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On the validity of sandwich shell theory

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Abstract

A three-dimensional elasticity solution for a sandwich shell subjected to axisymmetric load has been used here to establish the applicability of sandwich shell theory. Numerical results, presented for several non-dimensional parameters, have been examined keeping in view the assumptions made in sandwich shell theory. Based on this, conclusions have been drawn regarding the application of sandwich shell theory for a three-layered shell.

Key words : Sandwich shell, elasticity theory, axisymmetric load.

1. Introduction

The demand for efficient structures has grown, especially where insulation and erection costs are primary considerations. Due to its high strength-to-weight ratios and inherent insulation properties, sandwich construction provides a viable candidate for structural elements. Sandwich construction has been used extensively in aircraft and aerospace industries. The response of sandwich construction to various inputs is determined to a large extent by the geometrical and the material properties of the core relative to the facings. The extent of such effects as due to transverse shear and normal deformations, large amplitudes, and local instability can be investigated in terms of their relative properties. Constraints due to the continuity of the displacements and of the transverse shear and normal stresses at the interface of the core and facings have been accounted for in more accurate analysis.

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The first significant contribution to an understanding of the behaviour of sandwich shell was presented by Reissner[1]. He has presented a small deflection theory in which the isotropic facings are treated as membranes. The theory is applied to particular shells and the importance of accounting for transverse shear and transverse normal stress is investigated. The results show their effects to be significant when there exists an order of magnitude difference between the elastic constants of the core and the facings. On similar lines, Stein and Mayers[2] have presented sandwich shell theory, treating facings as membranes, for orthotropic core. Wang[3] has formulated sandwich shell theory by making use of the principle of complementary energy in the nonlinear elasticity theory. A large deflection theory for unsymmetrical shallow, doubly curved sandwich shells, in which bending stiffness of face sheets is considered, is presented by Fulton[4] through variational principles. It must be stated here that none of the above[1–4] formulations have considered the effect of the core compressibility. Schmidt[5] establishes the equations for the small deflection of sandwich shells with orthotropic cores, considering the bending stiffness of isotropic similar facings and core compressibility effects. This formulation has been extended to the large deflection analysis of multisandwich shells by Schmidt[6] himself and to the analysis of sandwich shells with laminated anisotropic facings by Martin[7]. Wempner[8–10] derives equations for sandwich shells with a weak core which account for moderately large deflections, orthotropy, and the bending stiffness of facings. Extensive review on the developments in the theory of sandwich structures has been given by Habip[11].

It should be noted that all the above formulations are extensions of the two-dimensional classical thin shell theory and have been formulated under the assumptions that face parallel stresses in the core are negligibly small and transverse shear stress distribution over the core thickness is constant. Recently, a long axisymmetrically loaded sandwich cylinder was analysed by the authors[15–17] using three-dimensional elasticity theory and thin shell theory and comparison of results were made with sandwich shell theories. It may be mentioned here that no attempt has been made to study the admissibility of assumptions made in the sandwich shell theories and also the effects of t_2/t (core to facing thickness ratio and $t_1 = t_3 = t$) and $\beta = E/E_2$ (facing elastic modulus to core elastic modulus ratio and $E_1 = E_3 = E$) on the behaviour of sandwich shell. In this paper, an attempt is made to study the above aspects of the sandwich shell using the elasticity solution[15].

2. Method of analysis

A brief descritpion of the method of solution as presented in (15) is given here. The problem of a sandwich circular cylindrical shell is treated as a three layered cylinder (Fig. 1) and the solution has been obtained using Love's stress functions approach. For this a stress function φ is to be selected so as to satisfy the differential equation

$$\nabla^2 \nabla^2 \varphi = 0 \tag{1}$$

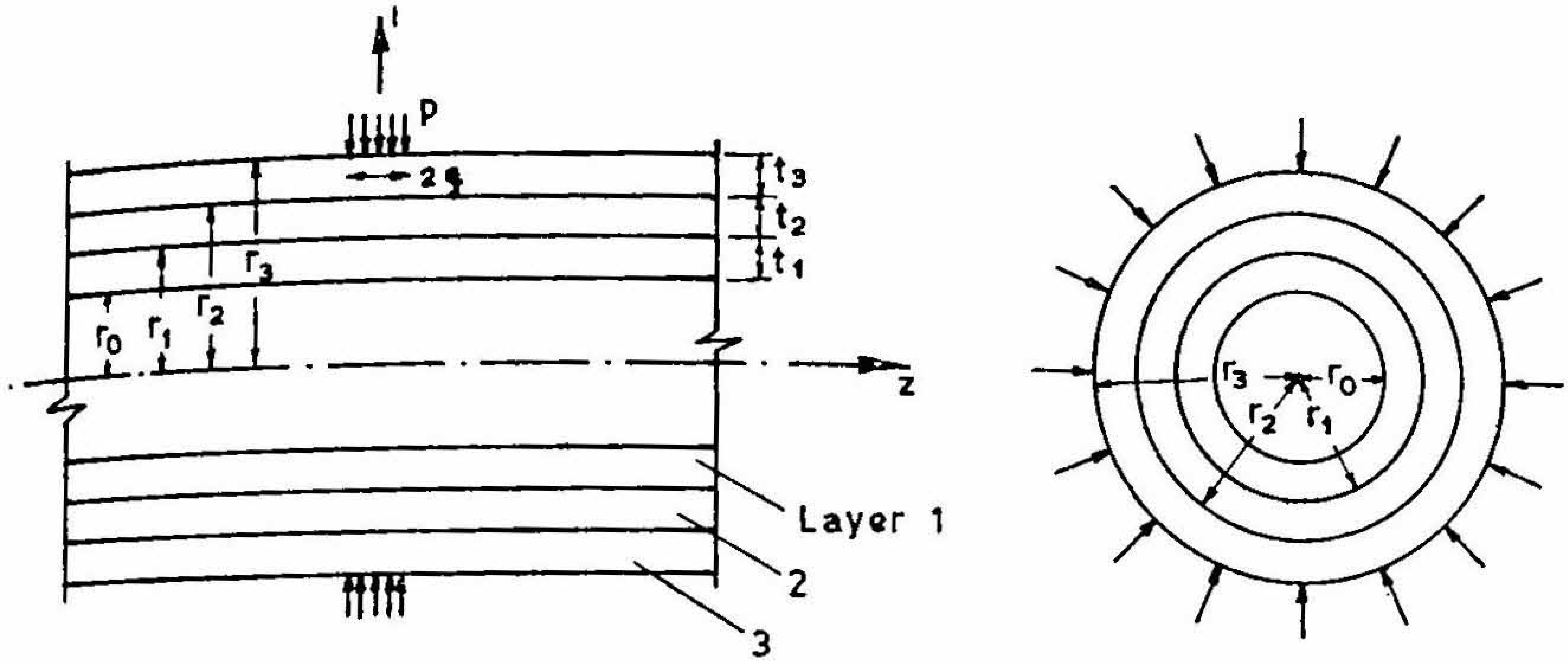

FIG. 1.

The stresses and displacements are determined from (18) :

$$\sigma_r = \frac{\partial}{\partial z}\left[\mu\,\nabla^2\varphi - \frac{\partial^2\varphi}{\partial r^2}\right]$$

$$\sigma z = \frac{\partial}{\partial z}\left[(2-\mu)\,\nabla^2\varphi - \frac{\partial^2\varphi}{\partial z^2}\right]$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left[\mu\,\nabla^2\varphi - \frac{1}{r}\frac{\partial\varphi}{\partial r}\right]$$

$$\tau_{rz} = \frac{\partial}{\partial z}\left[(1-\mu)\,\nabla^2\varphi - \frac{\partial^2\varphi}{\partial z^2}\right] \qquad (2)$$

$$u = -\frac{1}{2G}\frac{\partial^2\varphi}{\partial r\partial z}$$

$$w = \frac{1}{2G}\left[2(1-\mu)\,\nabla^2\varphi - \frac{\partial^2\varphi}{\partial z^2}\right] \qquad (3)$$

For a three-layered cylinder subjected to axisymmetric normal load on the outer surface (Fig. 1), the following boundary conditions can be written down.

$$\begin{aligned} &\text{at } r = r_3 \;;\quad \sigma_r = f(z) \quad;\quad \tau_{rz} = 0 \\ &\text{at } r = r_0 \;;\quad \sigma_r = 0 \quad;\quad \tau_{rz} = 0 \end{aligned} \qquad (4)$$

For perfect bond between the layers, the continuity conditions along a typical interface can be written as

$$\begin{aligned} \text{at } r = r_{j-1} \;;\; &(\sigma_r)_{j-1} = (\sigma_r)_j;\; (\tau_{rz})_{j-1} = (\tau_{rz})_j \\ &(u)_{j-1} = (u)_j \;;\; (w)_{j-1} = (w)_j \; (j = 2, 3) \end{aligned} \qquad (5)$$

The radial load acting on the outer boundary ($r = r_3$) can be expressed in terms of Fourier integral as

$$f(z) = \int_0^\infty q(a) \cos az \, da$$

$$\text{where} \quad q(a) = \frac{2}{\pi} \int_0^\infty f(z) \cos az \, dz$$

For a uniform band of pressure of intensity p/unit area distributed over a length (Fig. 1).

$$q(a) = -\frac{2p}{\pi a} \sin a\xi$$

The stress function φ for a typical i^{th} layer ($i = 1, 2, 3$) which satisfies eqn. can be taken as (18)

$$\varphi_i = \int_0^\infty \frac{1}{a^3} [A_i(a) I_0(ar) + B_i(a) ar I_1(ar)$$

$$+ C_i(a) K_0(ar) + D_i(a) ar K_1(ar)] \sin az \, da \quad (r_{i-1} \leq r \leq r_i)$$

where r_{i-1} and r_i are the inner and outer radius of the i^{th} layer. The stresses and displacements can be obtained from eqns. (2), (3) and (8). The constants $A_i(a)$, $B_i(a)$, $C_i(a)$ and $D_i(a)$ can later be determined using the boundary and continuity conditions (Eqns. (4) and (5)). The detailed procedure and the final equations can be found in (15).

3. Results and discussion

Though the solution obtained is general in that dissimilar facings (in terms of thickness and elastic properties) could be considered, for convenience the numerical results have been obtained here for similar facings (outer and inner facing has the same thickness and elastic properties). Numerical results have been obtained for the following parameters :

(i) $\mu = 0{\cdot}3$; $\eta_2 = 0{\cdot}01$ —$0{\cdot}06$; $t_2/t = 1$–100 ; $\beta = 550$
(ii) $\mu = 0{\cdot}3$; $\eta_2 = 0{\cdot}03$; $t_2/t = 5$ and 15 ; $\beta = 10$—2000

The above parameters are selected from the consideration of a sandwich shell in which the facings are thin and have higher elastic modulus compared to the core.

For a clear understanding of the behaviour of the facings and core, the variation of stresses and displacements over the thickness, at some typical sections, are presented in figs. 2–11. In figs. 12–14, the variation of percentage of membrane to total stress

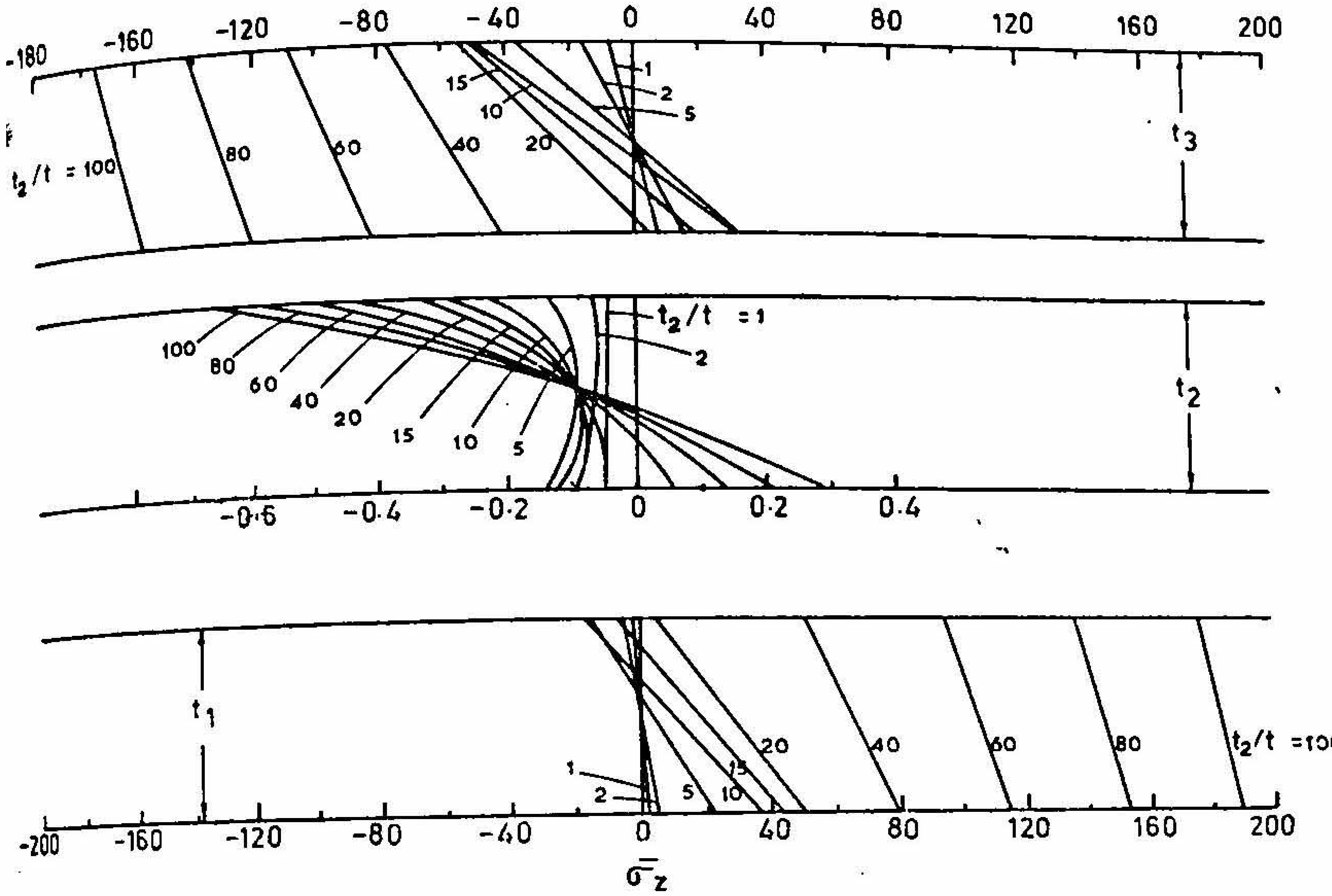

FIG. 2. Variation of longitudinal stress at $z/r_2 = 0\cdot0$ for different (t_2/t) values ($\beta = 550$, $\eta_3 = 0\cdot03$).

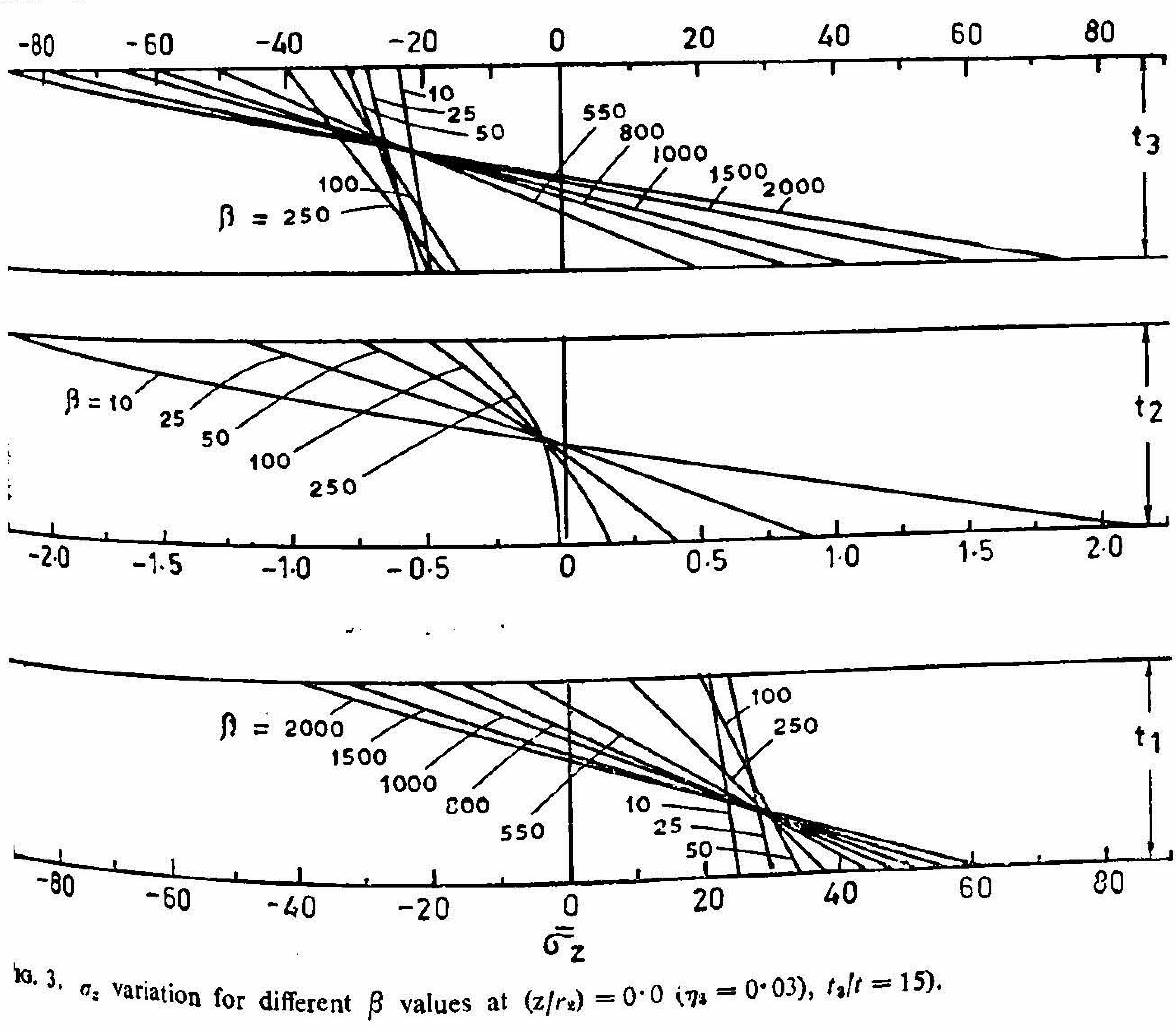

FIG. 3. σ_z variation for different β values at $(z/r_2) = 0\cdot0$ ($\eta_3 = 0\cdot03$), $t_2/t = 15$).

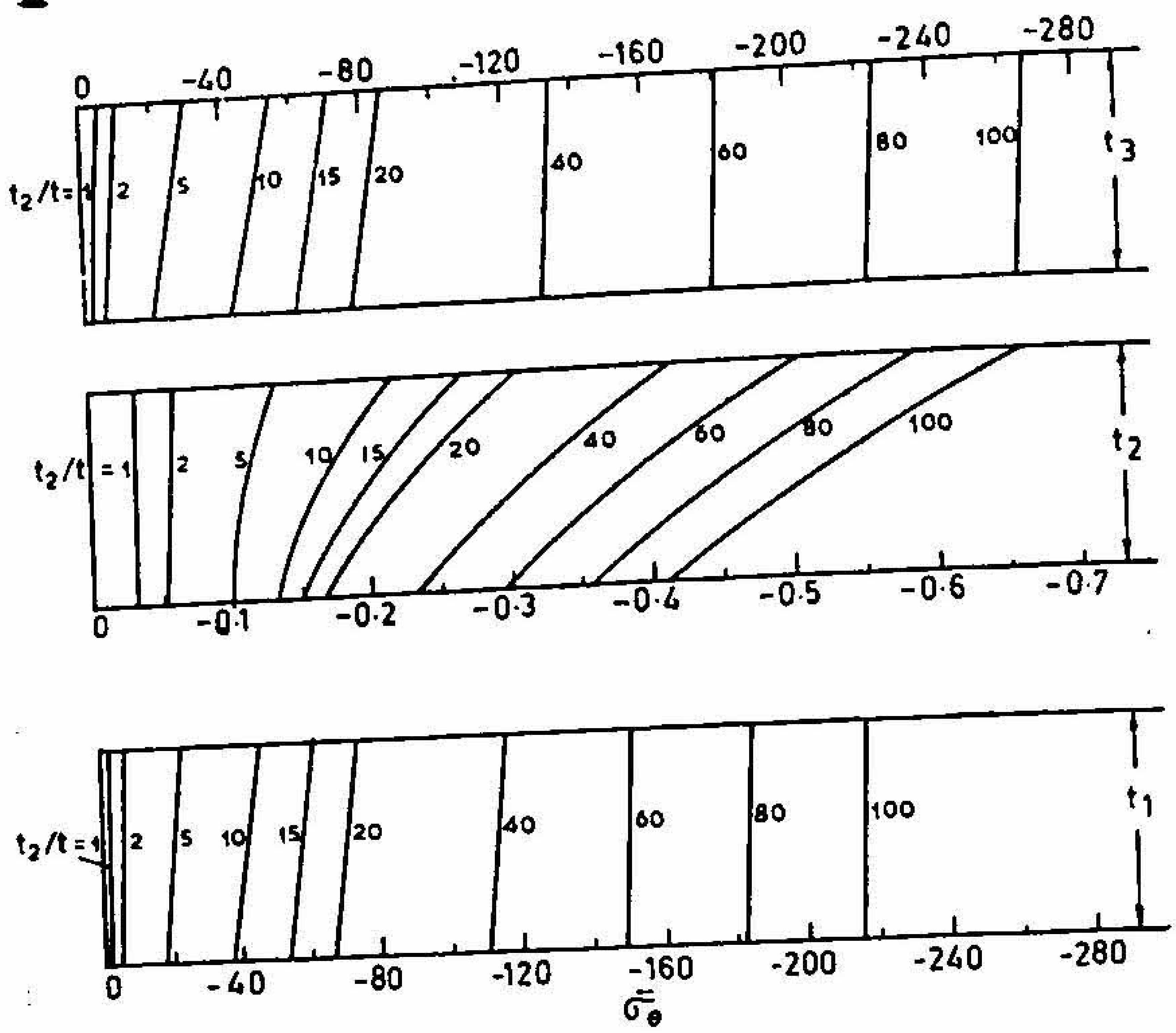

FIG. 4. Variation of tangential stress ($\bar{\sigma}_\theta$) at $z/r = 0 \cdot 0$ for different (t_2/t) values ($\beta = 550$) $\eta_2 = 0 \cdot 03$).

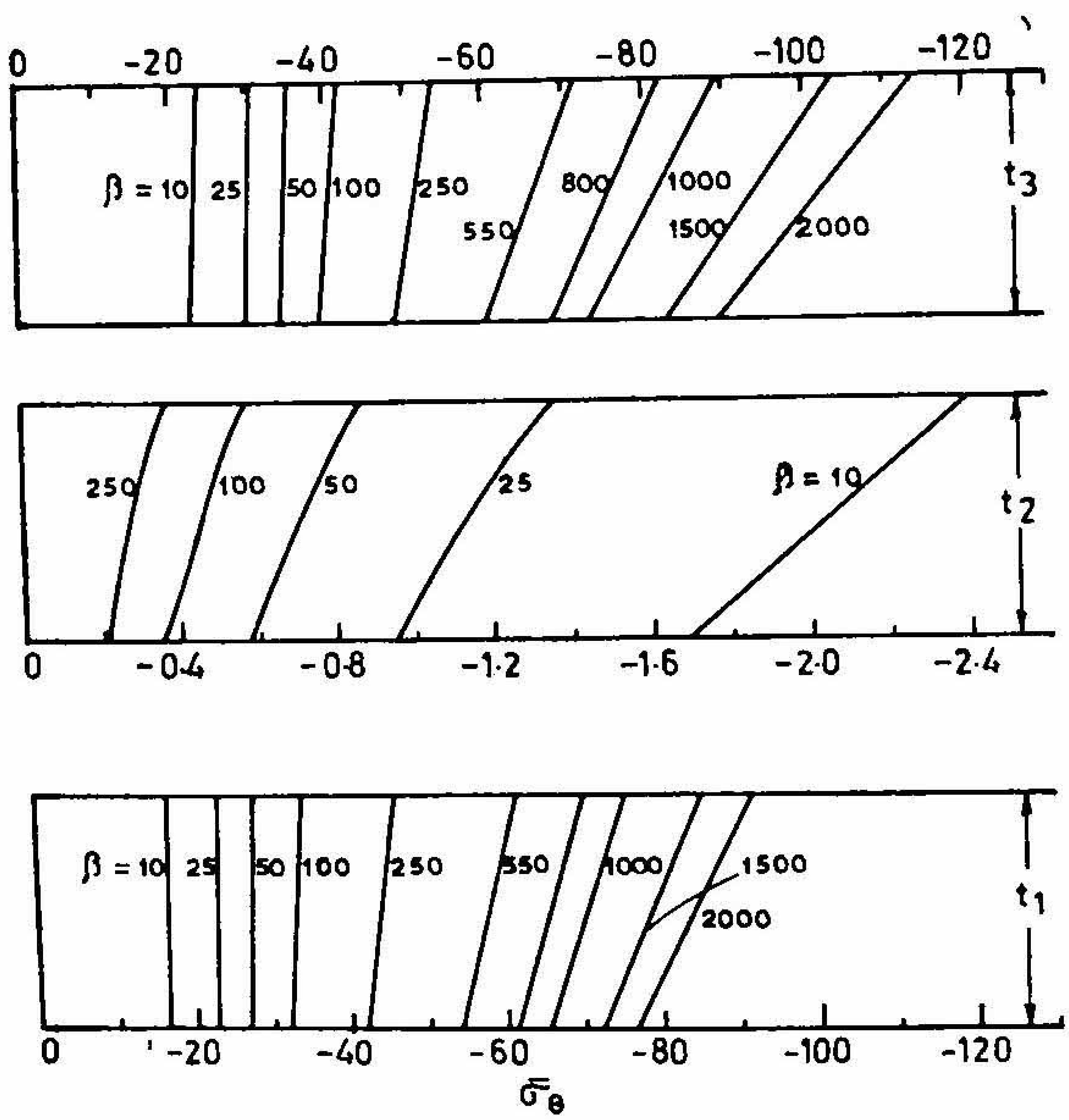

FIG. 5. $\bar{\sigma}_\theta$ variation for different β-values at $z/r_2 = 0 \cdot 0$ ($\eta_2 = 0 \cdot 03$, $t_2/t = 15$).

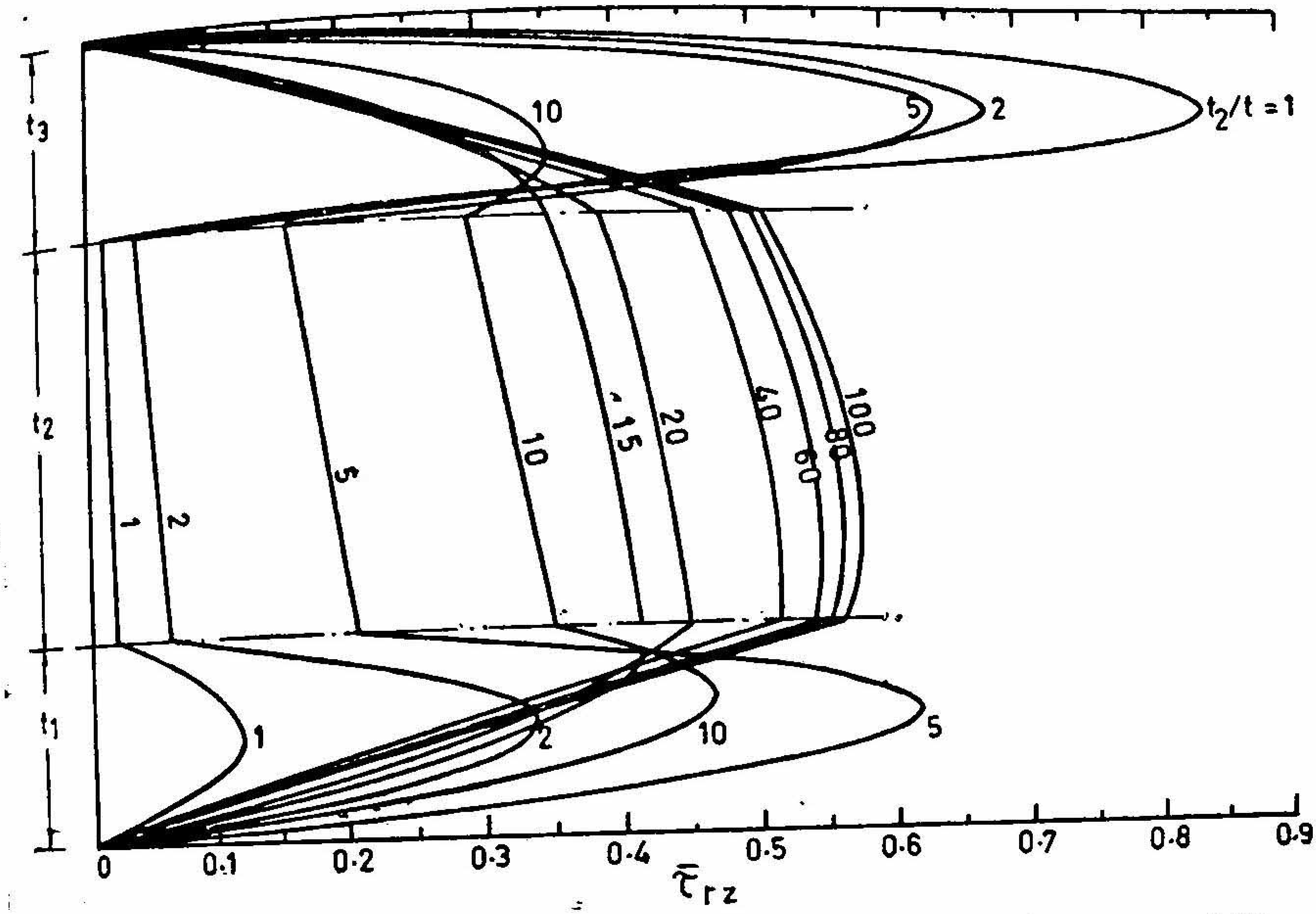

FIG. 6. Variation of shear stress at $(z/r_2) = 0{\cdot}06$ for different values of (t_2/t) $(\beta = 550,\ \eta_2 = 0{\cdot}03)$.

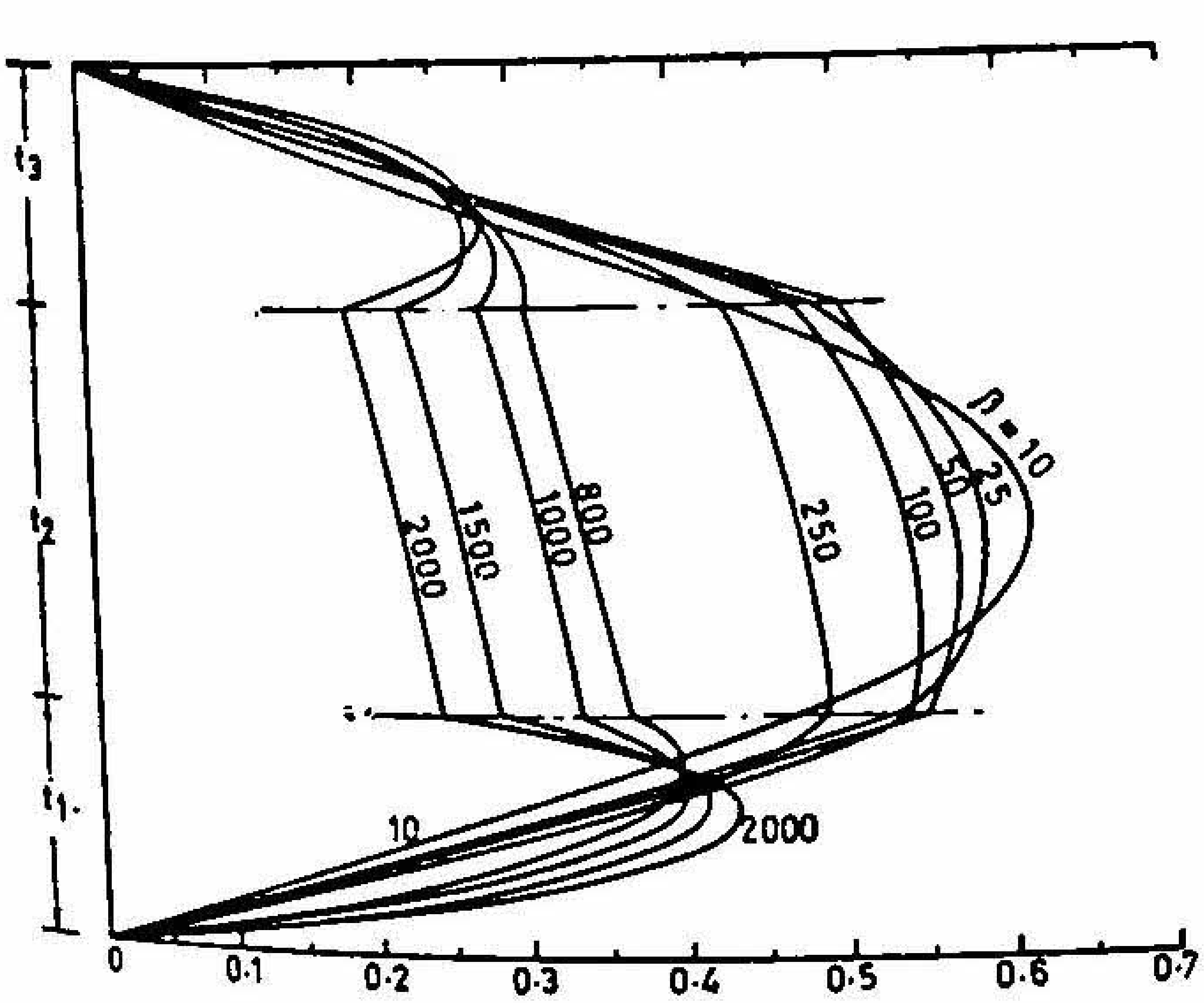

FIG. 7. τ_{rz} variation for different β values at $z/r_2 = 0{\cdot}06$ $(\eta_2 = 0{\cdot}03)$, $t_2/t = 15)$.

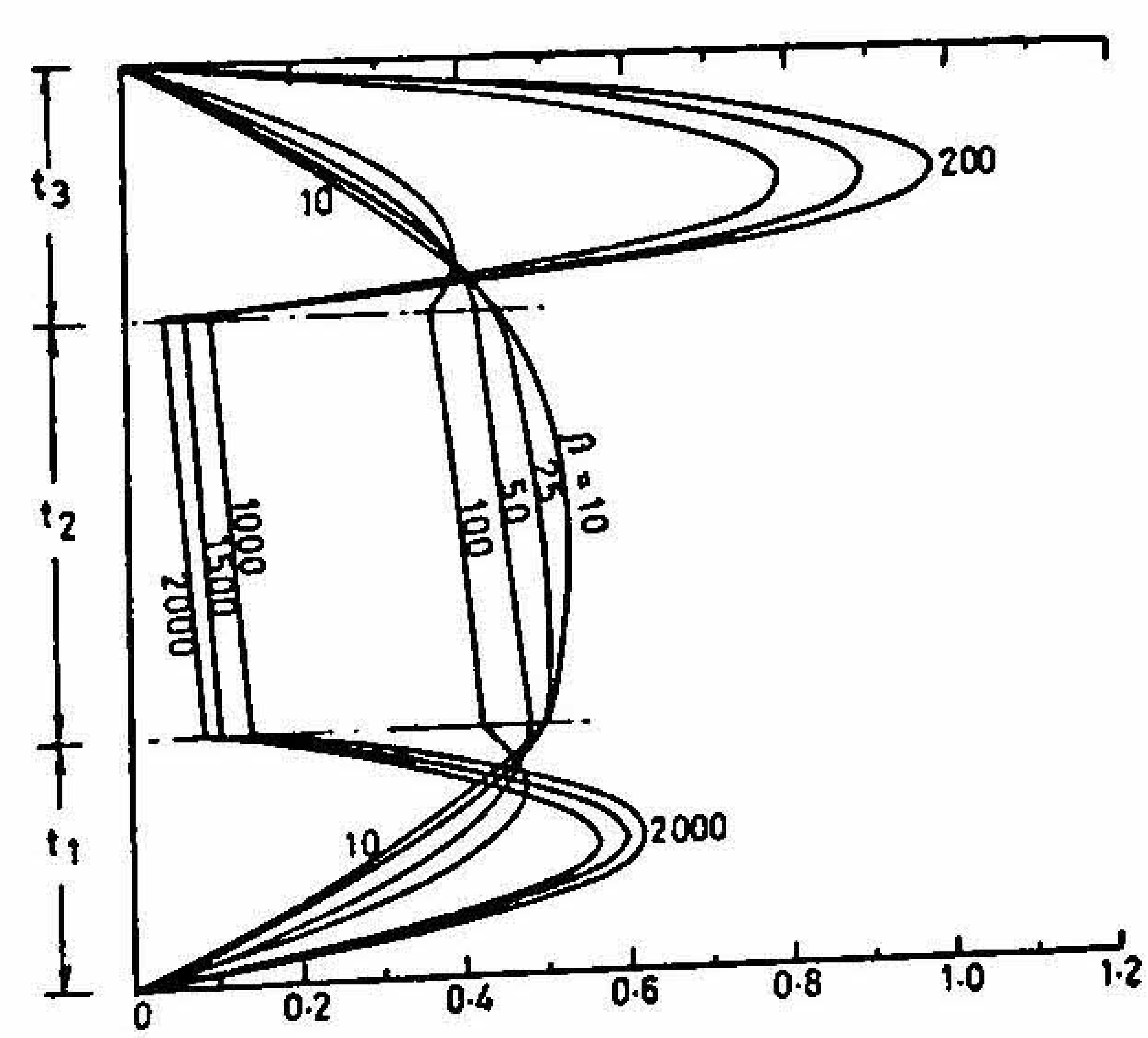

FIG. 8. τ_{rz} variation for different β values at $z/r_2 = 0{\cdot}06$ $(\eta_2 = 0{\cdot}03,\ t_2/t = 5)$.

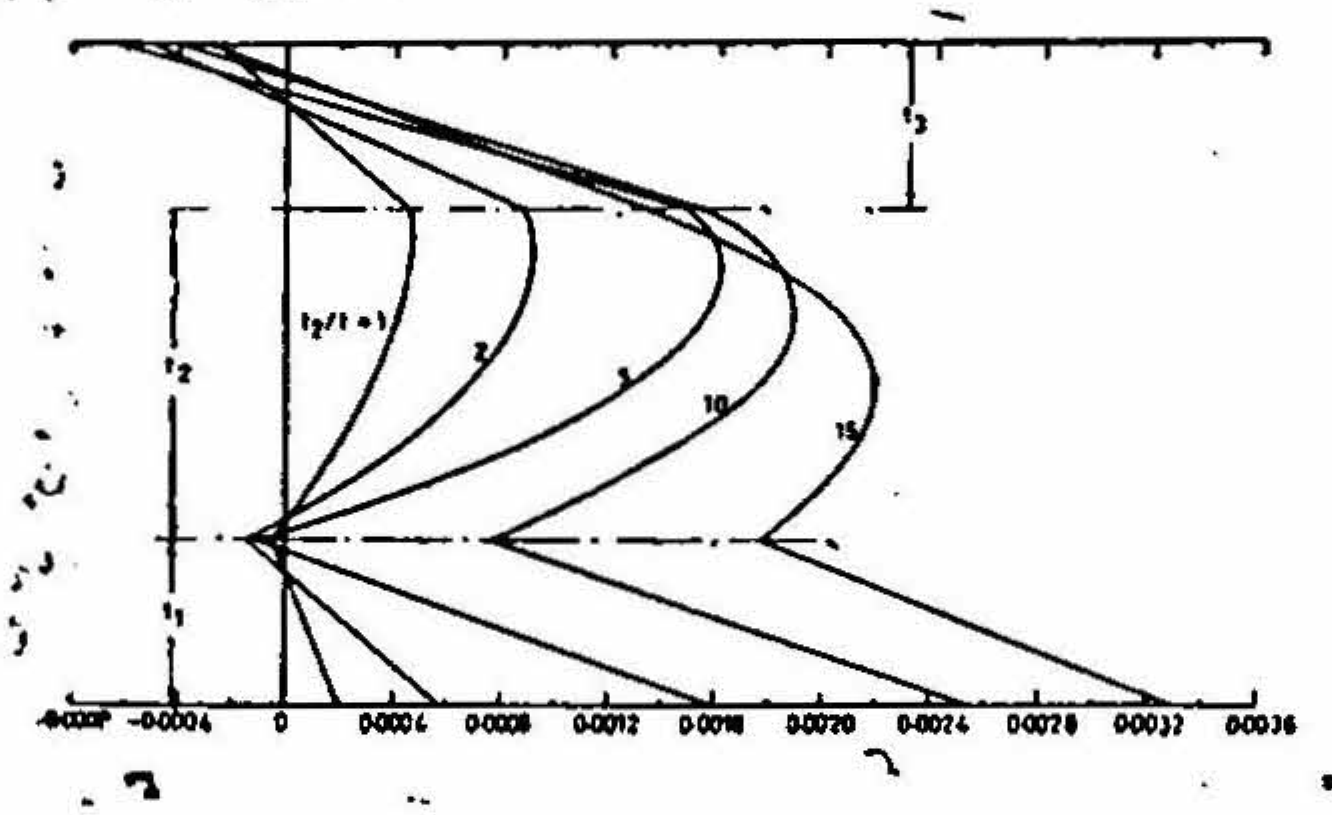

FIG. 9a. Variation of longitudinal displacement ($\bar{w}$) for different (t_2/t) values at $z/r_2 = 0{\cdot}06$ ($\eta_2 = 0{\cdot}03$, $\beta = 550$).

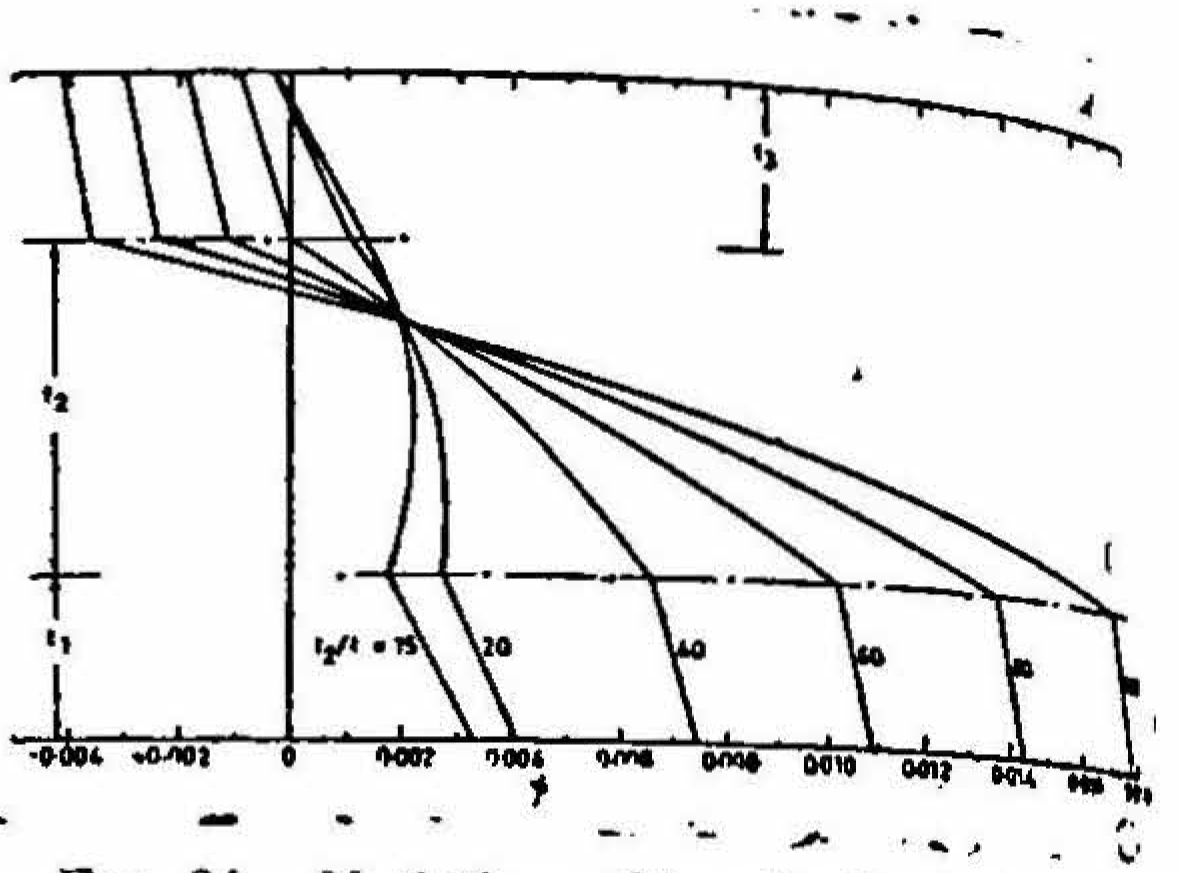

FIG. 9b. Variation of longitudinal displacement ($\bar{w}$) for different t_2/t values at $z/r_2 = 0{\cdot}06$ ($\eta_2 = 0{\cdot}03$, $\beta = 550$).

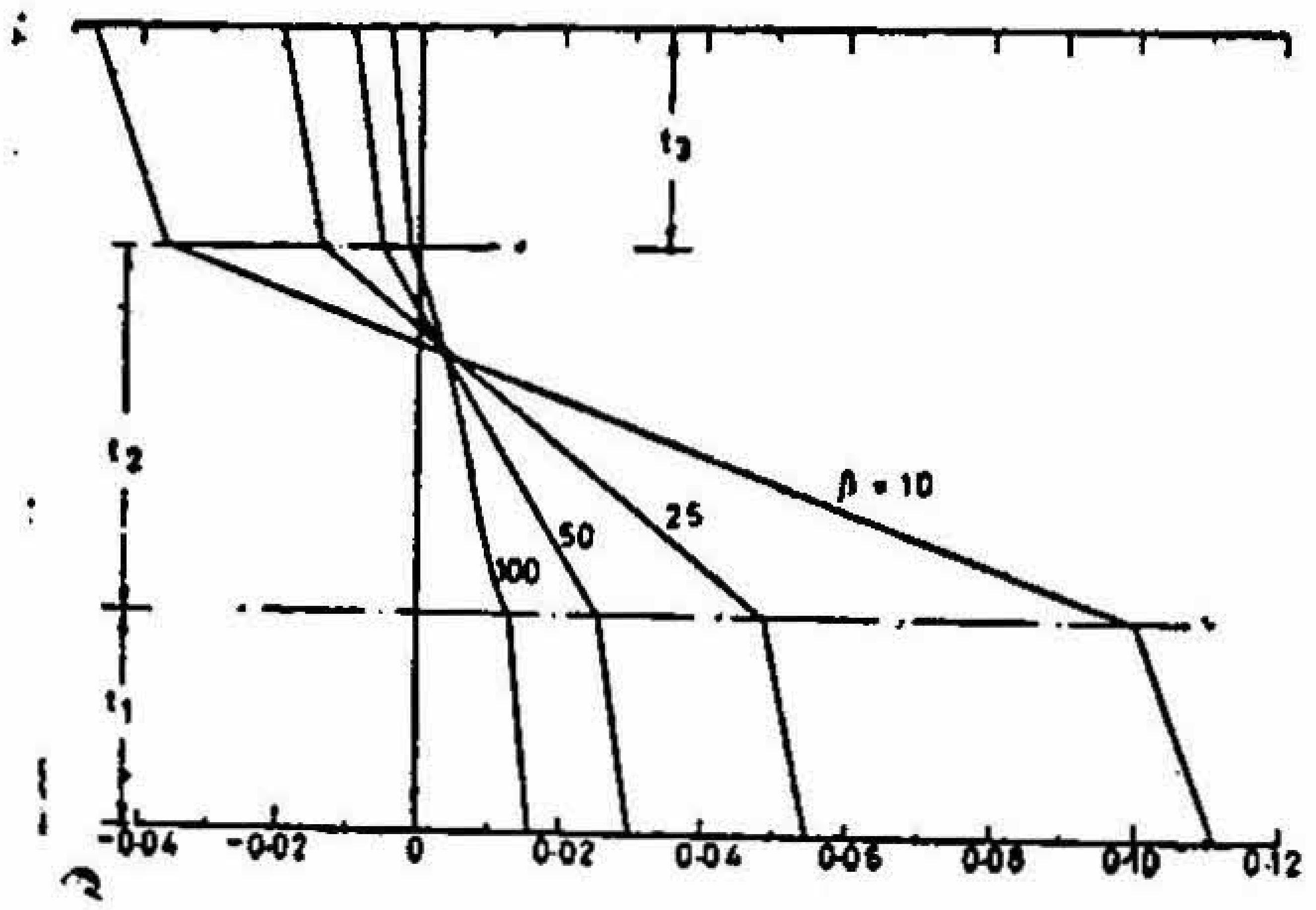

FIG. 10a. $\bar{w}$ variation for different values of β at $z/r_2 = 0{\cdot}06$ ($\eta_2 = 0{\cdot}03$, $t_2/t = 15$).

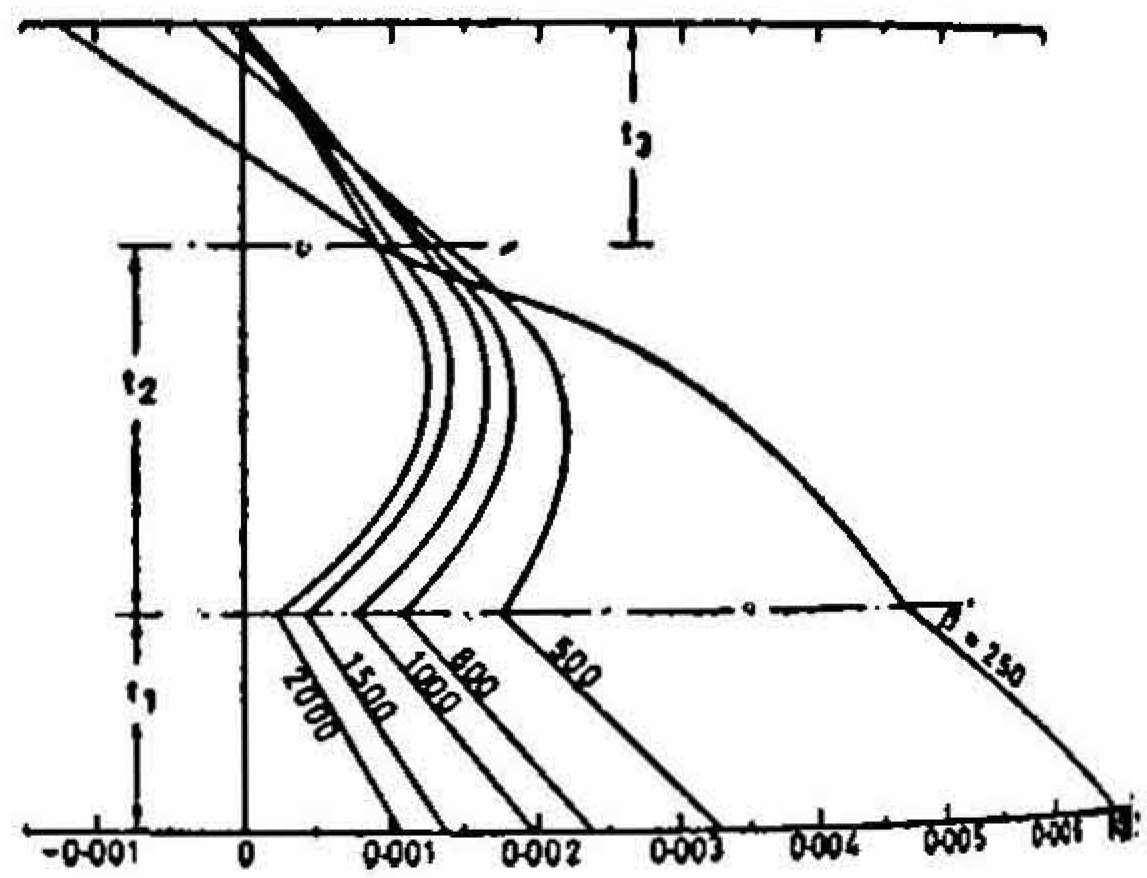

FIG. 10b. w variation for different values of β at $z/r_2 = 0{\cdot}06$ ($\eta_2 = 0{\cdot}03$, $t_2/t = 15$).

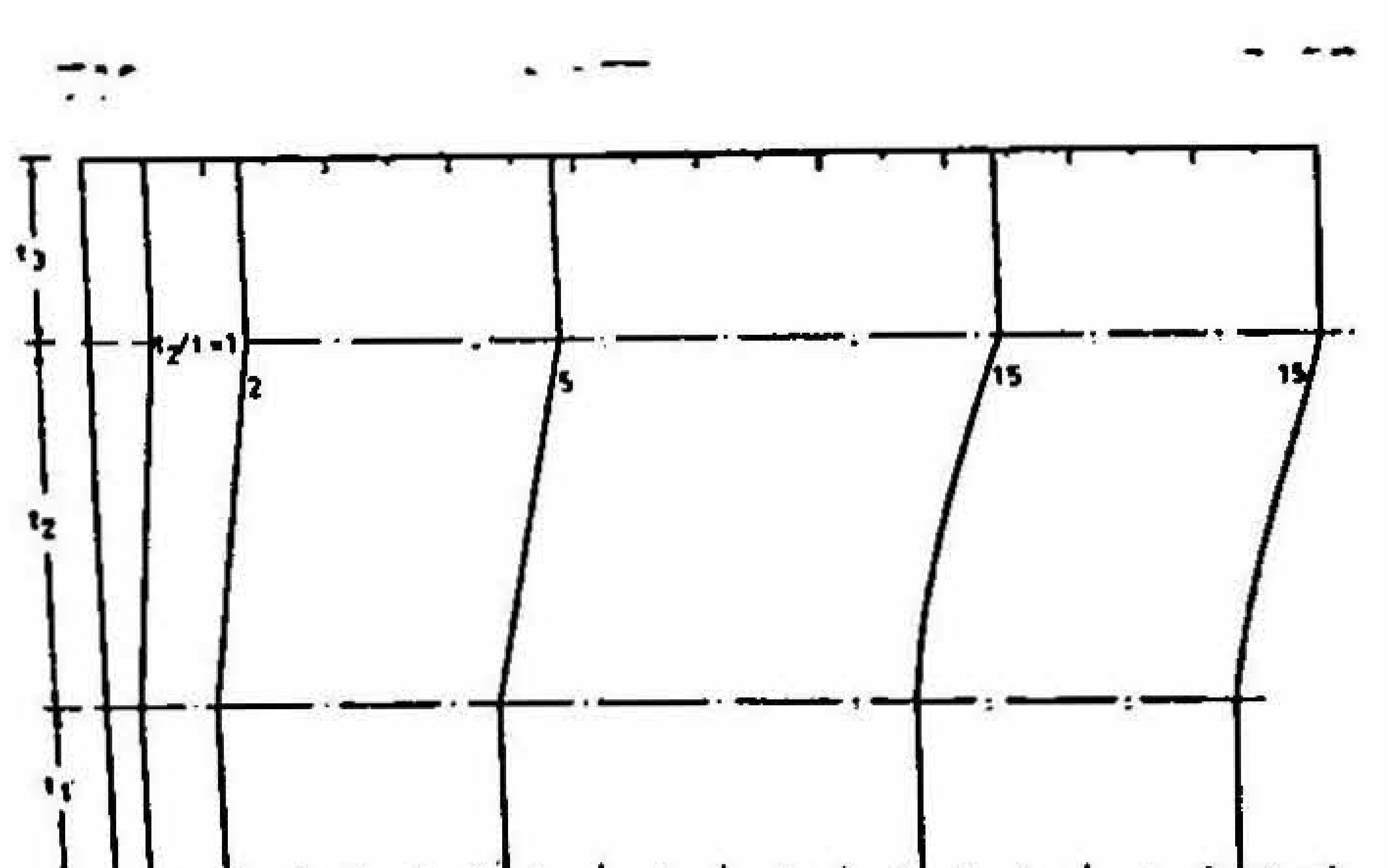

FIG. 11a. Variation of radial displacement ($\bar{u}$) for different values of t_2/t at $z/r_2 = 0{\cdot}0$ ($\eta_2 = 0{\cdot}03$, $\beta = 550$).

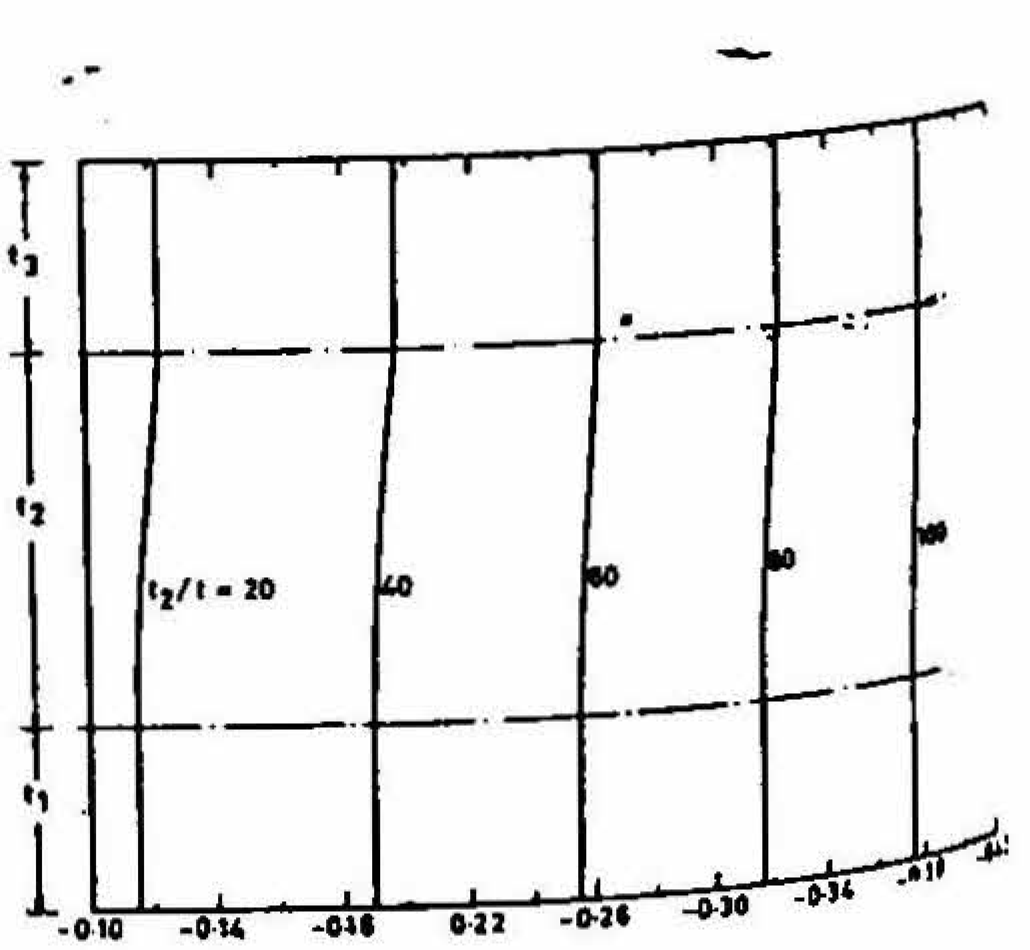

FIG. 11b. Variation of radial displacement ($\bar{u}$) for different t_2/t values at $z/r_2 = 0{\cdot}0$ ($\eta_2 = 0{\cdot}03$, $\beta = 550$).

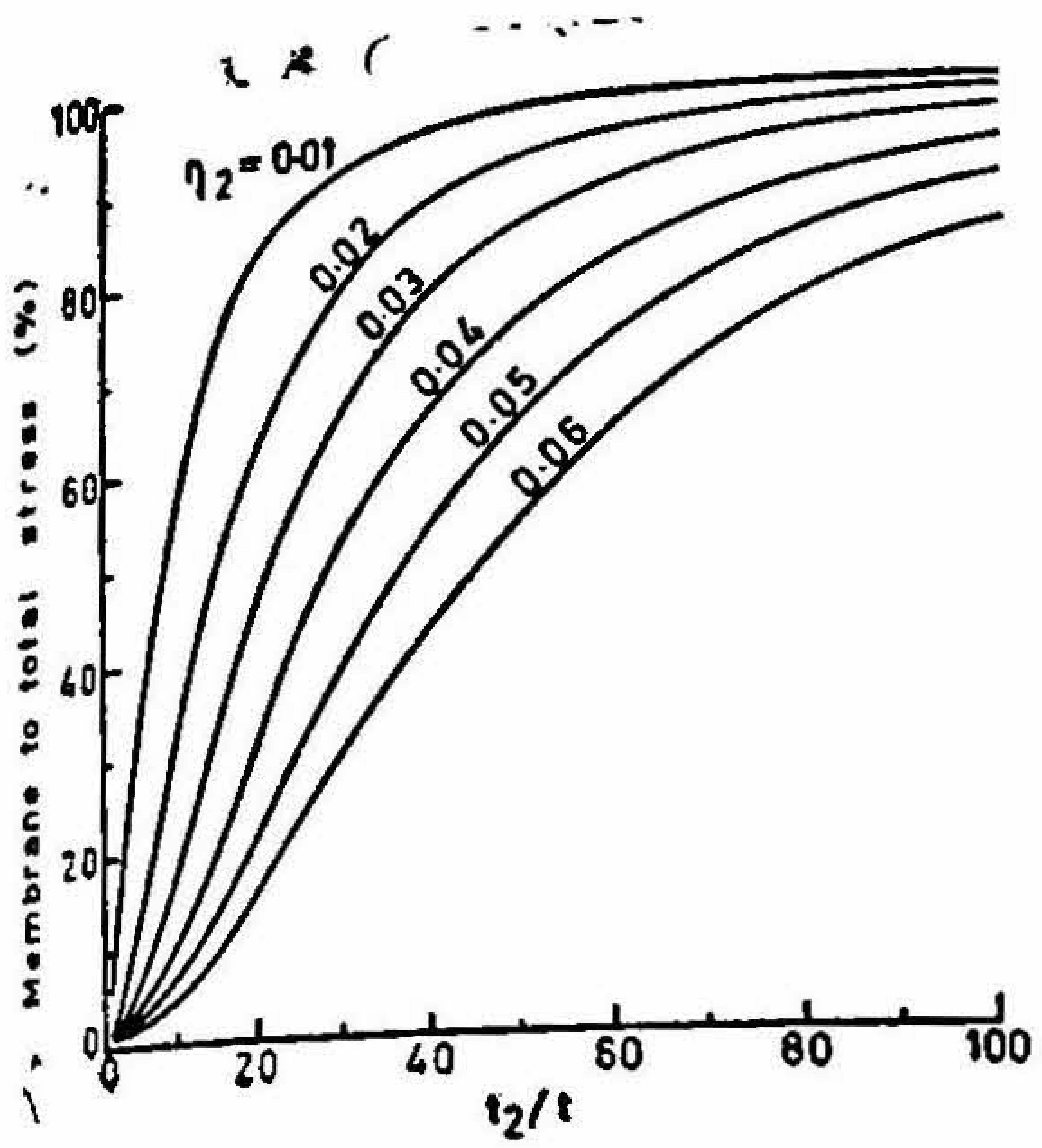

FIG. 12. Variation of percentage of membrane stress $(\bar{\sigma}_t)_m$ for different values of (t_2/t) and η_2 ($\beta = 550$).

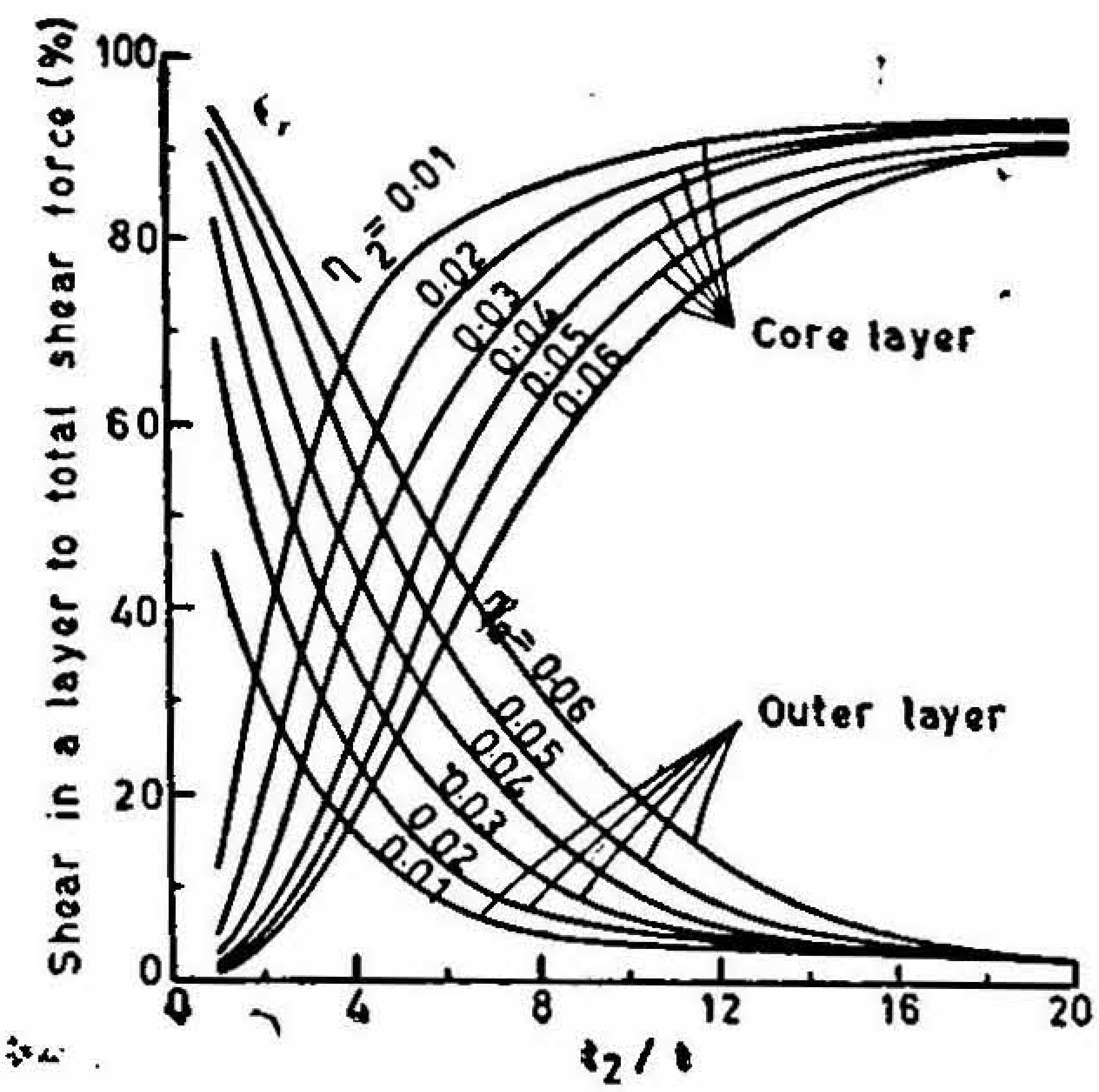

FIG. 13. Variation of percentage of shear force taken by core and outer facing for different (t_2/t) and η_2 values ($\beta = 550$).

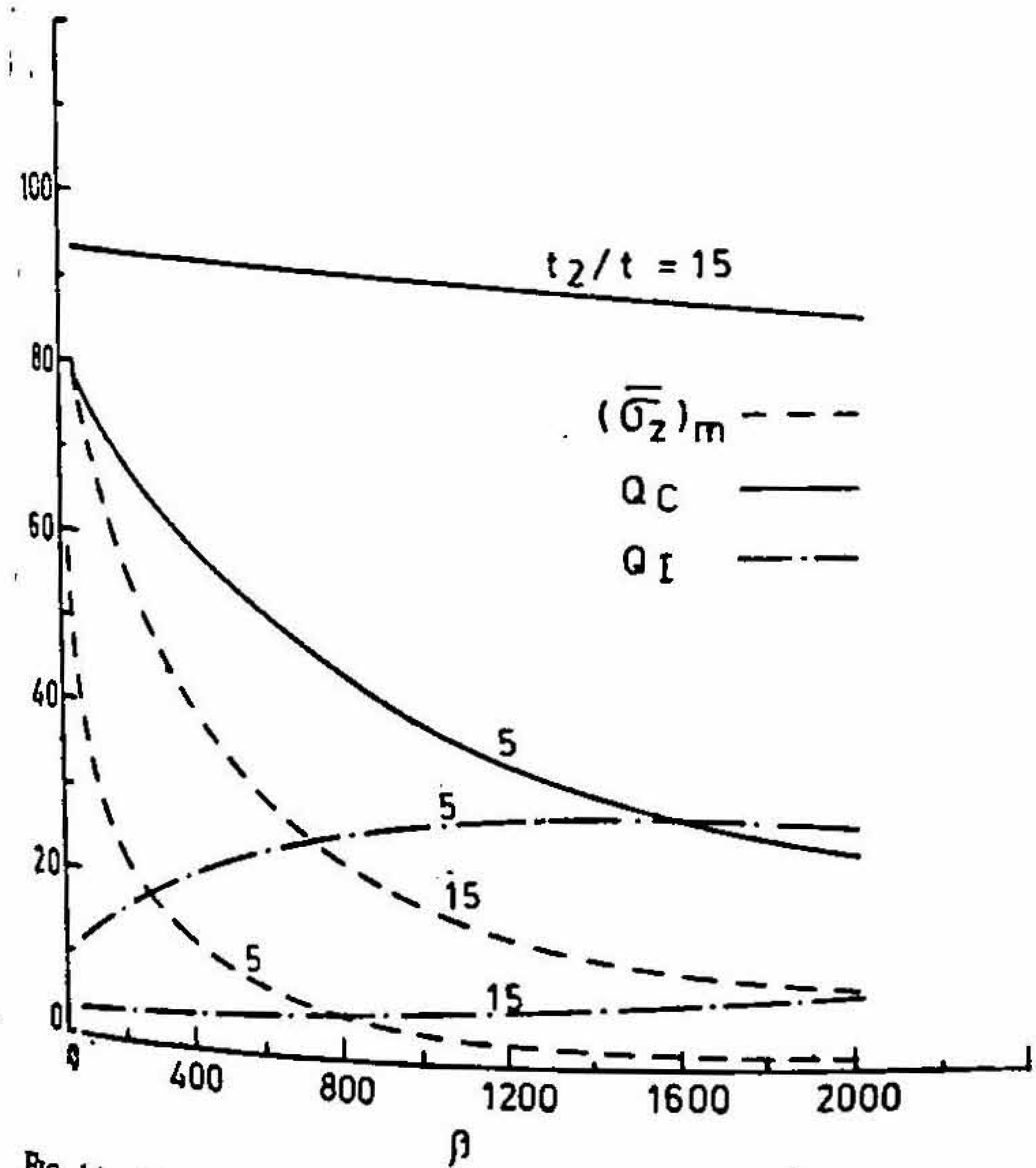

FIG. 14. Percentage shear force in core and inner facing at $z/r_2 = 0\cdot06$ and percentage of membrane stress $(\sigma_\theta)_m$ at $z/r_2 = 0\cdot0$ for $t_2/t = 5$ and 15 for different values of β ($\eta_2 = 0\cdot03$).

(σ_θ) and the percentage of shear force taken by each layer to total shear force, for the several parameters considered here, have been presented. The percentage of membrane stress to total stress $(\bar{\sigma}_z)_m$ is calculated from

$$(\bar{\sigma}_z)_m = \left[\frac{(\bar{\sigma}_z)_{r=r_1} + (\bar{\sigma}_z)_{r=r_2}}{2(\sigma_z)_{r=r_2}}\right] \times 100$$

In the above formula, absolute values of σ_z have been taken for the calculation of $(\bar{\sigma}_z)_m$. The shear force (Q_i) in each layer is first computed by taking the shear stress variation over the thickness of each layer. Then the percentage shear force taken by each layer $\bar{Q}_i$ is determined from

$$\text{Percentage shear force } \bar{Q}_i = \frac{Q_i \times 100}{\sum_{j=1}^{3} Q_j} \quad (i, j = 1, 2, 3,)$$

From figs. 2 and 3, it may be observed that the variation of longitudinal stress is linear in the facings while it is nonlinear in the core. Further, for β less then 25, the magnitude of $\bar{\sigma}_z$ in the core is comparable to that in the facings and hence σ_z cannot be neglected in the core. Also for a given value of β with increase of $t_{2/t}$ ratio, the external load is resisted predominantly by membrane action of the facings. A similar behaviour is observed when the value of β is decreased holding $t_{2/t}$ constant. It may be seen from figs. 4 and 5 that in general the variation of tangential stress ($\bar{\sigma}_\theta$) in the core is nonlinear and for small values of β its magnitude in the core is comparable with that in the facings.

Figures 6, 7 and 8 indicate that the shear stress ($\bar{\tau}_{rz}$) variation greatly depends on t_2/t and β values. For example, for a given value of t_2/t and with increase in β the shear stress distribution in the core becomes almost constant. Further, for a given value of β the shear stress distribution in the facings change from parabolic variation to linear variation when t_2/t is more than 15. From figs. 9*a* and *b*, it may be observed that the longitudinal displacement ($\bar{w}$) varies linearly for all values of t_2/t considered here. However, the variation in the core becomes linear only when t_2/t is greater than 40. Further, it may be seen from figs. 10*a* and *b* that the longitudinal displacement variation is linear in the facings for all values of β while in the core it becomes almost linear only when β is greater than 250. Figures 11(*a*) and (*b*) indicate that the radial displacement ($\bar{u}$) is almost constant over the thickness for the parameters considered here.

It may be seen from fig. 12 that for $\beta = 550$, the facings will have predominantly a membrane state of stress for small values of η_2 and large values of t_2/t ($\eta_2 < 0{\cdot}03$ and $t_2/t > 60$). Figure 13 indicates that for $\beta = 550$ the core takes about 80% to 90% of total shear when $\eta < 0{\cdot}03$ and $t_2/t > 12$. It may be observed from fig. 14 that for $\eta_2 = 0{\cdot}03$ and $t_2/t = 5$ and 15, the predominant membrane state of stress in the facings exists only for lesser values of β ($\beta < 100$) and the shear taken up by the core is almost independent of β for $t_2/t = 15$.

4. Conclusions

From the results presented here, the following conclusions can be drawn :

(i) The magnitude of $\bar{\sigma}_z$ and $\bar{\sigma}_\theta$ in the core are comparable with that of the facing for smaller values of β ($\beta < 25$) and hence these stresses cannot be neglected in the analysis as is usually done in the sandwich shell theories.

(ii) A near membrane state of stress in the facing is possible for certain range of t_2/t and β. It was also observed that for $t_2/t > 80$, a near membrane state of stress exists in facings irrespective of the range of values of β and η_2 considered here. For sandwich shell falling under this category the analysis can be greatly simplified by considering only the membrane force in the facing.

(iii) The core takes about 80% to 90% of the total shear for $t_2/t > 15$ irrespective of the range values of β considered here. Hence, it may be said that if $t_2/t > 15$ then a three-layered shell with a weak middle core can behave like a sandwich shell.

(iv) In general for the parameters considered here, the longitudinal displacement is essentially linear in the facings while it is significantly nonlinear in the core.

The above conclusions can be used to assess the application of sandwich shell theory for given dimensions of sandwich shell structure. Though the validity of the sandwich shell theory for circular cylindrical sandwich shell subjected to axisymmetric load has been established here, it is believed that these conclusions would also be useful for a general sandw'ch shell structure.

E_j	modulus of elasticity of j^{th} layer
G_2	shear modulus of core
I_0, I_1	modified Bessel functions of first kind, zero and first order, respectively
j	subscript to indicate any layer
K_0, K_1	modified Bessel functions of second kind, zero and first order, respectively
p	axisymmetric radial load per unit area
r_0, r_1, r_2	inner radius of layers 1, 2 and 3 respectively
r_3	outer radius of layer 3
r, z	cylindrical coordinate system
t_j	thickness of j^{th} layer ($j = 1, 2, 3$)
$\bar{u}, \bar{w}$	nondimensionalised radial and longitudinal displacement ($\bar{u} = (2G_2/pr_2)\,u$, $w = (2G_2/pr_2)\,w$)
a	variable of integration
$(\bar{\sigma}_r), (\bar{\sigma}_\theta), (\bar{\sigma}_z)$	nondimensionalised radial and tangential and longitudinal stress ($\bar{\sigma}_r = \sigma_{r/p}, \bar{\sigma}_\theta = \sigma_{\theta/p}, \sigma_z = \bar{\sigma}_{z/p}$)
$\bar{\tau}_{rz}$	nondimensionalised shear stress ($\bar{\tau}_{rz} = \tau_{rz/p}$)

ratio of modulus of elasticity of inner or outer layer (facing) with the middle layer (core)—

$\left(\frac{E_1}{E_2} = \frac{E_3}{E_2}\right)$

Poisson's ratio $\eta_2 = \frac{r_2 - r_1}{r_2}$

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