GRIFFITH CRACK IN AN INFINITE ELASTIC MEDIUM

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ABSTRACT

An asymmetrically loaded Griffith type strip crack in an infinite isotropic elastic medium is studied. Analysis is based on the finite Hilbert transform technique developed by Srivastava and Lowengrub[9] *and a result on singular integral equation for the change of order of Cauchy principal integrals referred in Ref* 2. *Closed form solutions are obtained for physically important quantities such as maximum shearing stress and normal component of displacement along the line of the crack.*

Key words: Transform method, Griffith crack, Dual integral equations, Hilbert transform technique, Maximum shearing stress, Closed form solution, Dislocation layer method.

1. INTRODUCTION

The two dimensional problem of an asymmetrically loaded Griffith crack in isotropic medium has been considered by Sneddon and Ejike.[1] Using Fourier transform technique Sneddon and Ejike[1] reduce the problem to that of solving four pairs of dual integral equations with sine, cosine kernel, and solve the dual integral equations by an "elementary method of dual integral equations"[10] based on Bessel functions. J. Tweed[6] has further investigated the problem considered by Sneddon and Ejike[1] for two coplanar Griffith cracks.

In this paper, we study the problem considered by Sneddon and Ejike.[1] We first reduce this problem to that of solving four pairs of dual integral equations and then give an exact solution using a modified technique of Srivastava and Lowengrub.[9] It is to be noted that the previous solutions presented in Sneddon and Ejike[1] and J. Tweed[6] have been confined to deriving the stress intensity factor and crack shape. In this study besides presenting simple closed form expression to normal component of displacement along the line of the crack, we also present solutions for complete stress field in the upper half plane at an arbitrary point. In a recent paper, R. W. Lardner[8] has studied this problem by using dislocation layer method.

In section 2, basic formulation of the governing equations of elasticity and general solution corresponding to an asymmetrical loading on the surface of the crack is derived, it is shown that the certain arbitrary functions entering into this general solution can be determined from the solution of four pairs of dual integral equations. In section 3, a simple expression for normal component of displacement along the line of the crack is presented. In the last section 4, closed form solutions for stress components corresponding to an asymmetrical loading on crack surface at an arbitrary point are derived.

2. Basic Formulation and Mathematical Analysis

A strip Griffith crack occupies the region $y = 0$, $|x| < c$ and is loaded internally in such a way that for $|x| \leqslant c$ we have the following conditions:

For the upper half plane:

$$\sigma_{yy}(x, 0+) = -p^{+}(x), \; \sigma_{xy}(x, 0+) = q^{+}(x) \tag{2.1}$$

while for the lower half plane:

$$\sigma_{yy}(x, 0-) = -p^{-}(x), \; \sigma_{xy}(x, 0-) = -q^{-}(x) \tag{2.2}$$

and $\sigma_{xx}, \sigma_{yy}, \sigma_{xy} \rightarrow 0$ as $\sqrt{x^2+y^2} \rightarrow \infty$. In the region of the x-axis outside of the crack, we assume that all the components of stress and displacement are continuous and normal component of displacement u_y satisfies the condition $u_y(x, 0+) > u_y(x, 0-)$.

Using the notation

$$\mathscr{F}_c[f(p, y), p \rightarrow x] = \frac{2}{\pi} \int_0^\infty f(p, y) \cos(px)\, dp$$

$$\mathscr{F}_s[f(p, y), p \rightarrow x] = \frac{2}{\pi} \int_0^\infty f(p, y) \sin(px)\, dp$$

to denote the Fourier sine and cosine transforms of $f(p, y)$ respectively, it can be shown[4] that the equations of elastic equilibrium have the solution of the form:

$$\begin{aligned} \frac{E}{(1+\eta)} u_x(x, y) &= \mathscr{F}_s[p^{-1}\{(1-\eta) G_{yy}(p, y) + \eta p^2 G(p, y)\}, p \rightarrow x] \\ &\quad - \mathscr{F}_c[p^{-1}\{(1-\eta) H_{yy}(p, y) + \eta p^2 H(p, y)\}, p \rightarrow x] \end{aligned} \tag{2.3}$$

$$\frac{E}{(1+\eta)} u_y(x, y)$$
$$= \mathscr{F}_s [\{(-2+\eta) H_y(p, y) + p^{-2}(1-\eta) H_{yyy}(p, y)\}, p \to x]$$
$$- \mathscr{F}_c [\{(-2+\eta) G_y(p, y) + p^{-2}(1-\eta) G_{yyy}(p, y)\}, p \to x] \tag{2.4}$$

where, the functions $G(p, y)$ and $H(p, y)$ both satisfy the equation $(d^2/dy^2 - p^2)^2 \phi(p, y) = 0$, $H_y(p, y)$ denotes $\partial H(p, y)/\partial y$, etc., E, η are the Young's modulus and Poisson's ratio of the material forming the infinite elastic body, and $G(p, y)$ and $H(p, y)$ have the following solutions:

For the upper half plane $y \geq 0$:

$$G(p, y) = [A^+(p) + yB^+(p)] e^{-py},$$
$$H(p, y) = [C^+(p) + yD^+(p)] e^{-py}. \tag{2.5}$$

For the lower half plane $y \leq 0$:

$$G(p, y) = [A^-(p) + B^-(p) y] e^{py},$$
$$H(p, y) = [C^-(p) + D^-(p) y] e^{py} \tag{2.6}$$

where, $A^{\pm}(p)$, $B^{\pm}(p)$, $C^{\pm}(p)$ and $D^{\pm}(p)$ are the arbitrary functions of p only.

Now following a method analogous to that of Sneddon and Ejike[1] and J. Tweed[6] and using the elementary results on integrals, it can be shown that the determination of arbitrary constants depends on the solution of four pairs of dual integral equations and four algebraic equations:

$$p^2 \triangle_2 A(p) = R[\triangle_2 p_e(u), \cos(pu)]$$
$$p^2 \triangle_2 C(p) = R[\triangle_2 p_0(u), \sin(pu)]$$
$$p[\triangle_2 B(p) - p\triangle_1 A(p)] = R[\triangle_1 q_0(u), \sin(pu)]$$
$$p[\triangle_2 D(p) - p\triangle_1 C(p)] = R[\triangle_1 q_e(u), \sin(pu)] \tag{2.7}$$

$$\left.\begin{aligned} &\mathscr{F}_c[p\triangle_1 D(p), p \to x] = Q_e(x), && 0 < x < c \\ &\mathscr{F}_c[\triangle_1 D(p), p \to x] = 0, && x > c \end{aligned}\right\} \tag{2.8}$$

$$\begin{aligned} &\mathscr{F}_c[p\{(1-2\eta)\triangle_2 B(p) + \triangle_1 A(p)\}, p \to x] = P_e(x), && 0 < x < c \\ &\mathscr{F}_c[\{(1-2\eta)\triangle_2 B(p) + \triangle_1 A(p)\}, p \to x] = 0, && x > c \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\mathscr{F}_s[p\{p\triangle_1 C(p) + (1-2\eta)\triangle_2 D(p)\}, p \to x] = p_0(x), && 0 < x < c \\ &\mathscr{F}_s[\{p\triangle_1 C(p) + (1-2\eta)\triangle_2 D(p)\}, p \to x] = 0, && x > c \end{aligned} \tag{2.10}$$

$$\mathscr{F}_S\,[p\,\{p\,\triangle_2 A\,(p) - 2\,(1-\eta)\ \triangle_1 B\,(p)\},\ p \to x] = -\ Q_0\,(x),\quad 0 < x < c$$
$$\mathscr{F}_s\,[\{p\,\triangle_2 A\,(p) - 2\,(1-\eta)\ \triangle_1 B\,(p)\},\ p \to x] = 0,\qquad x > c \tag{2.11}$$

where

$$Q_e\,(x) = -\ \triangle_2 q_e\,(x) + V\,[u\,\triangle_2 p_0\,(u),\ x],$$
$$P_e\,(x) = 2\,(1-\eta)\ \triangle_1 p_e\,(x) + (1-2\eta)\ V\,[u\,\triangle_1 q_0\,(u),\ x],$$
$$P_0\,(x) = 2\,(1-\eta)\ \triangle_1 p_0\,(x) + (1-2\eta)\ xV\,[\triangle_1 q_e\,(u),\ x],$$
$$Q_0\,(x) = 2\,(1-\eta)\ \triangle_2 q_0\,(x) - (1-2\eta)\ xV\,[\triangle_2 p_e\,(u),\ x],$$
$$\triangle_1 g\,(p) = g^+\,(p) + g^-\,(p),\quad \triangle_2 g\,(p) = g^+\,(p) - g^-\,(p),$$
$$R\,[T\,(u),\ f\,(p,u)] = \int_0^c T\,(u)\ f\,(p,u)\,du,$$
$$V\,[T\,(u),\ x] = \frac{2}{\pi}\int_0^c [T\,(u)/(u^2 - x^2)]\,du,$$
$$p_e^{\pm}\,(x) = \tfrac{1}{2}\,[p^{\pm}\,(x) \times p^{\pm}\,(-x)],\quad p_0^{\pm}\,(x) = \tfrac{1}{2}\,[p^{\pm}\,(x) - p^{\pm}\,(-x)],$$
$$q_e^{\pm}\,(x) = \tfrac{1}{2}\,[q^{\pm}\,(x) \times q^{\pm}\,(-x)],\quad q_0^{\pm}\,(x) = \tfrac{1}{2}\,[q^{\pm}\,(x) - q^{\pm}\,(-x)]. \tag{2.12}$$

Using a method analogous to that of Srivastava and Lowengrub,[9] the solutions to the dual integral equations (2.7) through (2.11) are found as follows:

$$p\,\triangle_1 D\,(p) = R\,[g_1\,(t),\ \sin\,(pt)],\ g_1\,(0) = 0,$$
$$p\,[p\,\triangle_1 B\,(p) + (1-2\eta)\ \triangle_2 A\,(p)] = R\,[g_2\,(t),\ \sin\,(pt)],\ g_2\,(0) = 0,$$
$$p\,[p\,\triangle_1 C\,(p) + (1-2\eta)\ \triangle_2 D\,(p)] = R\,[g_3\,(t),\ (1-\cos pt)],\ g_3\,(0) = 0,$$
$$p\,[2\,(1-\eta)\ \triangle_1 B\,(p) - p\,\triangle_2 A\,(p)] = R\,[g_4\,(t),\ (1-\cos pt)],\ g_4\,(0) = 0. \tag{2.13}$$

In addition, the conditions at infinity are to be satisfied by using the equations:

$$\int_0^c \triangle_2 p_e\,(u)\,du = 0,\quad \text{and}\quad \int_0^c \triangle_1 q_e\,(u)\,du = 0$$

where, the functions $g_k\,(t)$, $(k = 1, 2, 3, 4)$ are given by:

$$g_1\,(t) = -\ \mathscr{L}_1\,[\triangle_2 q_e\,(u),\ t] + \triangle_2 p_0\,(t),$$
$$g_2\,(t) = 2\,(1-\eta)\ \mathscr{L}_1\,[\triangle_1 p_e\,(u),\ t] + (1-2\eta)\ \triangle_1 q_0\,(t),$$
$$g_3\,(t) = 2\,(1-\eta)\ t^{-1}\,\mathscr{L}_1\,[u\,\triangle_1 p_0\,(u),\ t] + (1-2\eta)\ \triangle_1 q_e\,(t),$$
$$g_4\,(t) = 2\,(1-\eta)\ t^{-1}\,\mathscr{L}_1\,[u\,\triangle_2 q_0\,(u),\ t] + (1-2\eta)\ \triangle_2 p_e\,(t) \tag{2.14}$$

where

$$\mathcal{L}_1 [T(u), t] = \frac{2}{\pi} \frac{t}{\sqrt{(c^2 - t^2)}} \int_0^c \frac{T(u) \sqrt{(c^2 - u^2)}\, du}{(t^2 - u^2)} . \tag{2.15}$$

It is now clear that all the eight arbitrary constants $A^{\pm}(p)$, $B^{\pm}(p)$, $C^{\pm}(p)$ and $D^{\pm}(p)$ can be explicitly determined by using equations (2.7) and (2.13).

3. Normal Component of Displacement

Using equations (2.4), (2.5) and (2.6) and the result on the change of order of Cauchy principal integral referred in Tricomi[2] and some simple results on integrals it can be shown that:

$$\begin{aligned} &\frac{E}{(1+\eta)} u_y (x, o_{\pm}) \\ &\quad = (1-\eta) Z_1(x) + (2\eta - 1) Z_2(x)/2 \\ &\qquad \pm 2(1-\eta) \{I_3 [\triangle_1 p_e (u), x] + I_4 [\triangle_1 p_0 (u), x]\} \\ &\qquad + (2\eta - 1) \{I_1 [\triangle_2 q_e (u), x] - I_2 [\triangle_2 q_0 (u), x]\} \end{aligned} \tag{3.1}$$

where

$$Z_1(x) = \frac{1}{\pi} \int_0^c \left\{ \triangle_2 p_0 (u) \log \left| \frac{x+u}{x-u} \right| - \triangle_2 p_e (u) \log \left| \frac{x^2 - u^2}{x^2} \right| \right\} du,$$

$$Z_2(x) = \int_0^c \{\pm\, \triangle_1 q_e (u) H(x-u) \mp \triangle_1 q_0 (u) H(u-x)\}\, du \tag{3.2}$$

and (see Appendix)

$$I_1 [\triangle_2 q_e (u), x]$$

$$= \begin{cases} \dfrac{1}{2} \displaystyle\int_0^x \triangle_2 q_e (u)\, du, & 0 < x < c \\[2ex] \dfrac{1}{\pi} \displaystyle\int_0^c \triangle_2 q_e (u) \cos^{-1} [\sqrt{(x^2 - c^2)}/\sqrt{(x^2 - u^2)}]\, du, & x > c \end{cases} \tag{3.3}$$

$$I_2[\triangle_2 q_0(u), x] = \begin{cases} -\frac{1}{2}\int_0^x \triangle_2 q_0(u)\,du + \frac{1}{\pi}\int_0^c \triangle_2 q_0(u)\cos^{-1}(u/c)\,du, & 0 < x < c \\ \frac{1}{\pi}\int_0^c \triangle_2 q_0(u)\{\tan^{-1}[u\sqrt{(x^2-c^2)}/x\sqrt{(c^2-u^2)}] \\ \quad -\tan^{-1}[u/\sqrt{(c^2-u^2)}]\}\,du, & x > c \end{cases} \tag{3.4}$$

$$I_3[\triangle_1 p_e(u), x] = H(c-x)\frac{1}{2\pi}\int_0^c \triangle_1 p_e(u)\log|f_1(x,u)|\,du,$$

$$f_1(x,u) = [\sqrt{(c^2-x^2)} + \sqrt{(c^2-x^2)}]/[\sqrt{(c^2-x^2)} - \sqrt{(c^2-u^2)}] \tag{3.5}$$

$$I_4[\triangle_1 p_0(u), x] = H(c-x)\frac{1}{2\pi}\int_0^c \triangle_1 p_0(u)\log|f_2(x,u)|\,du,$$

$$f_2(x,u) = [u\sqrt{(c^2-x^2)} + x\sqrt{(c^2-u^2)}]/[u\sqrt{(c^2-x^2)} - x\sqrt{(c^2-u^2)}] \tag{3.6}$$

and H is the usual Heaviside unit function.

4. STRESS COMPONENTS IN THE UPPER HALF PLANE

The evaluation of the stress components in the upper half plane depends crucially upon the four functions $W_j(w)$, $(j = 1, 2, 3, 4)$ and given by the following equations:

$$W_1(w) = U[pA^+(p), w], \qquad W_2(w) = U[B^+(p), w]$$
$$W_3(w) = U[pC^+(p), w], \qquad W_4(w) = U[D^+(p), w] \tag{4.1}$$

where

$$\left.\begin{aligned} U[f(p), w] &= \frac{2}{\pi}\int_0^\infty pf(p)\,e^{-pw}\,dp \\ w + y &= ix, \end{aligned}\right\} \tag{4.2}$$

The functions $W_j(w)$ $(j = 1, 2, 3, 4)$ can be evaluated by using elementary results on integrals. The expressions for $W_j(w)$, $(j = 1, 2, 3, 4)$ are found as follows:

$$\begin{aligned}
W_1(w) &= H^{(1)}[\triangle_2 p_e(u), w] + H^{(2)}[\triangle_1 p_e(u), w], \\
W_2(w) &= H^{(1)}[\triangle_2 p_e(u), w] + G^{(1)}[\triangle_1 q_0(u), w] \\
&\quad + H^{(2)}[\triangle_1 p_e(u), w] + G^{(2)}[\triangle_2 q_0(u), w], \\
W_3(w) &= G^{(1)}[\triangle_2 p_0(u), w] + G^{(2)}[\triangle_1 p_0(u), w], \\
W_4(w) &= G^{(1)}[\triangle_2 p_0(u), w] - H^{(1)}[\triangle_1 q_e(u), w] \\
&\quad + G^{(2)}[\triangle_1 p_0(u), w] - H^{(2)}[\triangle_2 q_e(u), w]
\end{aligned} \tag{4.3}$$

where

$$H^{(1)}[T(u), w] = \frac{w}{\pi} \int_0^c \frac{T(u)\,du}{(w^2 + u^2)},$$

$$H^{(2)}[T(u), w] = \frac{w}{\pi\sqrt{(w^2 + c^2)}} \int_0^c \frac{\sqrt{(c^2 - u^2)}\,T(u)\,du}{(w^2 + u^2)},$$

$$\begin{aligned}
G^{(1)}[T(u), w] &= w^{-1} H^{(1)}[uT(u), w], \\
G^{(2)}[T(u), w] &= w^{-1} H^{(2)}[uT(u), w].
\end{aligned} \tag{4.4}$$

In the notations (4.1), using equations (2.4), (2.5) and Hooke's law of stress and strain, it can be shown that

$$[\sigma_{xx}(x, y+) + \sigma_{yy}(x, y+)] = 2Re\,[iW_4(\bar{w}) - W_2(\bar{w})] \tag{4.5}$$

$$\begin{aligned}
\tfrac{1}{2}[\sigma_{yy}(x, y+) - \sigma_{xx}(x, y+)] + i\sigma_{xy}(x, y+) \\
= -[W_1(\bar{w}) - W_2(\bar{w})] + i[W_3(\bar{w}) - W_4(\bar{w})] \\
+ y\frac{d}{dz}[W_4(\bar{w}) + iW_2(\bar{w})].
\end{aligned} \tag{4.6}$$

Using the results obtained in equations (4.3) into equations (4.5), and (4.6), it can be shown that:

$$\begin{aligned}
[\sigma_{xx}(x, y+) + \sigma_{yy}(x, y+)] \\
= Re\,\{I^{(1)}[i\triangle_2 p(u) - \triangle_1 q(u), z] + I^2[i\triangle_1 p(u) - \triangle_2 q(u), z]\}
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
[\sigma_{yy}(x, y+) - \sigma_{xx}(x, y+)] + i2\,\sigma_{xy}(x, y+) \\
= I^{(1)}[\triangle_1 q(u), z] + I^{(2)}[\triangle_2 q(u), z]
\end{aligned}$$

$$+ y \frac{d}{dz}\{I^{(1)}[\triangle_2 p(u) + i\triangle_1 q(u), z] + I^{(2)}[\triangle_1 p(u) + i\triangle_2 q(u), z]\} \tag{4.8}$$

where,

$$I^{(1)}[T(u), z] = \frac{1}{\pi}\int_{-c}^{c}\frac{T(u)\,du}{(u-z)},$$

$$I^{(2)}[T(u), z] = \frac{1}{\pi}\frac{1}{\sqrt{(c^2-z^2)}}\int_{-c}^{c}\frac{T(u)\sqrt{(c^2-u^2)}\,du}{(u-z)}. \tag{4.9}$$

It can be verified that the expressions obtained in equations (4.7) and (4·8) are in complete agreement with those obtained by R. W. Lardner[8] using dislocation layer method. Special cases of the type of point forces discussed in G. R. Irwin[5] and the constant normal and shear loadings on the crack surfaces discussed in S. M. Sharfuddin[11] can also be studied using equations (4.7) and (4.8).

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APPENDIX

Consider the expressions given by the following equations:

$$I_1[\triangle_2 q_e(u), x] = \frac{1}{\pi^2}\int_0^c \frac{t}{\sqrt{(c^2-t^2)}}\log\left|\frac{x+t}{x-t}\right| dt \int_0^c \frac{\sqrt{(c^2-u^2)}\,\triangle_2 q_e(u)\,du}{(t^2-u^2)} \tag{A.1}$$

$$I_2[\triangle_2 q_0(u), x] = \frac{1}{\pi^2}\int_0^c \frac{1}{\sqrt{(c^2-t^2)}}\log\left|\frac{x^2-t^2}{x^2}\right| dt \int_0^c \frac{u\sqrt{(c^2-u^2)}\,\triangle_2 q_0(u)\,du}{(t^2-u^2)} \tag{A.2}$$

$$I_3[\triangle_1 p_e(u), x] = \frac{1}{\pi}\int_0^c \frac{tH(t-x)}{\sqrt{(c^2-t^2)}}\,dt \int_0^c \frac{\sqrt{(c^2-u^2)}\,\triangle_1 p_e(u)\,du}{(t^2-u^2)} \tag{A.3}$$

$$I_4[\triangle_1 p_0(u), x] = \frac{1}{\pi}\int_0^c \frac{H(t-x)\,dt}{\sqrt{(c^2-t^2)}} \int_0^c \frac{u\sqrt{(c^2-u^2)}\,\triangle_1 p_0(u)\,du}{(t^2-u^2)} \tag{A.4}$$

Now we shall evaluate the expressions $I_k[\ \ , x]$, $(k = 1, 2, 3, 4)$ defined in equations (A.1)–(A.4) for the intervals $0 < x < c$ and $c < x < \infty$.

Applying the result referred in Tricomi to $D_x I_1[\triangle_2 q_e(u), x]$, $D_x = d/dx$ in the interval $0 < x < c$ for the change of order of Cauchy principal integrals and using the result

$$\frac{1}{\pi}\int_a^b \frac{t\,dt}{\sqrt{(t^2-a^2)(b^2-t^2)}\,(t^2-y^2)} = \begin{cases} [(a^2-y^2)(b^2-y^2)]^{-1/2}, & 0 \leq y < a \\ 0, & a < y < b \\ -[(y^2-a^2)(y^2-b^2)]^{-1/2}, & b < y < \infty \end{cases}$$

and noting that in the interval $c < x < \infty$ direct change of order of integration is permissible, we obtain the following equation:

$$D_x I_1[\triangle_2 q_e(u), x] = \begin{cases} \frac{1}{2}\triangle_2 q_e(x), & 0 < x < c \\ \dfrac{1}{\pi}\dfrac{x}{\sqrt{(x^2-c^2)}}\displaystyle\int_0^c \frac{\sqrt{(c^2-u^2)}\,\triangle_2 q_e(u)\,du}{(u^2-x^2)}, & x > c. \end{cases} \tag{A.5}$$

From equation (A.5) it is clear that:

$$I_1[\triangle_2 q_e(u), x] = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^x \triangle_2 q_e(u)\,du + k_1(u), & 0 < x < c, \text{(i)} \\ -\dfrac{1}{\pi}\displaystyle\int_0^c \triangle_2 q_e(u)\tan^{-1}\left[\frac{\sqrt{(x^2-c^2)}}{\sqrt{(c^2-u^2)}}\right] du + k_2(u), & c < x < \infty, \text{(ii)} \end{cases} \tag{A.6}$$

In the evaluation of (A.6) we have used the result

$$\int \frac{x\,dx}{(u^2 - x^2)\sqrt{(x^2 - c^2)}} = -\frac{1}{\sqrt{(c^2 - u^2)}} \tan^{-1}\left[\frac{\sqrt{(x^2 - c^2)}}{\sqrt{(c^2 - u^2)}}\right].$$

Observing the fact that as $x \to \infty$, $I_1[\triangle_2 q_e(u), x]$ of (A.1) tends to zero, and hence as x approaches in the limit to infinity (A.6) (ii) should also tend to zero. This implies that:

$$k_2(u) = \tfrac{1}{2} \int_0^c \triangle_2 q_e(u)\,du. \tag{A.7}$$

Also as x approaches in the limit to c from left and right (A.6) (ii) and (A.6) (i) should be equal, this implies that $k_1(u) = 0$. Thus finally it is found that:

$$I_1[\triangle_2 q_e(u), x] = \begin{cases} \dfrac{1}{2} \displaystyle\int_0^c \triangle_2 q_e(u)\,du, & 0 < x < c \\[2ex] \dfrac{1}{\pi} \displaystyle\int_0^c \triangle_2 q_e(u) \cos^{-1}[\sqrt{(x^2 - c^2)}/\sqrt{(x^2 - u^2)}]\,du, & x > c. \end{cases} \tag{A.8}$$

In a similar manner, using the result

$$\int \frac{dx}{(u^2 - x^2)\sqrt{(x^2 - c^2)}} = -\frac{1}{u\sqrt{(c^2 - u^2)}} \tan^{-1}[u\sqrt{(x^2 - c^2)}/x\sqrt{(c^2 - u^2)}]$$

it can be shown that

$$I_2[\triangle_2 q_0(u), x] = \begin{cases} -\dfrac{1}{2} \displaystyle\int_0^c \triangle_2 q_0(u)\,du + \dfrac{1}{\pi} \displaystyle\int_0^c \triangle_2 q_0(u) \cos^{-1}(u/c)\,du, & 0 < x < c \\[2ex] \dfrac{1}{\pi} \displaystyle\int_0^c \triangle_2 q_0(u) \left\{\tan^{-1}\left[\frac{u\sqrt{(x^2 - c^2)}}{x\sqrt{(c^2 - u^2)}}\right] - \tan^{-1}\left[\frac{u}{\sqrt{(c^2 - u^2)}}\right]\right\} du, & c < x < \infty \end{cases} \tag{A.9}$$

Also using the results

$$\int_x^c \frac{t\,dt}{(u^2 - t^2)\sqrt{(c^2 - t^2)}}$$

$$= -\frac{1}{2\sqrt{(c^2 - u^2)}} \log \left| \frac{\sqrt{(c^2 - x^2)} + \sqrt{(c^2 - u^2)}}{\sqrt{(c^2 - x^2)} - \sqrt{(c^2 - u^2)}} \right|,$$

$$\int_x^c \frac{dt}{(u^2 - t^2)\sqrt{(c^2 - t^2)}}$$

$$= -\frac{1}{2u\sqrt{(c^2 - u^2)}} \log \left| \frac{u\sqrt{(c^2 - x^2)} + x\sqrt{(c^2 - u^2)}}{u\sqrt{(c^2 - x^2)} - x\sqrt{(c^2 - u^2)}} \right|. \quad (A.10)$$

It can be shown that

$$I_3[\triangle_1 p_e(u), x] = H(c - x)\frac{1}{2\pi}\int_0^c \triangle_1 p_e(u) \log |f_1(x, u)|\, du,$$

$$f_1(x, u) = [\sqrt{(c^2 - x^2)} + \sqrt{(c^2 - u^2)}] / [\sqrt{(c^2 - x^2)} - \sqrt{(c^2 - u^2)}],$$

$$I_4[\triangle_1 p_0(u), x] = H(c - x)\frac{1}{2\pi}\int_0^c \triangle_1 p_0(u) \log |f_2(x, u)|\, du,$$

$$f_2(x, u) = [u\sqrt{(c^2 - x^2)} + x\sqrt{(c^2 - u^2)}]/[u\sqrt{(c^2 - x^2)} - x\sqrt{(c^2 - u^2)}] \quad (A.11)$$

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