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# Scattering of surface waves by a submerged fixed vertical plate in water of finite depth

S. K. GOSWAMI
Department of Mathematics, Presidency College, Calcutta 700 073, India.

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**Abstract**

The problem of scattering of surface waves by a submerged fixed vertical plate in water of finite depth is solved by reducing the problem to the approximate solution of an integral equation formed by using a suitable Green's function and applying Green's theorem over the fluid region. The reflection and transmission coefficients are obtained by neglecting the exponentially small terms.

Key words : Scattering, surface waves, fluid.

## 1. Introduction

The problem of scattering or generating of surface waves by one or more plates placed in the same vertical plane in deep water was solved by Lewin[1], Mei[2], Evans[3] and others, by various techniques but the corresponding problems concerning a plate in water of finite depth appear to have not been considered so far. However, for obstacles in the form of horizontal cylinder either partly immersed or completely submerged in water of finite depth have been considered by Rhodes-Robinson[4] and Davis and Hood[5].

In this paper the scattering problem of surface waves for a submerged fixed vertical plate in water of depth $h$ is considered. Applying Green's theorem in the fluid region, the velocity potential at any point is obtained in terms of the unknown difference of potentials across the plate. This unknown function is seen to satisfy a certain integral equation of the second kind. Its kernel is found to be small with large $h$ and of the order of $(b/h)^2$, the exponentially small terms for large $h$ are being neglected. Expanding the unknown function and the kernel in series involving different powers of $(b/h)$, an attempt is made to solve it approximately. The complex reflection and transmission coefficients are then obtained in series powers of $(b/h)$. As $h$ tends to infinity the results coincide with those for the infinite depth case given by Evans[3]. The depth correction terms for the reflection and transmission co-efficients are found to be of the order of $(b/h)^4$.

## 2. Statement of the problem

We consider the two-dimensional scattering of surface waves by a submerged fixed vertical plate in water of depth $h$ and use a coordinate system in which the $y$-axis is taken to be vertically downwards, the mean free surface is the plane $y = 0$, the position of the plate is given by $x = 0$, $a \leq y \leq b$. (*cf.* fig. 1).

Assuming the fluid to be inviscid and incompressible and the motion to be irrotational and simple harmonic in time with circular frequency $\sigma$ and small amplitude, a velocity potential exists and it can be expressed as Re $\{\Phi(x, y)\, e^{-i\sigma t}\}$, satisfying the equations

$$\nabla^2 \Phi = 0, \text{ in the fluid region,} \tag{2.1}$$

$$K\Phi + \frac{\partial \Phi}{\partial y} = 0, \text{ on the free surface } y = 0, \tag{2.2}$$

where $K = \sigma^2/g$,

$$\frac{\partial \Phi}{\partial x} = 0, \text{ on the plate } x = 0, \quad a < y < b, \tag{2.3}$$

because the plate is fixed and vertical,

$$\frac{\partial \Phi}{\partial y} = 0, \text{ at the bottom of the fluid.} \tag{2.4}$$

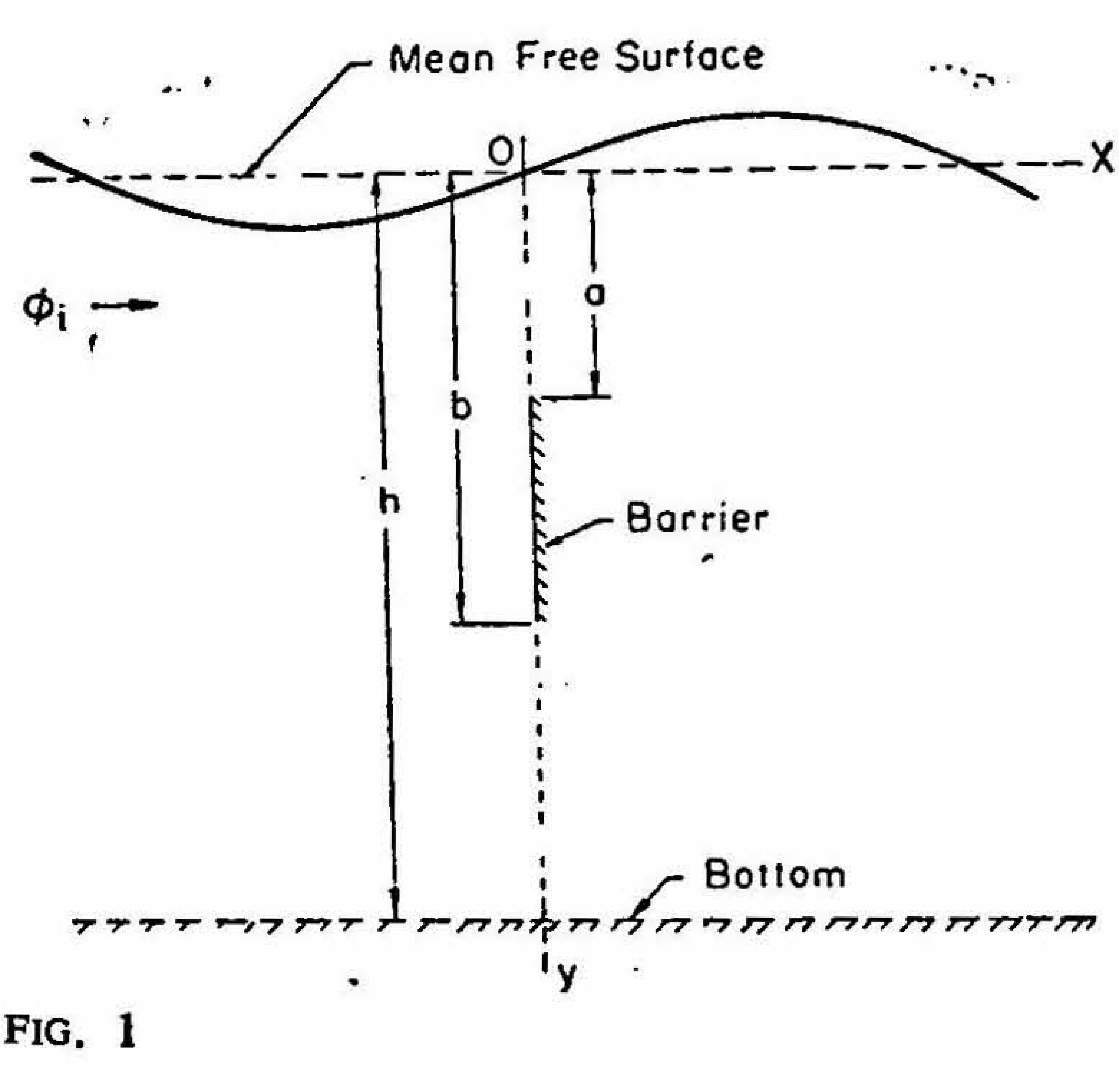


FIG. 1

Let a train of surface waves be made incident from negative infinity in the fluid ; it will then be partially reflected by the plate and transmitted over and below it.

If the incident wave is represented by

$$\varphi_i(x, y) = \frac{\cosh k_0(h - y)}{\cosh k_0 h} e^{ik_0 x}, \tag{2.5}$$

where the wave number $k_0$ is defined by

$$k_0 \tanh k_0 h = K, \tag{2.6}$$

then the total field is $\Phi = \varphi_i + \varphi$, where $\varphi$ also satisfies the Laplace's equation (2.1) the free surface condition (2.2).

In addition, we assume that $\varphi$ and its derivatives are uniformly bounded as $(x^2 + y^2)^{1/2} \to \infty$, and

$$\varphi, \{x^2 + (y - a)^2\}^{1/2} \operatorname{grad} \varphi, \text{ and } \{x^2 + (y - b)^2\}^{1/2} \operatorname{grad} \phi$$

are bounded as $\{x^2 + (y - a)^2\}^{1/2} \to 0$ or $\{x^2 + (y - b)^2\}^{1/2} \to 0$, the latter conditions being the so-called 'edge conditions' which are to be satisfied near the upper and lower edges of the plate.

The reflection and transmission coefficients $R$ and $T$ satisfy the following relations obtained by considering the behaviour at positive and negative infinity,

$$\varphi(x, y) \sim (T - 1)\, \varphi_i(x, y) \text{ as } x \to +\infty, \tag{2.7}$$

$$\varphi(x, y) \sim R\, \varphi_i(-x, y) \text{ as } x \to -\infty, \tag{2.8}$$

The expressions (2.7) and (2.8) ensure that $\varphi(x, y)$ represents an outgoing wave at infinity, *i.e.*, satisfies the 'radiation' conditions at infinity.

## 3. Formation of the integral equation

The generalized Green's function $G(x, y\,; \xi, \eta)$ satisfying

$$\nabla^2 G \equiv \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = -2\pi\delta(x - \xi)\,\delta(y - \eta), \quad y \geq 0, \tag{3.1}$$

$$KG + \frac{\partial G}{\partial y} = 0, \quad \text{on } y = 0, \tag{3.2}$$

$$\frac{\partial G}{\partial y} = 0, \quad \text{on } y = h, \tag{3.3}$$

$G$, grad $G$, being bounded at a large distance, and $G$ representing an outgoing wave at infinity, is given by

$$G(x, y\,;\xi, \eta) = -\log\frac{r}{r'} + 2\int_0^\infty \frac{\sinh ky \sinh k\eta}{k\cosh kh} e^{-kh} \cos k(x-\xi)\, dk$$

$$- 2\int_0^\infty \frac{\cosh k(h-y)\cosh k(h-\eta)}{\cosh kh\,(K\cosh kh - k\sinh kh)} \cos k(x-\xi)\, dk$$

$$+ 4\pi i \frac{\cosh k_0(h-y)\cosh k_0(h-\eta)}{2k_0 h + \sinh 2k_0 h} \cos k_0 (x-\xi), \tag{3.4}$$

where

$$r^2 = (x-\xi)^2 + (y-\eta)^2\,;\; r'^2 = (x-\xi)^2 + (y+\eta)^2.$$

This expression can be deduced from the results given by Thorne[7].

Applying Green's theorem to $\varphi(x, y)$ and $G(x, y\,;\xi, \eta)$ in the fluid region bounded by $y = 0$, $x = -X$, $y = h$, $x = X$, two straight lines enclosing the plate, and a circle with centre at $(\xi, \eta)$ and small radius $\epsilon$, and making $X \to \infty$ and $\epsilon \to 0$, we obtain

$$-2\pi\,\varphi(\xi, \eta) = -\int_a^b f(y) \frac{\partial G}{\partial x}(0, y\,;\xi, \eta)\, dy, \tag{3.5}$$

where

$$f(y) \equiv \varphi(0+, y) - \varphi(0-, y).$$

By (2.3), (2.5) and (3.5), we have

$$-2\pi i k_0 \frac{\cosh k_0(h-\eta)}{\cosh k_0 h} = \frac{d^2}{d\eta^2}\int_a^b f(y)\, G(0, y\,;0, \eta)\, dy, \quad a < \eta < b, \tag{3.6}$$

Therefore,

$$2\pi i \frac{\sinh k_0(h-\eta)}{\cosh k_0 h} = \frac{d}{d\eta}\int_a^b f(y)\, G(0, y\,;0, \eta)\, dy + A, \quad a < \eta < b, \tag{3.7}$$

where $A$ is a complex constant.

By (3.6) and (3.7) and neglecting exponentially small terms for large $h$, we obtain

$$\int_a^b \psi(y) \frac{2y\,dy}{y^2 - \eta^2} = KA - 2\int_a^b \psi(y)\, H(y, \eta\,;\, kh)\, dy, \quad a < \eta < b, \tag{3.8}$$

where

$$\psi(y) = Kf(y) + f'(y),$$

so that

$$f(y) = e^{-Ky} \int_a^y e^{Ku} \psi(u)\, du, \tag{3.9}$$

and, following Rhodes-Robinson[4],

$$\left.\begin{aligned} H(y, \eta\,; Kh) &= \int_0^\infty \frac{(K + k)\, e^{-kh} \sinh ky \cosh k\eta}{K \cosh kh - \sinh kh}\, dk \\ &= \frac{1}{2} \sum_{r=1}^{\infty} a_{2r} K^{2r} \frac{(y + \eta)^{2r-1} + (y - \eta)^{2r-1}}{(2r - 1)!\,(Kh)^{2r}} \\ &\quad + \frac{1}{2} \sum_{r=1} \sum_{s=1}^{\infty} \beta_{2r,s} K^{2r} \frac{(y + \eta)^{2r-1} + (y - \eta)^{2r-1}}{(2r - 1)!\,(Kh)^{2r+s}}, \\ \text{where} \qquad & \\ a_{2r} = \int_0^\infty \frac{e^{-k}\, k^{2r-1}}{\cosh k}\, dk\ &;\ \beta_{2r,s} = \int_0^\infty \frac{\tanh^s k}{\cosh^2 k} \cdot k^{2r+s-1}\, dk, \end{aligned}\right\} \tag{3.10}$$

and $a_{2r}$; $\beta_{2r,s}$ may be expressed in terms of Bernoulli Numbers [*c.f.* Gradshteyn, I. S./ Ryzhik, I. M. (1980)].

Here (3.8) is a Cauchy type singular integral equation and following Mikhlin[8],

$$\psi(\eta) = \frac{4}{\pi^2 \sqrt{(\eta^2 - a^2)(b^2 - \eta^2)}} \int_a^b \psi(y)\, dy \int_a^b H(y, v\,; Kh)$$

$$\times \frac{\sqrt{(v^2 - a^2)(b^2 - v^2)}}{v^2 - \eta^2}\, v dv + \frac{C - D\eta^2}{\sqrt{(\eta^2 - a^2)(b^2 - \eta^2)}}, \qquad a < \eta < b, \tag{3.11}$$

where $C$, $D$ are two complex constants.

By (3.6), (3.9) and using the expression

$G(x, y\,; \xi, \eta)$

$$= 2 \int_0^\infty \frac{(K \sin ky - k \cos ky)\,(K \sin k\eta - k \cos k\eta)}{k\,(K^2 + k^2)}$$

$$\times e^{-k|x-\xi|}\, dk + 2\pi i e^{-K(y+\eta)+iK|x-\xi|}$$

$$+ 2 \int_0^\infty \frac{(K \sinh ky - k \cosh ky)\,(K \sinh k\eta - k \cosh k\eta)}{k\,(K - k)\,(K \cosh kh - k \sinh kh)}\, e^{-kh} \cos k\,(x - \xi)\, dk$$

$$+ 4\pi i \frac{\cosh k_0 (h - y) \cosh k_0 (h - \eta)}{2k_0 h + \sinh 2k_0 h} \cos k_0 (x - \xi)$$

$$- 2\pi i\, e^{-K(y+\eta)} \cos K\,(x - \xi),$$

and neglecting exponentially small terms for large $h$, and putting $\eta = a$, we obtain

$$\int_a^b \psi(y)\Big[\int_0^\infty \frac{\sin ky\,(K\sin ka - k\cos ka)}{K^2 + k^2}\,kdk + \pi i K e^{-Ka}\sinh ky - H'(y, a\,;\,Kh)\Big]dy = \pi i K e^{-Ka}, \tag{3.12}$$

where, following Rhodes-Robinson[4],

$$\left.\begin{aligned} H'(y, a\,;\,Kh) &= \mathop{\psi}_0^\infty \frac{\sinh ky\,(K\sinh ka - k\cosh ka)}{(K-k)(K\cosh kh - k\sinh kh)}\,ke^{-kh}\,dk \\ &= \frac{1}{2}\sum_{r=0}^\infty\sum_{s=0}^\infty\sum_{t=1}^\infty \frac{\gamma_{r,s,2t}\cdot K^{2t}}{(2t)\,!\,(Kh)^{r+s+2t+2}}\,[K((y+a)^{2t} - (y-a)^{2t}) \\ &\quad - 2t\,((y+a)^{2t-1} + (y-a)^{2t-1})], \\ \text{where} \\ \gamma_{r,s,2t} &= \int_0^\infty \frac{\tanh^s k}{\cosh k}\,e^{-k}\,k^{r+s+2t+1}\,dk, \end{aligned}\right\} \tag{3.13}$$

and $\gamma_{r,s,2t}$ also may be expressed in terms of Bernoulli Numbers.

## 4. Calculations for $\psi(\eta)$

If we write

$$\psi(\eta) = \psi_0(\eta) + \frac{1}{Kh}\psi_1(\eta) + \frac{1}{(Kh)^2}\psi_2(\eta) + \cdots, \tag{4.1}$$

$$\left.\begin{aligned} C &= C_0 + \frac{1}{Kh}C_1 + \frac{1}{(Kh)^2}C_2 + \cdots, \\ D &= D_0 + \frac{1}{Kh}D_1 + \frac{1}{(Kh)^2}D_2 + \cdots, \end{aligned}\right\} \tag{4.2}$$

then by (3.11), (3.12), (3.13), we obtain

$$\left.\begin{aligned} &\psi_0(\eta) = \frac{D_0(d_0^2 - \eta^2)}{\sqrt{(\eta^2 - a^2)(b^2 - \eta^2)}}\,;\psi_1(\eta) = \psi_2(\eta) = \psi_3(\eta) = 0\,; \\ &\text{and} \\ &\psi_4(\eta) = \frac{1}{2}D_0 K^4 \alpha_4\left(d_0^2 - \frac{a^2+b^2}{2}\right)\frac{(d_4^4 - \eta^4) + B(d_0^2 - \eta^2)}{\sqrt{(\eta^2 - a^2)(b^2 - \eta^2)}}\,; \\ &\cdots\text{etc.} \end{aligned}\right\} \tag{4.3}$$

where

$D_0$, $B$, $d_0^2$, $d_4^4$, etc., are constants to be determined.

Since, $f(b) = 0$, by (3.9)

$$\int_a^b \psi(u)\, e^{Ku}\, du = 0, \qquad \ldots. \tag{4.4}$$

and therefore,

$$\int_a^b \psi_0(y)\, e^{Ky}\, dy = 0 \; ; \; \int_a^b \psi_4(y)\, e^{Ky}\, dy = 0 \; ; \text{ etc.,}$$

i.e..

$$\left.\begin{aligned} &\int_a^b \frac{(d_0^2 - y^2)\, e^{Ky}\, dy}{\sqrt{(y^2 - a^2)(b^2 - y^2)}} = 0 \; ; \\ &\int_a^b \frac{(d_4^4 - y^4)\, e^{Ky}\, dy}{\sqrt{(y^2 - a^2)(b^2 - y^2)}} = 0 \; ; \\ &\qquad\qquad\qquad\qquad \ldots.\text{etc.} \end{aligned}\right\} \tag{4.5}$$

By (3.12), (3.13), (4.1), (4.3), we obtain

$$\left.\begin{aligned} &D_0 = \frac{2i}{\alpha - \beta - i\gamma} \\ &\text{where} \\ &\alpha = \int_{-a}^{a} \frac{(d_0^2 - y^2)\, e^{-Ky}\, dy}{\sqrt{(a^2 - y^2)(b^2 - y^2)}} \; ; \; \beta = \int_b^{\infty} \frac{(d_0^2 - y^2)\, e^{-Ky}\, dy}{\sqrt{(y^2 - a^2)(y^2 - b^2)}} \\ &\text{and} \\ &\gamma = \int_a^b \frac{(d_0^2 - y^2)\, e^{-Ky}\, dy}{\sqrt{(y^2 - a^2)(b^2 - y^2)}} \end{aligned}\right\} \tag{4.6}$$

and

$$\left.\begin{aligned} &B = -\frac{\alpha' - \beta' - i\gamma' + \dfrac{2}{K^3}(1 - Ka)\, e^{Ka}}{\alpha - \beta - i\gamma} \\ &\text{where} \\ &\alpha' = \int_{-a}^{a} \frac{(d_4^4 - y^4)\, e^{-Ky}\, dy}{\sqrt{(a^2 - y^2)(b^2 - y^2)}} \; ; \; \beta' = \int_b^{\infty} \frac{(d_4^4 - y^4)\, e^{-Ky}\, dy}{\sqrt{(y^2 - a^2)(y^2 - b^2)}} \\ &\text{and} \\ &\gamma' = \int_a^b \frac{(d_4^4 - y^4)\, e^{-Ky}\, dy}{\sqrt{(y^2 - a^2)(b^2 - y^2)}} \end{aligned}\right\} \tag{4.7}$$

and $d_0^2$, $d_4^4$ given by (4.5).

## 5. Reflection and transmission coefficients

As $\xi \to +\infty$ and $-\infty$ respectively, and neglecting exponentially small terms for large $h$, we have by (3.5) and (3.9) the complex transmission and reflection coefficients

$$\left.\begin{aligned} T &= 1 + \tfrac{1}{2} \int_a^b \psi(y)\, e^{-Ky}\, dy \\ R &= -\tfrac{1}{2} \int_a^b \psi(y)\, e^{-Ky}\, dy \end{aligned}\right\}. \qquad (5.1)$$

If we write

$$\left.\begin{aligned} T &= T_0 + \frac{1}{Kh} T_1 + \frac{1}{(Kh)^2} T_2 + \dots, \\ R &= R_0 + \frac{1}{Kh} R_1 + \frac{1}{(Kh)^2} R_2 + \cdots, \end{aligned}\right\} \qquad (5.2)$$

we have since $R + T = 1$

$$T_0 = 1 - R_0\ ;\ T_1 = -R_1\ ;\ T_2 = -R_2\ ; \quad \text{etc.} \qquad (5.3)$$

Then by (4.1), (4.3), (4.6), (4.7) and (5.2), we obtain

$$R_0 = \frac{K\gamma}{K\gamma + i(K\alpha - K\beta)}\ ;$$

$$R_1 = R_2 = R_3 = 0\ ;$$

and

$$R_4 = -i\frac{\alpha_4}{2}\left(K^2 d_0^2 - \frac{K^2 a^2 + K^2 b^2}{2}\right) \times \frac{K\gamma\,(K^3\alpha' - K^3\beta') - K^3\gamma'\,(K\alpha - K\beta) + 2K\gamma\,(1 - Ka)\,e^{Ka}}{[K\gamma + i(K\alpha - K\beta)]^2}.$$

It should be noted that $R_0$, $T_0$ give the corresponding results for the infinite depth case (Evans[3]).

If we take $2\,Ka = Kb = 1$, and $Kh = 10$,

$$R_0 = 0{\cdot}00\,06\,35\,37 - i\,0{\cdot}02\,51\,98\,62$$

$$R_4 = 0{\cdot}00\,00\,0\,123 - i\,0{\cdot}00\,00\,24\,37$$

and

$$(b/h)^4\,|\,R_4\,| \sim 0\,(10^{-9})$$

*i.e.*, the depth effects on reflection and transmission coefficients are found to be significant only after the eight places of decimal point.

## 6. Discussion

Reduction of scattering problems to integral equation is a simple technique provided the integral equation can be solved successfully. The original scattering problem involving a submerged fixed vertical barrier in deep water has recently been considered by Goswami[9] by simply using the integral equation method, although it was solved by other methods by various workers from time to time (*c.f.* Goswami[9]). Because of complexity in the solution of the corresponding integral equations which will arise in the problems involving water of finite depth, there seems to be not many contributions in this line, although there are some contributions in deep water cases (which are themselves complicated).

As the depth of the bottom tends to infinity the transmission and reflection coefficients obtained in this paper coincide with the expressions for the corresponding problem in deep water treated by Evans[3]. The depth correction terms for the reflection and transmission coefficients are found to be of the order of $(b/h)^4$. As $a \to 0$ the results coincide with those for the partially immersed vertical barrier obtained earlier by Goswami[6].

The problem discussed here has an interesting application in naval warfare. The presence of an enemy submarine submerged in not too deep water in the neighbourhood of a ship may perhaps be ascertained simply by observing the transmission coefficients at a large distance from the ship where the vertical plate may be regarded as a crude approximation for the submarine. However, in this model, the ship is to be at a large horizontal distance from the submarine and the waves are generated by some mechanism in the ship, and no other type of disturbance in the water is assumed to exist.

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