THE INTERACTION OF AN INHOMOGENEITY WITH A CONCENTRATED FORCE IN COUPLE STRESS THEORY

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ABSTRACT

Using complex variable methods, the problem of interaction between an inhomogeneity and a concentrated force in two dimensional linear couple stress theory has been studied in this paper. The concentrated force could be situated in the matrix or in the inhomogeneity. Edge dislocation type singularities can also be considered. The effect of a concentrated force on a circular inhomogeneity in an infinite medium has been discussed in detail. Stresses could be bounded at infinity. Numerical results are in conformity with the fact that the effect of couple stresses is negligible when the ratio of the smallest dimension of the body to the characteristic length is large.

Key Words: Interaction; Inhomogeneity; Couple stress Theory.

INTRODUCTION

The problem of two-dimensional circular inhomogeneity in an infinite region with uniaxial tension at infinity and with couple stresses accounted by Mindlin's couple stress theory [1, 2] was solved by Weitsman [3] and Hartranft and Sih [4]. The size of the inserted material in [3] and [4] is the same as that of the cavity in the infinite region. The solutions in [2, 3, 4] depend on the choice of some suitable functions and this does not seem to be a systematic approach towards other inhomogeneity problems. Huigol [5] solved the two-dimensional problem of a concentrated force in an infinite medium using complex variable formulation developed by Mindlin [6] and Muskhelishvili [7]. In the present paper complex variable methods have been employed to study the problem of interaction between an inhomogeneity and a concentrated force (or edge dislocation with Burger's vector). The size of the inhomogeneity could be different from the size of the cavity and the stresses could be bounded at infinity.

When Mindlin's [2] two-dimensional linear couple stress theory is considered, the basic equations to be solved are

$$\nabla^4 U = 0, \tag{1}$$

$$\nabla^2 (V - l^2 \nabla^2 V) = 0. \tag{2}$$

The solutions of (1) can be expressed in terms of two analytic functions $\phi(z)$ and $\chi(z)$ [7].

$$2U = \bar{z}\varphi(z) + z\overline{\phi(z)} + \chi(z) + \overline{\chi(z)}. \tag{3}$$

V and U are not independent and satisfy the relation

$$V - l^2 \nabla^2 V = 8(1-\nu) l^2 \operatorname{Im}\{\Phi(z)\} \tag{4}$$

where $\varphi'(z) = \Phi(z)$ and Im stands for the imaginary part of a complex quantity.

The solutions of (2) are not available in terms of analytic functions.

Although the theory developed below is applicable even if there are more than one concentrated forces and more than one inhomogeneities, the results in this paper are given for the case when only one concentrated force is applied in the presence of one inhomogeneity.

1. Consider a two-dimensional isotropic infinite elastic medium with a cavity in a state of plane strain. The boundary of the cavity will be denoted by L. This infinite region is called matrix. Let a concentrated force $X + iY$ be applied at an interior point $z = z_0$ ($\bar{z} = x + iy$) of the matrix. If an elastic body of dimensions slightly larger than those of the cavity but remaining within the limits of proportional elasticity is embedded in the matrix then because of the misfit in size stresses would develop everywhere. This embedded material is called inhomogeneity if the elastic constants of matrix and embedded material are different and inclusion if their elastic constants are the same.

Let the inhomogeneity in the absence of matrix undergo a prescribed deformation $(\epsilon_2 x, \ \epsilon y)$ which in the presence of matrix will attain a different equilibrium configuration. If body forces and body couples are absent but couple stresses are taken into account then the following conditions should hold at the equilibrium boundary L.

$$u^+ - u^- = \epsilon_1 x = g_1(t); \ V^+ - V^- = \epsilon_2 y = g_2(t) \tag{5.1}$$

$$\tau^+_{rr} + i\tau^+_{r\theta} = \tau^-_{rr} + i\tau^-_{r\theta} \tag{5.2}$$

$$\mu_r{}^+ = \mu_r{}^- \tag{5.3}$$

$$\omega^+_{r\theta} = \omega^-_{r\theta} \tag{5.4}$$

where t is a point on the boundary L; the superscripts $+$ and $-$ stand for the matrix and inhomogeneity respectively, u and v are displacement components in Cartesian coordinates, τ_{rr}, $\tau_{r\theta}$, etc., are the components of the asymmetric Cosserat stress tensor in polar coordinates, μ_r is the component of the Cosserat couple-stress tensor in polar coordinates and $\omega_{r\theta}$ is the component of rotation produced by the anti-symmetric part of the shear stresses.

The components of Cosserat stress tensor, displacements and rotation may be expressed in terms of analytic functions $\varphi(z)$ and $\psi(z) = \chi'(z)$ and the real valued function $V(z, \bar{z})$ [5].

The boundary conditions (5.1)–(5.4) when rewritten in terms of $\phi(z)$, $\psi(z)$ and $V(z, \bar{z})$ become

$$k_2 G_1 \varphi^+(t) - G_1 t \overline{\phi'^+(t)} - G_1 \overline{\psi^+(t)} + 2i\, G_1 \frac{\partial V^+}{\partial \bar{t}}$$

$$= k_1 G_2 \varphi^-(t) - G_2 t\, \overline{\varphi'^-(t)} - G_2 \overline{\psi^-(t)} + 2i\, G_2 \frac{\partial V^-}{\partial \bar{t}}$$

$$+ 2G_1 G_2 \{g_1(t) + i g_2(t)\} \tag{6.1}$$

$$\varphi^+(t) + t\, \overline{\phi'^+(t)} + \overline{\psi^+(t)} - 2i \frac{\partial V^+}{\partial \bar{t}}$$

$$= \varphi^-(t) + t\, \overline{\varphi'^-(t)} + \overline{\psi^-(t)} - 2i \frac{\partial V^-}{\partial \bar{t}} \tag{6.2}$$

$$\mathrm{Re}\left(e^{i\theta} \frac{\partial V^+}{\partial t}\right) = \mathrm{Re}\left(e^{i\theta} \frac{\partial V^-}{\partial t}\right) \tag{6.3}$$

$$l_1^2\, G_1\, V^+ = l_2^2\, G_2\, V^-. \tag{6.4}$$

$\phi^+(t)$, $\varphi^-(t)$, etc., are the boundary values of the functions $\phi(z)$, etc., from the right and from the left respectively as the boundary L is traversed in the anti-clockwise direction. The elastic constants and characteristic lengths of inhomogeneity and matrix are denoted by the subscripts 1 and 2 respectively; $k = 3 - 4\nu$, ν being Poisson ratio, G is the shear modulus of elasticity and l denotes the characteristic length. Re stands for the real part of a complex quantity.

$\phi(z)$ and $\psi(z)$ are to be determined from (6.1) and (6.2). If the elastic constants of matrix and inhomogeneity are entirely different and the boundary L is any general boundary then there does not seem to be any systematic way of determining $\varphi(z)$ and $\psi(z)$ from (6.1) and (6.2). However, if it is assumed that the Poisson ratios of matrix and inhomogeneity are different but their shear moduli are the same then $\phi(z)$ and $\psi(z)$ can be determined from the following Hilbert problems which can be easily derived from (6.1) and (6.2).

$$\phi^+(t) - \frac{1+k_1}{1+k_2}\varphi^-(t) = \frac{G(\epsilon_1+\epsilon_2)}{(1+k_2)}\, t + \frac{G(\epsilon_1-\epsilon_2)}{(1+k_2)}\,\bar{t} \text{ on } L \qquad (7)$$

$$\psi^+(t) - \psi^-(t) = \overline{\phi^-(t)} - \overline{\varphi^+(t)} + \bar{t}\left(\phi'^-(t) - \varphi'^+(t)\right) + 2i\frac{\partial V^-}{\partial \bar{t}} - 2i\frac{\partial V^+}{\partial \bar{t}} \text{ on } L. \qquad (8)$$

Assuming zero stresses at infinity, the solution of (7) is given by

$$\varphi^+(z) = -\frac{G(\epsilon_1-\epsilon_2)}{2\pi i(1+k_2)}\int_L \frac{\bar{t}\,dt}{t-z} - \frac{C}{z-z_0}, \; z\epsilon \text{ matrix} \qquad (9)$$

$$\phi^-(z) = -\frac{G(\epsilon_1+\epsilon_2)}{(1+k_2)}z - \frac{G(\epsilon_1-\epsilon_2)}{2\pi i(1+k_2)}\int_L \frac{\bar{t}\,dt}{t-z} - \frac{(1+k_2)}{(1+k_1)}\frac{C}{(z-z_0)}, \; z\epsilon \text{ inhomogeneity} \qquad (10)$$

where

$$C = (X+iY)/2\pi(1+k_2).$$

The solution of (8) is given by

$$\psi(z) = \frac{1}{2\pi i}\int_L \frac{(\overline{\phi^+(t)} - \overline{\varphi^-(t)})}{t-z}dt + \frac{1}{2\pi i}\int_L \frac{t\left(\phi'^+ - \phi'^-(t)\,dt\right)}{t-z} + \frac{1}{\pi}\int \frac{\left(\frac{\overline{\partial V^+}}{\partial \bar{t}} - \frac{\overline{\partial V^-}}{d\bar{t}}\right)dt}{t-z} + \frac{D}{z-z_0} + \frac{8(1-\nu_2)\,l_2^2\,C}{(z-z_0)^3} - \bar{z}_0\frac{C}{(z-{}_0)^2} \qquad (11)$$

where

$$D = k_2\,\bar{C}.$$

If the concentrated force is situated at an interior point $z = z_0$ in the inhomogeneity then appropriate changes in the elastic constants, characteristic lengths, etc., are to be made in (9), (10) and (11).

For an edge dislocation in the matrix with Burger's vector $(F_x, 0, 0)$ $C = D = iG_2 F_x/\pi(1+k_2)$ and for an edge dislocation with Burger's vector $(0, F_y, 0)$, $C = D = G_2F_y/\pi(1+k_2)$.

Concentrated force introduces singularity in $V(z, \bar{z})$. The solutions of (2) are to be suitably modified to account for this singularity. Let

$$V^+(z, \bar{z}) = A_1/(z - z_0) + \bar{A}_1/(\bar{z} - \bar{z}_0) + \sum_{n=0}^{\infty} (\bar{b}_n z^{-n} + \bar{b}_n \bar{z}^{-n}) + V_0^+(z, \bar{z}) \tag{12}$$

$$V^-(z, \bar{z}) = A_2/(z - z_0) + \bar{A}_2/(\bar{z} - \bar{z}_0) + \sum_{n=0}^{\infty} (a_n z^n + \bar{a}_n \bar{z}^n) + V_0^-(z, \bar{z}). \tag{13}$$

$V_0^+(z, \bar{z})$ and $V_0^-(z, \bar{z})$ are the solutions of equation

$$V - l^2 \nabla^2 V = 0 \tag{14}$$

in appropriate regions and depend upon the equation of the contour L. The constants A_1 and A_2 can be guessed easily and

$$A_1 = 4i(1 - \nu_2) l_2^2 C, \qquad A_2 = 4i(1 - \nu_1) l_1^2 (1 + k_2) C/(1 + k_1). \tag{15}$$

The unknowns b_n, a_n and those involved in $V_0^+(z, \bar{z})$ and $V_0^-(z, \bar{z})$ are to be determined with the help of boundary conditions (6.3) and (6.4) and the condition (4).

2. We now consider the two-dimensional problem of circular inhomogeneity in an infinite medium in the presence of a concentrated force $X + iY$ (edge dislocation with Burger's Vectors can also be considered) acting at some interior point $z = z_0$ of the matrix. Because of all-round symmetry z_0 can be taken to be a real quantity. Let the equation of the contour L be denoted by $|z| = R$. Both the Poisson ratios and shear moduli of inhomogeneity and matrix are taken to be different and as before they will be denoted by the subscripts 1 and 2 for inhomogeneity and matrix respectively. The boundary conditions are given by (5.1)–(5.4).

Let us introduce a new function $\Omega(z)$ as follows ([7], Chapter 20)

$$\Omega(z) = \bar{\Phi}(R^2/z) - R^2 z^{-1} \bar{\Phi}'(R^2/z) - R^2 z^{-2} \bar{\psi}(R^2/z) \tag{16}$$

and so

$$\bar{\psi}(z) = R^2 z^{-2} \Phi(z) - R^2 z^{-2} \bar{\Omega}(R^2/z) - R^2 z^{-1} \Phi'(z) \tag{17}$$

where

$$\bar{\psi}(z) = \psi'(z)$$

For large $|z|$

$$\Phi(z) = \Gamma - C z^{-1} + 0(z^{-2}), \quad \bar{\psi}(z) = \Gamma' + k_2 \bar{C} z^{-1} + 0(z^{-2}) \tag{18}$$

$$\Omega(z) = \bar{\Phi}(0) + 0(z^{-2}). \tag{19}$$

Near $z = 0$

$$\Omega(z) = -\bar{\Gamma}' z^{-2} - k_2 C z^{-1} + a \text{ hoiomorphic function} \tag{20}$$

In terms of $\Omega(z)$ and $\Phi(z)$, boundary conditions (6.1) and (6.2) become

$$\{\Phi(t) - \Omega(t)\}^{+} = \{\Phi(t) - \Omega(t)\}^{-} + 2i \overline{\frac{\partial^2 V^+}{\partial t \partial \bar{t}}} - 2i \frac{\bar{t}}{t} \overline{\frac{\partial^2 V^+}{\partial t^2}}$$
$$- 2i \overline{\frac{\partial^2 V^-}{\partial t \partial \bar{t}}} + 2i \frac{\bar{t}}{t} \overline{\frac{\partial^2 V^-}{\partial t^2}} \quad \text{on } L \tag{21}$$

and

$$G_1 \{k_2 \Phi^+(t) - \Omega^-(t)\} - G_2 \{k_1 \Phi^-(t) - \Omega^+(t)\}$$
$$= G_1 G_2 (\epsilon_1 + \epsilon_2) - G_1 G_2 (\epsilon_1 - \epsilon_2) \bar{t}/t + 2i G_2$$
$$\times \left\{ \overline{\frac{\partial^2 V^-}{\partial t \partial \bar{t}}} - \frac{\bar{t}}{t} \overline{\frac{\partial^2 V^-}{\partial t^2}} \right\} - 2i G_1 \left\{ \overline{\frac{\partial^2 V^-}{\partial t \partial \bar{t}}} - \frac{\bar{t}}{t} \overline{\frac{\partial^2 V^+}{\partial t^2}} \right\} \text{ on } L. \tag{22}$$

In (22), the discontinuity in the derivatives (with respect to θ) of displacements has been considered in place of discontinuity in the displacements.

For a circular boundary L, $V(z, \bar{z})$ may be written as

$$V^+(z, \bar{z}) = A_1'/(z - z_0) + A_1'/(\bar{z} - z_0) + \sum_{n=0}^{\infty} (b_n z^{-n} \bar{b}_n + \bar{z}^{-n})$$
$$+ \sum_{n=1}^{\infty} K_n(r/l_2)(c_n \sin n\theta + d_n \cos n\theta) \tag{23}$$

$$V^-(z, \bar{z}) = A_2'/(z - z_0) + \bar{A}_2'/(\bar{z} - z_0) + \sum_{n=0}^{\infty} (a_n z^n + \bar{a}_n \bar{z}^n)$$
$$+ \sum_{n=1}^{\infty} I_n(r/l_1)\{c_n' \cos n\theta + d_n' \cos n\theta\}. \tag{24}$$

K_n and I_n are Bessel functions of second kind and order n; c_n, d_n, c_n' and d_n' are real constants to be determined together with a_n and b_n,

$$A_1' = 4i(1 - \nu_2) l_2^2 C, \quad A_2' = 4i(1 - \nu_1) G_1 l_1^2 (1 + k_2) C/$$
$$(G_1 + G_2 k_1).$$

Using (23) and (24), the solution of Hilbert problem in (21) may be written as

$$\Phi(z) - \Omega(z) = h(z) + \sum_{n=2}^{\infty} \{(1-n) R^{n-2} m_2 K_{n-1}(m_2)(c_n - i d_n) z^{-n}/2$$
$$+ (1-n) R^{n-2} m_1 I_{n-1}(m_1)(c_n' - i d_n') z^{-n}/2\}$$
$$+ 2i \sum_{n=2}^{\infty} n(n-1) \bar{a}_n R^{2n-2} z^{-n} + 4i R_0 z_0^{-2} (\bar{A}_1 - \bar{A}_2) z/$$
$$(R_0 - z)^3, \ |z| > R \tag{26}$$

and

$$\Phi(z) - \Omega(z) = h(z) + 2i \sum_{n=1}^{\infty} n(1+n) R^{-2n-2} b_n z^n$$
$$+ \sum_{n=1}^{\infty} (1+n) R^{-n-2} z^n [\{m_2 K_{n-1}(m_2) + 2n K_n(m_2)\}(c_n + i d_n)$$
$$+ \{m_1 I_{n-1}(m_1) - 2n I_n(m_1)\}(c_n' + i d_n')]/2, \ |z| < R. \tag{27}$$

where

$$m_1 = R/l_1, \ m_2 = R/l_2, \ R_0 = R^2/z_0,$$
$$h(z) = -\frac{\bar{C}}{z - z_0} + \frac{R^2 \bar{C}(R_0 - z_0)}{z_0^2 (z - R_0)^2} - \frac{k_2 \bar{C}}{z - R_0} + \frac{k_2 \bar{C}}{z}$$
$$- \frac{8R_0(1-\nu_2) l_2^2}{z_0^2} \frac{\bar{C} z}{(z - R_0)^3} - \bar{\Phi}(0). \tag{28}$$

$\Omega(t)$ can be eliminated from (22) with the help of (26) and (27) and the resulting Hilbert problem when solved for $\Phi(z)$ gives

$$\Phi(z) = -\alpha_1 \cdot \Big[\frac{R_0 \bar{C}(z_0 - R_0)}{z_0 (R_0 - z)^2} + \frac{R_0 k_2 C}{z(z - R_0)}$$
$$+ \frac{4R_0(1-\nu_2) l_2^2 \bar{C}}{z_0^2} \times \frac{z}{(z - R_0)^3}\Big] - \frac{C}{z - z_0}$$
$$- \frac{i\alpha_1}{2} \sum_{n=2}^{\infty} (1-n) R^{n-2} m_2 K_{n-1}(m_2)(d_n + i c_n) z^{-n}$$
$$- G_1 G_2 (\epsilon_1 - \epsilon_2) R^2/(G_2 + G_1 k_2) z^2, \qquad |z| > R \tag{29}$$

where

$$\alpha_1 = (G_2 - G_1)/(G_2 + G_1 k_2)$$
$$\Phi(z) = -G_1(1 + k_2) C/(G_1 + G_2 k_1)(z - z_0) + \alpha_2 \bar{\bar{\Phi}}(0)$$
$$- G_1 G_2 (\epsilon_1 + \epsilon_2)/(G_1 + G_2 k_1) - i\alpha_2 \sum_{n=1}^{\infty} (1+n) R^{-n-2} z^n$$
$$\times \{m_1 I_{n-1}(m_1) - 2n I_n(m_1)\}(d_n' - i c_n'), \quad |z| < R. \tag{30}$$

where

$$\alpha_2 = (G_2 - G_1)/(G_1 + G_2 k_1)$$

$\Phi(0)$ is determined from the equation

$$(G_1 + G_2 k_1)\, \Phi(0) = G_1 (1 + k_2)\, C/z_0 + (G_2 - G_1)\, \bar{\Phi}(0) - G_1 G_2 (\epsilon_1 + \epsilon_2). \tag{31}$$

The unknowns a_n, b_n, etc., can be determined from the conditions (4), (5.3) and (6.4) and are given below:

$$d_n + ic_n = \{2n (f_1 - f_2) P_n - 2n f_4 Q_n + 2n f_2 T_n\}/\triangle,\ n \geqslant 1 \tag{32}$$

$$d_n' + ic_n' = \{2n (f_1 - f_3) P_n + 2n f_3 Q_n - 2n f_1 T_n\}/\triangle,\ n \geqslant 1 \tag{33}$$

$$2nb_n = - S_2 R^n K_{n-1}(m_2)(d_n + ic_n)/m_2 + 2n T_n R^n,\ n \geqslant 1 \tag{34}$$

$$2na_n = S_1 \{m_1 I_{n-1}(m_1) - 2n I_n(m_1)\}(d_n' - ic_n')/R^n m_1^2,\ n \geqslant 1 \tag{35}$$

where

$$f_1 = m_1^2 m_2^2 K_{n-1}(m_2),\ S_1 = - 4n (1 + n)(1 - \nu_1)\, \alpha_2$$

$$S_2 = 4n (1 - n)(1 - \nu_2)\, \alpha_1,$$

$$f_2 = m_1 m_2 \{m_1^2 + S_1 (1 + g)\} I_{n-1}(m_1) + m_2 \{n (g - 1) m_1^2 - 2n (1 + g) S_1\} I_n(m_1)$$

$$f_3 = m_1^2 (m_2^2 - S_2) K_{n-1}(m_2) + n\, m_1^2 m_2 K_n(m_2)$$

$$f_4 = m_1 m_2 (m_1^2 + S_1) I_{n-1}(m_1) - n\, m_2 (m_1^2 + 2S_1) I_n(m_1)$$

$$P_n = n (\bar{A}_2/R - \bar{A}_1/R) R_1^{n+1},\ n \geqslant 1\,;\ \ Q_n = (\bar{A}_1/R - g\bar{A}_2/R) \times R_1^{n+1},\ n \geqslant 1$$

$$n (1 - n) T_n = i\, S_2 l_2 m_2^{-1} \{(n - 1)\, \bar{C} R_1^{n-1} (1 - R_1^2) + k_2 C R_1^{n-1} + 4n (1 - n)(1 - \nu_2)\, \bar{C} R_1^{n+1} m_2^{-2}\}, \qquad n \geqslant 3$$

$$T_1 = 0,\ T_2 = 4i (1 - \nu_2)\, l_2 R_1 m_2^{-1} \alpha_2 \{\bar{C} (1 - R_1^2) + k_2 C - 8 (1 - \nu_2)\, \bar{C} R_1^2 m_2^{-2}\} + 4i (1 - \nu_2)\, l_2^2 G_1 G_2 (\epsilon_1 - \epsilon_2)/(G_1 + G_2 k_1)$$

$$R_1 = R/z_0, \qquad g = m_1^2 G_2/m_2^2 G_1$$

$$\triangle = [m_1 S_1 \{- m_2^2 g + S_2 (1 + g)\} + S_2 m_1^3]\, I_{n-1}(m_1) K_{n-1}(m_2) - n^2 m_2 \{m_1^2 (g - 1) - 2S_1 (1 + g)\} I_n(m_1) K_n(m_2) - n\, m_1 m_2 \{m_1^2 + S_1 (1 + g)\} K_n(m_2) I_{n-1}(m_1)$$

$$+ \{-2S_1(1+g)(m_2^2 - S_2) + 2S_1 m_2^2 + g\, m_1^2 m_2^2 - S_2\, m_1^2(g-1)\}\, n\, I_n(m_1)\, K_{n-1}(m_2)$$

$$\mathrm{Re}\,(a_0) = 4(1-\nu_1)\, l_1^2\, \alpha_2\, \mathrm{Im}\,\{\bar{\Phi}(0)\}$$

$$\mathrm{Re}\,(b_0) = g\,\{\mathrm{Re}\,(a_0) - \mathrm{Re}\,(A_2/z_0)\} + \mathrm{Re}\,(A_1/z_0).$$

Having obtained $\Phi(z)$, $\Omega(z)$ can be determined from (26) and (27) and $\bar{\psi}(z)$ from (17). It may be noted that $\Phi(z)$ and $\Omega(z)$ so determined satisfy the conditions (18)–(20). If the concentrated force is situated at a point on the boundary L then $h(t)$ has a pole of third order at $t = R$ and the solution of Hilbert problem in (22) can not be found.

The results for a concentrated force applied at a point $z = z_0$ ($|z_0| > R$) in an infinite medium containing a circular hole of radius R at the origin can be obtained from the results given above by putting $l_1 = 0$ and $G_1 = 0$. The unknowns in this case are given as follows:

$$a_n = 0, \quad n \geqslant 1, \quad d_n' + ic_n' = 0, \quad n \geqslant 1, \quad A_2 = 0$$

$$d_n + ic_n = -8i(1-\nu_2)\,\bar{C}\;n^3 R^{n+1}\, l_2/\triangle_1 + 2n^2\, l_2 - 1\; T_n^*/\triangle_1, \quad n \geqslant 1 \tag{36}$$

where T_n^* is obtained from T_n by putting $G_1 = 0$,

$$b_n = 2(1-n)(1-\nu_2)\, R^n\, K_{n-1}(m_2)\,(d_n + ic_n)/m_2 + T_n^*\, R^n, \; n \geqslant 1. \tag{37}$$

and

$$\triangle_1 = n\, K_{n-1}(m_2)\,\{4n(1-n)(1-\nu_2) - m_2^2\} - n^2\, m_2 K_n(m_2).$$

The results for a circular rigid inclusion in an infinite medium can be obtained by taking $l_1 = 0$ and $G_1 = \infty$.

3. Consider next that the concentrated force is situated at a point $z = z_0$ in the interior of the inhomogeneity. Because of the all-round symmetry z_0 can be taken to be a real quantity. Although the method of solution remains as above but various quantities change considerably.

Conditions (18)–(20) shall now be as follows:
For large $|z|$

$$\Phi(z) = \Gamma - D_0\, z^{-1} + 0\,(z^{-1}), \quad \bar{\psi}(z) = \Gamma' + D_1/z + 0\,(z^{-2}) \tag{38}$$

$$\Omega(z) = \bar{\Phi}(0) + (z^{-2}) \tag{39}$$

where D_0 and D_1 are some complex constants.

Near $z = 0$

$$\Omega(z) = -\bar{\Gamma}' z^{-2} - \bar{D}_1 z^{-1} + a \text{ holomorphic function} \tag{40}$$

$V^+(z, \bar{z})$ and $V^-(z, \bar{z})$ are given by (23) and (24) but the constants are different. Let

$$V^+(z, \bar{z}) = A_{10}/(z - \bar{z}_0) + \bar{A}_{10}/(\bar{z} - z_0)$$
$$+ \sum_{n=0}^{\infty} (b_{n0} z^{-2} + \bar{b}_{n0} \bar{z}^{-n})$$
$$+ \sum_{n=1}^{\infty} K_n(r/l_2)(c_{n0} \sin n\theta + d_{n0} \cos n\theta). \quad |z| > R \tag{41}$$

$$V^-(z, \bar{z}) = A_{20}/(z - z_0) + \bar{A}_{20}/(z - \bar{z}_0)$$
$$+ \sum_{n=0}^{\infty} (a_n z^n + \bar{a}_n \bar{z}^n)$$
$$+ \sum_{n=1}^{\infty} I_n(r/l_1)(c'_{n0} \sin n\theta + d'_{n0} \cos n\theta). \quad |z| < R \tag{42}$$

$\Phi(z) - \Omega(z)$ is given by (26) and (27) where c_n, d_n, etc., are to be replaced by c_{n0}, d_{n0}, etc., and $h(z)$ is to be replaced by $h_1(z)$.

$$h_1(z) = -\frac{C_1}{z - z_0} + \frac{R^2 \bar{C}_1 (z - z_0)}{z_0^2 (z - R_0)^2} - \frac{\bar{C}_1}{z_0} \frac{R_0}{(z - R_0)} - \frac{k_1 C_1}{z - R_0}$$
$$- \frac{8R^2(1 - \nu_1) l_1^2}{z_0 3} \frac{\bar{C}_1 z}{(z - R_0)^3} + \frac{\bar{D}_1}{z} - \bar{\Phi}(0), \tag{43}$$

$C_1 = (X + iY)/2\pi(1 + k_1)$; D_1 and $\Phi(0)$ are to be determined.

$\Phi(z)$ can be determined as before.

$$(G_2 + G_1 k_2)\Phi(z) = -G_2(1 + k_1) C_1/(z - z_0) + (G_2 - G_1) D_1/$$
$$z - G_1 G_2 (\epsilon_1 - \epsilon_2) R^2/z^2 - i(G_2 - G_1) \sum_{n=2}^{\infty} (1 - n) R^{n-2}$$
$$\times m_2 K_{n-1}(m_2)(d_{n0} + i c_{n0}) z^{-n}/2, \; |z| > R \tag{44}$$

and

$$\Phi(z) = -\alpha_2 \left[\frac{R^2 \bar{C}_1 (z - z_0)}{z_0^2 (z - R_0)^2} - \frac{\bar{C}_1}{z_0} \frac{R_0}{z - R_0} - \frac{k_1 C_1}{z - R_0} \right.$$
$$\left. - \frac{4R_0(1 - \nu_1) l_1^2 \bar{C}_1}{z_0^2} \frac{z}{(z - R_0)^3} - \bar{\Phi}(0) \right] - C_1/(z - z_0)$$

$$-G_1G_2(\epsilon_1+\epsilon_2)/(G_1+G_2k_1)+ia_2\sum_{n=1}^{\infty}(1+n)R^{-n-2}z^n$$
$$\times\{m_1 I_{n-1}(m_1)-2n I_n(m_1)(d'_{n0}-ic'_{n0})/2,\quad |z|<R \tag{45}$$

From (45) and (4)

$$\Phi(0)=-a_2\{-\bar{C}_1/R_0+\bar{C}_1/z_0+k_1C_1/R_0-\bar{\Phi}(0)\}$$
$$+C_1/z_0-G_1G_2(\epsilon_1+\epsilon_2)/(G_1+G_2k_1) \tag{46}$$

and

$$A_{10}=4i(1-\nu_2)l_2^2G_2(1+k_1)C_1/(G_2+G_1k_2),$$
$$A_{20}=4i(1-\nu_1)l_1^2C_1. \tag{47}$$

In order that $\bar{\psi}(z)$ should be holomorphic near $z=0$ the coefficient of z^{-1} in $\Omega(z)$ for large z must be zero. This condition determines the constant D_1.

$$D_1=k_2(1+k_1)/(1+k_2).$$

The constants a_{n0}, b_{n0}, etc., can be determined from the boundary conditions (6.3), (6.4) and the condition (4).

$$d_{n0}+ic_{n0}=[2nf_4\{P_{n0}-Q_{n0}-(1+g)\bar{U}_{n0}\}$$
$$+2nf_2\{\bar{U}_{n0}-P_{n0}+T_{n0}\}]/\triangle,\quad n\geqslant 1 \tag{48}$$

$$d_{n0}'+ic_{n0}'=[2nf_3\{Q_{n0}-P_{n0}+(1+g)\bar{U}_{n0}\}$$
$$+2nf_1\{Q_{n0}-T_{n0}-\bar{U}_{n0}\}]/\triangle,\quad n\geqslant 1 \tag{49}$$

$$2nb_{n0}=-S_2R^nK_{n-1}(m_2)(d_{n0}+ie_{n0})/m_2+2nT_{n0}R^n,$$
$$n\geqslant 1 \tag{50}$$

$$2n\,a_{n0}=S_1\{m_1I_{n-1}(m_1)-2nI_n(m_1)\}(d_{n0}'-ic_{n0}')/R^n m_1^2$$
$$+2nU_{n0}/R^n,\qquad n\geqslant 1. \tag{51}$$

Some of the quantities in (48)–(51) which are not defined earlier are as follows:

$$P_{n0}=n(\bar{A}_{20}/R-\bar{A}_{10}/R)R_1^{n+2},\qquad Q_{n0}=(\bar{A}_{10}/R-g\bar{A}_{20}/R)\times R_1^{n+1},\quad n\geqslant 1$$

$$T_{n0}=0,\quad n\geqslant 3,\quad T_{10}=-4i(1-\nu_2)l_2m_2^{-1}a_1\bar{D}_1$$

$$T_{20}=4i(1-\nu_2)l_2^2G_1G_2(\epsilon_1-\epsilon_2)/(G_2+G_1k_2)$$

$$n(1+n)U_{n0}=iS_1l_1m_1^{-1}\{(n+1)\bar{C}_1(R_1^2-1)/R_1^{n+1}+k_1C_1/R_1^{n+1}$$
$$-4n(i+n)\bar{C}_1m_1^{-2}/R_1^{n-1}\},\qquad n\geqslant 1$$

$$\mathrm{Re}(b_{00})=0,\qquad \mathrm{Re}(a_{00})=\mathrm{Re}(A_2/z_0)-g\,\mathrm{Re}(A_1/z_0).$$

TABLE I

Stresses at the equilibrium boundary. $G_1 = G_2$, $\nu_1 = \nu_2 = 0\cdot 25$, $\epsilon_1 = -\epsilon_2$ *and* $Y = 0$. *The values in the numerator and denominator refer to couple stress effects and without couple stresses, respectively.*

$P = (1 + k)/2G\,\epsilon_1$, $X_0 = X/2G\,\epsilon_1$, $R_0 = z_0/R\ (\mid z_0 \mid > R)$

θ	O			$\pi/2$			π		
R_0	1·01		10·0	1·01		10·0	1·01		10·0
X_0	0	5	15	0	5	15	0	5	15
τ_{rr}/P	−2·1411	−1·3218	−0·9741	2·1411	1·3728	1·9276	−2·1411	−0·9821	−0·9723
$R/l = 1\cdot 0$	−1·0000	−0·1807	0·1808	1·0000	1·3899	0·7730	−1·0000	1·5527	0·1827
τ_{rr}/P	−1·6713	−0·8520	−0·4911	1·6713	2·0150	1·4449	−1·6713	0·8256	−0·4892
$R/l = 5\cdot 0$	−1·0000	−0·1807	0·1808	1·0000	1·3899	0·7730	−1·0000	1·5527	0·1827
τ_{rr}/P	−1·2857	−0·1807	0·1808	1·2837	1·3881	0·7730	−1·2837	1·5505	0·1827
$R/l = 25\cdot 0$	−1·0000	−0·1807	0·1808	1·0000	1·3899	0·7730	−1·0000	1·5527	0·1827
$\tau_{r\theta}/P$	0·0000	0·0000	0·0000	0·0000	7·4803	0·1534	0·0000	0·0000	0·0000
$R/l = 1\cdot 0$	0·0000	0·0000	0·0000	0·0000	1·5836	0·1409	0·0000	0·0000	0·0000
$\tau_{r\theta}/P$	0·0000	0·0000	0·0000	0·0000	1·8194	0·1419	0·0000	0·0000	0·0000
$R/l = 5\cdot 0$	0·0000	0·0000	0·0000	0·0000	1·5836	0·1409	0·0000	0·0000	0·0000

$\tau_{r\theta}/P$	0·0000	0·0000	0·0000	0·0000	1·5930	0·1409	0·0000	0·0000	0·0000
$R/l = 25\cdot0$	0·0000	0·0000	0·0000	0·0000	1·5836	0 1409	0·0000	0·0000	0·0000
$\tau_{\theta\theta}^{+}/P$	—1·8589	—2·6467	—2·0815	1·8589	4·2187	3·0179	—1·8589	—0 9085	—2·0814
$R/l = 1\cdot0$	—3·0000	—3·7878	3·2354	3·0000	4·2015	4·1725	—3·0000	—3·4412	—3·2364
$\tau_{\theta\theta}^{+}/P$	—2·3287	—3·1165	—2 5645	2·3287	3·5765	3·5006	—2·3287	—2·7141	—2·5645
$R/l = 5\cdot0$	—3·0000	—3·7878	—3·2364	3·0000	4·2015	4·1725	—3·0000	—3·4412	—3·2364
$\tau_{\theta\theta}^{+}/P$	—2·7163	—3·7878	—3·2364	2·7163	4·2034	4·1725	—2·7163	—3·4390	—3·2364
$R/l = 25\cdot0$	—3·0000	—3·7878	—3·2364	3·0000	4·2015	4·1725	—3·0000	—3·4412	—3·2364
$\tau_{\theta\theta}^{-}/P$	2·1411	1·3553	1·9185	—2·1411	0·2187	—0·9821	2·1411	3·0935	1·9186
$R/l = 1\cdot0$	1·0000	0·2122	0·7636	—1·0000	0·2015	0·1725	1·0000	0·5588	0·7636
$\tau_{\theta\theta}^{-}/P$	1·6713	0 8835	1·4355	—1·6713	—0·4235	—0·4994	1·6713	1·2859	1·4355
$R/l = 5\cdot0$	1·0000	0·2122	0·7636	—1·0000	0·2015	0·1725	1·0000	0·5588	0·7636
$\tau_{\theta\theta}^{-}/P$	1·2837	0·2122	0·7636	—1·2837	0·2034	0·1725	1·2837	0·5610	0·7636
$R/l = 25\cdot0$	1·0000	0·2122	0·7636	—1·0000	0·2015	0·1725	1·0000	0·5588	0·7636

4. Till now in sections 1, 2 and 3 stresses at infinity were taken to be zero. If the stresses are bounded at infinity then it seems to be more convenient and systematic to obtain the solution as the superposition of two solutions. The first solution corresponds to the problem considered above in sections 2 and 3 with the boundary conditions (6.1)–(6.4). The second solution corresponds to the problem of circular inhomogeneity in an infinite medium with no discontinuity in the displacements in (6.1) and no concentrated force in the medium but bounded stresses at infinity. A systematic approach towards obtaining this second solution, is through the construction of two new functions ([8], equation (6) and (7)). An advantage in this approach is that the behaviour of $\Phi(z)$ for large $|z|$ and small $|z|$ is easily determined (refer [8], equation (17)). But this approach in [8] is not suitable for determining the singularities of $\Phi(z)$.

If $\Phi(z)$ so obtained by the superposition of two solutions is denoted by $\Phi_S(z)$, then

$$\Phi_S(z) = \Phi(z) + (G_2M_1 + M_2)/(G_2 + G_1k_2) + \alpha_1 M_3/z^2, \qquad |z| > R. \tag{52}$$

and

$$\Phi_S(z) = \Phi(z) + (G_1M_1 + M_2)/(G_1 + G_2k_1), \qquad |z| < R \tag{53}$$

where $\Phi(z)$ in (52) and (53) are given by (29) and (30) respectively or by (44) and (45) respectively; M_1 M_2 and M_3 depend on the conditions at infinity and are given below.

For a uniaxial tension p in the y direction

$$8M_1\{G_1 + G_2(1 - 2\nu_1)\} = -p\{G_1(1 - \nu_2) - G_2(1 - \nu_1)\} \tag{54}$$

$$G_2M_1 + M_2 = p(G_2 + G_1k_2)/8, \qquad M_3 = pR^2/4. \tag{55}$$

For the biaxial tensions q and p in the x and y directions respectively $M_3 = (p - q)R^2/4$ and M_1 and M_2 are given by (54) and (55) with p replaced by $p + q$.

When the principal stresses N_1 and N_2 act at infinity and the angle between N_1 and the x-axis is δ then $M_3 = (N_1 - N_2)e^{2i\delta}/4$ and M_1 and M_2 are given by (54) and (55) with p replaced by $N_1 + N_2$.

$\Omega(z)$ can be determined as before with $\Phi(0)$ replaced by $\Phi_S(0)$. By taking $\epsilon_1 = \epsilon_2 = 0$, $C = 0$ and appropriate stresses at infinity, the results given in [4] are obtained.

Stresses have been calculated at the equilibrium boundary for the case when $G_1 = G_2$, $\nu_1 = \nu_2 = 0 \cdot 25$ $\epsilon_1 = -\epsilon_2$, concentrated force acts along positive x axis and zero stresses at infinity. Numerical results which are presented in table 1 are in conformity with the fact that the effect of couple stresses is negligible when the ratio of the smallest dimension of the body to the characteristic length is large. This observation is independent of the point of application of the concentrated force and its magnitude.

REFERENCES

[1] Mindlin, R. D. and Tiersten, H. F. — Effects of Couple stresses in linear elasticity. *Archives for Rational Mechanics and Analysis*, 1962, **11**, 415–448.

[2] Mindlin, R. D. .. Influence of couple stresses on stress concentrations. *Experimental Mechanics*, 1963, **3**, 1–7.

[3] Weitsman, Y. .. Couple-Stress effects on stress concentrations around a cylindrical inclusion in a field of uniaxial tension. *Jour. of Appl. Mech.*, Vol. 32, No. 1., *Trans. ASME*, Vol. 87, Series E, June 1965, pp. 424–427.

[4] Hartranft, R. J. and Sih, G. C. — The effect of couple-stresses on the stress concentration of a circular inclusion. *Jour. of Appl. Mech.*, Vol. 32, No. **1**, *Trans. ASME*, Vol. 87, Series E, June 1965, pp. 429.

[5] Huigol, R. R. .. On the concentrated force problem for two-dimensional elasticity with couple stresses. *Int. Jour. Engg. Sci.*, 1967, **5**, 81–93.

[6] Mindlin, R. D. .. *Proceeding of the International Symposium on Applications of the Theory of Functions in Continuum Mechanics*, Tbilisi, USSR 1963 pp. 256–259.

[7] Muskhelishvili, N. I. — Some basic problems of the mathematical theory of elasticity, Translated by J. R. M. Radok, Groningen, 1953.

[8] England, A. H. .. An arc crack around a circular elastic inclusion. *Jour. of App Mech. Trans. ASME.*, Vol. 33, No. 1, Series E, 1966, pp. 637–640.