k_0}[C_1 J_0(k_0\rho) + C_2 Y_0(k_0\rho)] \,.\, \exp(-i\beta_0 nl) \\
&= C[F_0(k_0\rho)] \exp(-i\beta_0 nl) \\
H_{\phi 2}^{um} &= [C_1 J_1(k_0\rho) + C_2 Y_1(k_0\rho)] \cdot \exp(-i\beta_0 nl) \\
&= C' F_1(k_0\rho) \exp(-i\beta_0 nl)
\end{aligned} \tag{16}
$$

since at $\rho = a$, $E_{z2} = 0$ and hence

$$C_2 = -C_1 \frac{J_0(k_0 a)}{Y_0(k_0 a)}$$

and where

$$
\begin{aligned}
F_0(k_0\rho) &= J_0(k_0 a) Y_0(k_0\rho) - J_0(k_0\rho) Y_0(k_0 a) \\
F_1(k_0\rho) &= J_1(k_0\rho) Y_0(k_0 a) - J_0(k_0 a) Y_1(k_0\rho) \\
C &= \frac{C_1}{Y_0(k_0 a)} \frac{\mu_0 i\omega}{k_0}; \; C' = \frac{C_1}{Y_0(k_0 a)}
\end{aligned} \tag{16 a}
$$

The superscript *um* denotes field components in uniformly corrugated guide.

A wave travels in medium 1 in the vicinity of the corrugated medium such that the successive gap voltages are given by

$$V_m = V_0 \exp(-im\beta_0 l) \tag{17}$$

Assuming the field to be uniformly distributed in the gap at $\rho = b$, the boundary conditions at the mouth of the gap ($\rho = b$) are given by

$$E_{nz} \exp(-i\beta_0 z) = \begin{cases} \dfrac{-V_0}{s} \; 0 < z < s \\ o \; s < z < l \end{cases} \tag{18}$$

where $z = 0$ is at the beginning of the first cell. At $\rho = b$,

$$E_{nz2}(b, \phi) = A_n \; F_0(k_0 b)$$

$$= \frac{1}{l} \int_0^s \left(-\frac{V_0}{s}\right) \exp\ (i\beta_n z)\ dz$$

$$= -\frac{V_0}{sl} \frac{1}{i\beta_n} \left[\exp\ (i\beta_n s) - 1\right]$$

Therefore,

$$E_{z2}(\rho, \phi, z) = -\frac{V_0}{l} \sum_n \frac{\exp\ (i\beta_n s) - 1}{i\beta_n s} \frac{F_0(k_0\rho)}{F_0(k_0 b)} \exp\ (-i\beta_n z)$$

$$= -\frac{V_0}{l} \sum_n \left[\frac{F_0(k_0\rho)}{F_0(k_0 b)} \frac{\sin\ \beta_n s/2}{\beta_n s/2}\right] . \exp\ (i\beta_n s/2) \exp\ (-i\beta_n z) \qquad (19)$$

Since

$$\frac{\exp\ (i\beta_n s/2) - 1}{i\beta_n s} = \frac{\sin\ (\beta_n s/2)}{\beta_n s/2} \exp\ (i\beta_n s/2)$$

the Fourier gap coefficient C_n is given by

$$C_n = \frac{F_0(k_0\rho)}{F_0(k_0 b)} \frac{\sin\ (\beta_n s/2)}{\beta_n s/2} \qquad (20)$$

where $F_0(k_0\rho)/F_0(k_0 b)$ indicates the nature of variation of the field in the radial direction. It may be remarked that the assumption of uniform field distribution at the mouth of the gap introduces a very small error as long as $s < \lambda_0$

For a particular value of ρ, the gap coefficient

$$C_n \propto \frac{\sin(B_n s/2)}{\beta_n s/2} \qquad (21)$$

i.e., $f(\beta_n s/2)$

A plot of $f(\beta_n\, s/2)$ vs $\beta_n\, s/2$ shows that the function $f(\beta_n\, s/2)$, *i.e.*, the gap coefficient C_n is a very slowly varying function of the spacing s between discs. Hence, in the case of nonuniform spacing between discs which changes ϵ^0 to $\epsilon(z)$, the gap coefficient C_n may be considered to be the same as given by (20) for the case of uniform spacing without introducing appreciable error provided the modulation index δ which is always < 1 is maintained at a small value.

7. Power carried by surface waves

The power carried by E_0 wave inside (P_z^i) and outside (P_z^o) the modulated medium are given by

Med 1 :

$$P_z^o = \frac{1}{2}\,\text{Re} \int_{0}^{2\pi} \int_{\rho=b}^{\infty} E_{\rho 1} H_{\phi}^{*} \; \rho d\rho \, d\phi$$

$$= -\,\text{Re}\left[\frac{4A_1^2}{\pi}\chi\chi' \int_{\rho=b}^{\infty} K_1^{\,2}(k_1\rho)\;\rho d\rho\right]$$

$$= -\frac{4A_1^2}{\pi\omega\epsilon_0\,\epsilon(z)}\;\frac{b^2}{2}\left[\frac{K_0^2(k_1 b)}{k_1^2} + \frac{2}{k_1^2 b} K_1(k_1 b)\,K_0(k_1 b) - K_1^2(k_1 b)\right]\cdot f_n(\beta) \tag{22}$$

Med 2 :

$$P_z^i = \frac{1}{2}\,\text{Re} \int_{\phi=0}^{2\pi} \int_{\rho=a}^{b} E_{\rho 2} H_{\phi 2}^{*} \; \rho d\rho d\phi$$

$$= \pi\,\text{Re} \int_{\rho=a}^{b} \chi\chi' \, A_2^2\left[I_1(k_2\rho) + \frac{I_0(k_2 a)}{K_0(k_2 a)} K_1(k_2\rho)\right]^2 \rho d\rho$$

$$= -\frac{4A_1^2}{\pi\omega\epsilon_0\epsilon(z)}\cdot\frac{K_1^2(k_1 b)}{\left[I_1(k_2 b) + \dfrac{I_0(k_2 a)}{K_0(k_2 a)} K_1(k_2 b)\right]^2}$$

$$\times\Bigg[\left\{\frac{1}{2}a^2\left(\frac{I_0^2(k_2 a)}{a^2} - I_1^2(k_2 a) - \frac{2I_0(k_2 a)\,I_1(k_2 a)}{k_2 a^2}\right)\right.$$

$$\left. -\frac{1}{2}b^2\left(\frac{I_0^2(k_2 b)}{b^2} - I_1^2(k_2 b) - \frac{2I_0(k_2 b)\,I_1(k_2 b)}{k_2 b^2}\right)\right\}$$

$$+\frac{I_0^2(k_2 a)}{K_0^2(k_2 a)}\left\{\frac{1}{2}b^2\left(\frac{K_0^2(k_2 b)}{b^2} - K_1^2(k_2 b) + \frac{2K_1(k_2 b)\,K_0(k_2 b)}{k_2 b^2}\right)\right.$$

$$\left.-\frac{1}{2}a^2\left(\frac{K_0^2(k_2 a)}{a^2} - K_1^2(k_2 a) + \frac{2K_1(k_2 a)\,K_0(k_2 a)}{k_2 a^2}\right)\right\}$$

$$+\frac{I_0(k_2 a)}{K_0(k_2 a)}\cdot\frac{(b-a)}{k_2}\Bigg]\cdot f_n(\beta) \tag{23}$$

where,

$$f_n(\beta) = \Big\{\mp 2\sum_n C_n(\beta)\cos\theta \sum_n C_n(\beta)\sin\theta\cdot\frac{\pi\delta}{L}\sin\frac{2\pi z}{L}$$

$$\mp\left(1-\delta\cos\frac{2\pi z}{L}\right)\left(z\frac{d\beta}{dz}+\beta\right)\frac{\pi}{L}\Big[\sum_n C_n(\beta)\cdot\cos\cdot\theta\sum_n C_n(\beta)\cos\theta$$

$$-\sum_n C_n(\beta)\sin\theta\sum_n C_n(\beta)\sin\theta\Big]$$

$$\mp\left(1-\delta\cos\frac{2\pi z}{L}\right)\Big[\sum_n C_n(\beta)\cos\theta\sum_n\frac{dC_n(\beta)}{dz}\sin\theta$$

$$+ \sum_n C_n(\beta) \sin \theta \sum_n \frac{dC_n(\beta)}{dz} \cos \theta \Big]$$

$$\mp \left(1 - \delta \cos \frac{2\pi z}{L}\right)\Big[\sum_n C_n(\beta) \cos \theta \sum_n C_n(\beta) \frac{2n\pi}{L} \cos \theta$$

$$\sum_n C_n(\beta) \sin \theta \sum_n C_n(\beta) \frac{2n\pi}{L} \sin \theta \Big]\Big\} \tag{24}$$

The power P_z^0 carried by surface waves outside the corrugated medium with respect to the total power $P_z^T = P_z^i + P_z^0$ can be studied as a function of $k_1 b$ or $k_2 b$. Since k_1 and k_2 involve the modulation index factor δ, the percentage of power flow $P_z^0/P_z^T\%$ can be studied as $f(\delta)$ and compared with that in the case of a uniformly corrugated ($\delta = 0$) guide.

8. Relative magnitudes of Floquet harmonics

The magnitudes of space harmonics depend on $f_n(\beta)$. The relative absolute values

$$\left| \underset{n \neq 0}{f_n(\beta)} \Big/ \underset{n = 0}{f_0(\beta)} \right|$$

indicates the magnitude of power contained in the spatial harmonics ($n = \pm 1, \ \pm 2, \ \pm 3, \ldots$) compared to that contained in the fundamental ($n = 0$).

Since $\delta < 1$, β_n in a slowly varying function of z. Hence, C_n does not vary significantly with z over a period L, the order of smallness of variation may be estimated as (WKBJ approximation)

$$\frac{dC_n}{d\beta} \sim \frac{C_n}{L}, \qquad \frac{d^2 C_n}{d\beta^2} = \frac{C_n}{L^2}$$

where,

$$\frac{d^2 C_n}{d\beta^2} \ll \frac{dC_n}{d\beta} \tag{25}$$

Hence, $f_n(\beta)$ in (24) can be simplified. The relative absolute values of harmonics with respect to the fundamental are given by

$$\left| \frac{f_{-1}(\beta)}{f_0(\beta)} \right| = \frac{\begin{aligned}&\Big\{ C_{-1}^2(\beta) \Big[\frac{\pi\delta}{L} \sin \frac{2\pi z}{L} \sin 2\theta_- + \frac{\pi}{L}\left(1 - \delta \cos \frac{2\pi z}{L}\right) \\ &\times \left(z \frac{d\beta}{dz} + \beta\right) \cos 2\theta_- + \left(1 - \delta \cos \frac{2\pi z}{L}\right) \sin 2\theta_- - \frac{2\pi}{L} \\ &\times \left(1 - \delta \cos \frac{2\pi z}{L}\right) \cos 2\theta_- \Big]\Big\}\end{aligned}}{\begin{aligned}&\Big\{ C_0^2(\beta) \Big[\frac{\pi\delta}{L} \sin \frac{2\pi z}{L} \sin \frac{2\pi\beta z}{L} + \frac{\pi}{L}\left(1 - \delta \cos \frac{2\pi z}{L}\right) \\ &\times \left(z \frac{d\beta}{dz} + \beta\right) \cos \frac{2\pi\beta z}{L} + \left(1 - \delta \cos \frac{2\pi z}{L}\right) \sin \frac{2\pi\beta z}{L} \Big]\Big\}\end{aligned}}$$

$$\left|\frac{f_{+1}(\beta)}{f_0(\beta)}\right| = \frac{\left\{ C_{+1}^2(\beta)\left[\frac{\pi\delta}{L}\sin\frac{2\pi z}{L}\sin 2\theta_+ + \frac{\pi}{L}\left(1-\delta\cos\frac{2\pi z}{L}\right)\times\left(z\frac{d\beta}{dz}+\beta\right)\cos 2\theta_+ + \left(1-\delta\cos\frac{2\pi z}{L}\right)\sin 2\theta_+ + \frac{2\pi}{L}\times\left(1-\delta\cos\frac{2\pi z}{L}\right)\cos 2\theta_+\right]\right\}}{\left\{ C_0^2(\beta)\left[\frac{\pi\delta}{L}\sin\frac{2\pi z}{L}\sin\frac{2\pi\beta z}{L} + \frac{\pi}{L}\left(1-\delta\cos\frac{2\pi z}{L}\right)\times\left(z\frac{d\beta}{dz}+\beta\right)\cos\frac{2\pi\beta z}{L} + \left(1-\delta\cos\frac{2\pi z}{L}\right)\sin\frac{2\pi\beta z}{L}\right]\right\}}$$

etc. (26)

where,

$$\theta_- = \frac{\pi\beta z}{L} - \frac{2\pi z}{L}$$

associated with backward harmonics

$$\theta_+ = \frac{\pi\beta z}{L} + \frac{2\pi z}{L}$$

associated with forward harmonics

$$\frac{C_{-1}}{C_0} = \frac{\beta_0 \sin\left(\frac{\beta_0 s}{2} - \frac{\pi s}{l}\right)}{\left(\beta_0 - \frac{2\pi}{l}\right)\sin(\beta_0 s/2)} \tag{27}$$

$$\frac{C_{+1}}{C_0} = \frac{\beta_0 \sin\left(\frac{\beta_0 s}{2} + \frac{\pi s}{l}\right)}{\left(\beta_0 + \frac{2\pi}{l}\right)\sin(\beta_0 s/2)}$$

etc., since

$$\beta_n = \beta_0 + \frac{2\pi n}{l}, \quad (n = 0, \ \pm 1, \ \pm 2 \ldots)$$

which plays the role of wave numbers denoting the phase progressing along the z-direction. The factor β_0 correspond to the fundamental value of the unmodulated structure having s and b values corresponding to ε^0 used in computing $\varepsilon(z)$ whereas β associated with θ_- and θ_+ are the values corresponding to the modulated structure.

9. Power lost by surface waves

The power lost in the modulated medium (med 2) is

$$P_L = \frac{\eta}{2} \int\int_s H_{\phi 2} H_{\phi 2}^* \, ds \tag{28}$$

Since at $\rho = b,\ H_{\phi 1} = H_{\phi 2}$,

therefore

$$\frac{A_2}{A_1} = -i_2 \frac{K_0(k_1 b)}{I_0(k_2 b) - \frac{I_0(k_2 a)}{K_0(k_2 a)} K_0(k_2 b)} \tag{29}$$

The magnetic field component in the second medium

$$a \leqslant \rho \leqslant b$$

is therefore

$$H_{\phi 2} = 2i\chi A_1 \frac{K_0(k_1 b)\ [I_0(k_2 b) K_0(k_2 a) - I_0(k_2 a) K_0(k_2 b)]}{[I_0(k_2 b) K_0(k_2 a) - I_0(k_2 a) K_0(k_2 b)]} \tag{30}$$

The intrinsic impedance of the second medium is given by

$$\eta_2 = \left[\frac{i\omega\mu}{\sigma + j\omega\epsilon}\right]^{1/2} = 376{\cdot}7 \left[\epsilon_r + \frac{\sigma}{i\omega\epsilon_0}\right]^{1/2}$$

which for a low loss dielectric (assuming the second medium to be of low loss) $\sigma \ll \omega\epsilon_0$ at microwave frequencies, reduces to

$$\eta_2 = 376{\cdot}7\ [\epsilon^0 (1 - \delta)]^{-1/2} \tag{31}$$

since $\epsilon_r + \sigma/i\omega\epsilon_0 \simeq \epsilon_r$ which for a modulated medium may be written as $\epsilon_0 (1 - \delta)$.

Hence substituting (30) and (31) in (28), the power lost in the modulated corrugated medium is obtained as follows

$$P_{L2} = \eta_2 \int_a^b \int_0^{2\pi} H_{\phi 2} H^*_{\phi 2}\ \rho\, d\rho\, d\phi$$

$$= -4A_1^2 \chi\chi^* \left\{\frac{K_0(k_1 b)}{I_0(k_2 b) K_0(k_2 a) - I_0(k_2 a) K_0(k_2 b)}\right\}^2$$

$$\times \Bigg[K_0^2(k_2 a) \left\{ -\frac{1}{2} b^2 \left(\frac{I_0^2(k_2 b)}{b^2} - \frac{2I_0(k_2 b) I_1(k_2 b)}{k_2 b^2} - I_1^2(k_2 b)\right)\right.$$

$$\left. + \frac{1}{2} a^2 \left(\frac{I_0^2(k_2 a)}{a^2} - \frac{2I_0(k_2 a) I_1(k_2 a)}{k_2 a^2} - I_1^2(k_2 a)\right)\right\}$$

$$+ I_0^2(k_2 a) \left\{\frac{1}{2} b^2 \left(\frac{k_0^2(k_2 b)}{b^2} + \frac{2K_0(k_2 b) K_1(k_2 b)}{k_2 b^2} - K_1^2(k^2 b)\right)\right.$$

$$\left. - \frac{1}{2} a^2 \left(\frac{K_0^2(k_2 a)}{a^2} + \frac{2K_0(k_2 a) K_1(k_2 a)}{k_2 a^2} - K_1^2(k_2 a)\right)\right\}$$

$$+ \frac{(b-a)}{k_2} I_0(k_2 a) K_0(k_2 a)\Bigg] \times \frac{376{\cdot}7}{2} (\epsilon(z))^{-1/2} \tag{32}$$

here,

$$\chi\chi^* = \left(1 - \delta\cos\frac{2\pi z}{L}\right) \sum_n C_n(\beta) \sum_n C_n(\beta).$$

10. Attenuation Constant

The attenuation constant α is given by

$$\alpha = \frac{P_L}{2L\,P_z^T} \text{ nepers}/m \tag{33}$$

Since β is a slowly varying function of z over a period L, we can approximate $d\beta | dz \sim \beta/L$. Computing for the fundamental ($n = 0$) at $z = L$, the expression for attenuation (33) becomes

$$\alpha = \frac{\left[188\,\omega\,\epsilon_0\sqrt{\varepsilon^0}(1-\delta)\cdot\left(\frac{K_0(k_2b)}{I_0(k_2b)K_0(k_2a) - I_0(k_2a)K_0(k_2b)}\right)^2 \times\left\{K_0^2(k_2a)\,.\,f(k_2a,k_2b) + I_0^2(k_2a)\,g(k_2a,k_2b) + \frac{(b-a)}{k_2}I_0(k_2a)K_0(k_2a)\right\}\right]}{\left[\frac{1}{2}b^2\left(\frac{K_0^2(k_1b)}{k_1^2} + \frac{2}{k_1^2 b}K_1(k_1b)K_0(k_1b) - K_1^2(k_1b)\right) + \frac{K_1^2(k_1b)}{\left\{I_1(k_2b) + \frac{I_0(k_2a)}{K_0(k_2a)}K_1(k_2b)\right\}^2} \times\left\{f(k_2a,\ k_2b) + \frac{I_0^2(k_2a)}{K_0^2(k_2a)}\cdot g(k_2a,\ k_2b) + \frac{(b-a)}{k_2} \times\frac{I_0(k_2a)}{K_0(k_2a)}\right\}\cdot 4\beta\cos\ 2\pi\beta\right]}$$

where,

$$f(k_2a,\ k_2b) = \left[\frac{1}{2}a^2\left(\frac{I_0^2(k_2a)}{a^2} - \frac{2I_0(k_2a)I_1(k_2a)}{k_2a^2} - I_1^2(k_2a)\right) - \frac{1}{2}b^2\left(\frac{I_0^2(k_2b)}{b^2} - \frac{2I_0(k_2b)I_1(k_2b)}{k_2b^2} - I_1^2(k_2b)\right)\right]$$

$$g(k_2a, k_2b) = \left[\frac{1}{2}b^2\left(\frac{K_0^2(k_2b)}{b^2} + \frac{2K_0(k_2b)K_1(k_2b)}{k_2b^2} - K_1^2(k_2b)\right) - \frac{1}{2}a^2\left(\frac{K_0^2(k_2a)}{a^2} + \frac{2K_0(k_2a)K_1(k_2a)}{k_2a^2} - K_1^2(k_2a)\right)\right]. \tag{34}$$

11. Surface impedance

In dealing with surface waves it is often convenient to specify the properties of guiding structure in terms of surface impedance $Z_s = R_s + i^{"s}$ since then the behavi of the surface wave in the surrounding medium can be discussed without the neces of knowing the constitution of the guiding structure. The surface impedance is evalua at the surface which forms the interface between the structure and the surround medium and is defined by $Z_s = \left(\frac{E_{z1}}{H_{\varphi 1}}\right)$ at $\rho = b$ which yields

$$Z_s = -\frac{1}{\omega\epsilon_0\epsilon(z)}\frac{K_0(k_1 b)}{K_1(k_1 b)}$$

$$= \frac{i}{\omega\epsilon_0\epsilon(z)}\frac{H_0^{(1)}(ik_1 b)}{H_1^{(1)}(ik_1 b)} \tag{35}$$

Since in the case of axial cylindrical wave the small argument approximation $|ik_1 b|$ $\ll 1$ holds good,

$$H_0^{(1)}(ik_1 b) = J_0(q) + iY_0(q)$$

$$\approx 1 + i\frac{2}{\pi}(ln\, q/2 + 0\cdot577)$$

$$\approx 1 + i\frac{2}{\pi}\, ln\ 0\cdot 89q$$

$$= i\frac{2}{\pi}\, ln\ 0.89\ k_1 b$$

$$H_1^{(1)}(ik_1 b) = J_1(q) + iY_1(q)$$

$$\approx \frac{q}{2} - i\frac{2}{\pi q} \approx -i\frac{2}{\pi q}$$

where

$$q = ik_1 b$$

Hence

$$Z_s = \frac{k_1 b}{\omega\epsilon_0\epsilon(z)}\, ln\ 0\cdot 89\ k_1 b$$

$$= \frac{(a_1 - ib_1)\, b}{\omega\epsilon_0\epsilon(z)}\,[ln\ 0\cdot 89\, b + ln\ (a_1 - ib_1)]$$

$$= \frac{b}{\omega\epsilon_0\epsilon(z)}\,[a_1\, ln\, 0\cdot 89b\ \sqrt{a_1^2 + b_1^2} - b_1 \tan^{-1} b_1/a_1$$

$$- ia_1 \tan^{-1} b_1/a_1 - ib_1\ ln\, 0\cdot 89b\sqrt{a_1^2 + b_1^2}]$$

Therefore,

$$R_s = \frac{b}{\omega\epsilon_0\epsilon(z)}\left[a_1 \ln 0{\cdot}89b\sqrt{a_1^2+b_1^2} - b_1 \tan^{-1}\frac{b_1}{a_1}\right]$$

$$X_s = -\frac{b}{\omega\epsilon_0\epsilon(z)}\left[a_1 \tan^{-1}\frac{b_1}{a_1} + b_1 \ln 0{\cdot}89b\sqrt{a_1^2+b_1^2}\right]$$

which can be reduced to the following dimensionless form

$$\frac{2\pi b}{\lambda_0}\frac{R_s}{Z_0} = \frac{1}{\epsilon(z)}\left[a_1 b \ln 0{\cdot}89\sqrt{a_1^2b^2+b_1^2b^2} - b_1 b \tan^{-1}\frac{b_1 b}{a_1 b}\right] \tag{36}$$

$$\frac{2\pi b}{\lambda_0}\frac{X_s}{Z_0} = -\frac{1}{\epsilon(z)}\left[a_1 b \tan^{-1}\frac{b_1 b}{a_1 b} + b_1 b \ln 0{\cdot}89\sqrt{a_1^2b^2+b_1^2b^2}\right] \tag{37}$$

since

$$\lambda_0 f = (\mu_0\epsilon_0)^{-1/2} = (\epsilon_0 Z_0)^{-1}$$

12. Determination of a_1 and b_1

Since

$$k_1 = a_1 - ib_1$$

$$k_1^2 = -k_0^2 - \gamma^2,\ \gamma = i\beta$$

$$a_1^2 - b_1^2 + k_0^2 = \beta^2 - \alpha^2$$

and

$$a_1 b_1 = \alpha\beta$$

Therefore

$$a_1^4 + (\alpha^2 - \beta^2)\,a_1^2 + a_1^2 k_0^2 - \alpha^2\beta^2 = 0$$

which yields

$$a_1 = [\tfrac{1}{2}\{(\beta^2 - \alpha^2 - k_0^2) \pm \sqrt{(\alpha^2-\beta^2+k_0^2)^2 + 4\alpha^2\beta^2}\}]^{1/2}$$

$$b_1 = \frac{\alpha\beta}{a_1} \tag{38}$$

Hence the nature of the boundary surface can be determined from (36) and (37) and thus the rate of decay of the field outside the surface from a knowledge of the surface reactance can be determined. Since the resistance is associated with the attenuation of the surface wave, it is possible to determine the surfaces for which the attenuation is low.

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