Spherical shock due to point explosion with varying energy

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Abstract

Similarity solutions of the equations governing the motion of a perfect gas behind a spherical wave of a point explosion with a shock surface as wave front propagating outwards in non-uniform atmosphere at rest with varying energy of the flow are investigated. These solutions are applicable both in weak and strong shocks. In brief, the isothermal case has been also posed in the last but one section.

1 Introduction

Taylor[1] and Taylor[2] have obtained solutions for spherically symmetric strong shocks only. Det Ray[3] has discussed an exact analytic solution of equations for a point explosion assuming that the total energy of the expanding wave is constant. Taking the same assumption, Singh and Srivastava[4] have studied the point explosion problem for strong shocks in magnetogasdynamics. Rogers[5] and Singh[6] have considered the piston problem in uniform atmosphere with increasing energy in the absence and presence of the magnetic field respectively for strong shock only.

In the present paper the self-similar model of weak shock wave, produced by point explosion or uniform expanding piston in non-uniform atmosphere, has been adopted. The total energy of the flow between the shock front and inner expanding surface is time-dependent. The total energy in the flow is the sum of the kinetic and internal heat energies of the gas; in practice there will be losses due to dissipative effects while there will also be a gain in the internal heat energy as the shock front advances and encloses more of the quiescent gas. However, this increase has effectively been omitted in the infinitely strong shock case, i.e., the pressure, and therefore internal

heat energy of the gas, ahead of the shock front, is negligible. The total energy increases with the time due to the pressure exerted on the gas by an expanding spherical surface. Thus the flow is headed by a shock front and has an expanding surface as an inner boundary. In a real interplanatory situation when the energy of the explosion increases with time or even when the energy release can be reasonably considered to be instantaneous, the solutions correspond to the explosion wave being driven ahead of the ejected gases which form the sphere known as contact discontinuity surface or inner expanding surface.

It is always possible to adopt the present self-similar model to include a 'driven' wave produced by a flare energy release E that is time-dependent. Hence

$$E = E_c t^q, \qquad (q \geqslant 0) \tag{1.1}$$

where E_c is a constant.

Following Summers[7], if $q > 0$, then E increases with time and the solutions then correspond to a blast wave produced by intense, prolonged solar flare activity when the wave is driven by fresh erupting solar plasma for some time and its energy tends to increase as it propagates from the sun.

Ahead of the shock, the density distribution ρ_0 is taken to vary as

$$\rho_0 = \rho_\theta r^a, \qquad (-3 < a \leqslant 0) \tag{1.2}$$

where r is the radial distance from the point of explosion and ρ_c is a constant. The pressure distribution is taken as

$$p_0 = p_c r^\delta, \qquad (\delta < 0) \tag{1.3}$$

where p_c is a constant.

The numerical integration has been done only for adiabatic case taking different values of q. Viscosity, gravitational forces and magnetic field, etc., have been neglected.

2. Equations of the problem and boundary conditions

The governing equations of motion are

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} + \frac{2\rho u}{r} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0, \tag{2.2}$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} = \frac{\gamma p}{\rho} \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} \right), \tag{2.3}$$

where ρ, u and p are density, velocity and pressure of the gas at a radial distance r from the centre at time t; γ is the ratio of specific heats.

This motion will be supposed to be bounded on the outside by a shock surface at $r = R(t)$, which will move outward with velocity

$$V = dR/dt.$$

The Rankine–Hugoniot conditions at the shock surface are

$$u_1 = \frac{2V}{\gamma + 1}\left[1 - \frac{1}{M^2}\right], \tag{2.4}$$

$$\rho_1 = \frac{\rho_0 (\gamma + 1) M^2}{(\gamma - 1) M^2 + 2}, \tag{2.5}$$

$$p_1 = \frac{p_0}{\gamma + 1}[2\gamma M^2 - (\gamma - 1)], \tag{2.6}$$

where

$$M^2 = \frac{V^2 \rho_0}{\gamma p_0}, \tag{2.7}$$

and suffixes 1 and 0 denote the values just behind and ahead of the shock front respectively.

Next let us seek solutions of equations (1)–(3) in the form

$$u = \frac{r}{t}\, U(\eta), \tag{2.8}$$

$$\rho = r^k\, t^\lambda\, \Omega(\eta), \tag{2.9}$$

$$p = r^{k+2}\, t^{\lambda-2}\, P(\eta), \tag{2.10}$$

where

$$\eta = r^a\, t^b, \tag{2.11}$$

while k, λ, a and b are constants.

The total energy E inside a shock wave of radius R is given by

$$E = 4\pi \int_{r^*}^{R} \left(\tfrac{1}{2}\rho\, u^2 + \frac{p}{\gamma - 1}\right) r^2\, dr = E_c\, t^a, \tag{2.12}$$

where r^* is the radius of the inner expanding surface. In terms of the variable η we get

$$E = \frac{4\pi}{a} \int_{\eta^*}^{\eta_0} \left[\frac{1}{2} U^2(\eta)\, \Omega(\eta)\, \eta^{[(k+5/a)-1]} + \frac{1}{\gamma - 1} P(\eta)\, \eta^{[(k+5/a)-1]}\right]$$

$$\times\, t^{[\lambda-2-(b/a)(k+5)]}\, d\eta = E_c\, t^a; \tag{2.13}$$

where η_0 and η^* are the values of η at the shock front and at the inner expanding surface respectively.

We choose the external shock surface to be given by $\eta_0 = $ constant, which fixes the velocity of the shock surface as

$$V = -\frac{b}{a}\frac{R}{t}.$$

(2.14)

The above expression (2.13) is independent of time if

$$\lambda - 2 - \frac{b}{a}(k + 5) = q.$$

(2.15)

Now using equations (2.8)–(2.11) and (2.14) into the equations (2.4)–(2.7), following Deb Ray[3], we get

$$\frac{a + 2 - \delta}{-2} = a/b,$$

(2.16)

$$\frac{\delta - k - 2}{-\lambda + 2} = \frac{a}{b} = \frac{k - a}{\lambda}.$$

(2.17)

From (2.15), (2.16) and (2.17) we have

$$\delta = -\frac{6 - (a + 2)q}{2 + q},$$

(2.18)

also

$$\frac{a}{b} = \frac{-(a + 5)}{2 + q}.$$

(2.19)

Without any loss of generality, let us put $k = a$. Then from (2.17) $\lambda = 0$. Hence we have in fact

$$k = a \; ; \; \lambda = 0 \; ; \; a = -(a + 5) \; ; \; b = 2 + q.$$

(2.20)

From equation (2.11) we easily get

$$R = (\eta_0)^{-1/a+5} t^{(2+a/a+5)}.$$

(2.21)

The above expression discloses that the physically significant range of q is 0 to $3 + a$. Taking the different values of a and q within their ranges, we can find the values of δ.

3. Solutions of equations of motion

The condition inside the wave will be determined from the solutions of the equations (2.1), (2.2) and (2.3).

$$\frac{\partial \rho}{\partial t} = -V\frac{r}{R}\frac{\partial \rho}{\partial r} + a\rho\frac{V}{R},$$

(3.1)

$$\frac{\partial p}{\partial t} = -V\frac{r}{R}\frac{\partial p}{\partial r} + \delta p\frac{V}{R},$$

(3.2)

$$\frac{\partial u}{\partial t} = - V \frac{r}{R} \frac{\partial u}{\partial r} + \left[\frac{q - (\alpha + 3)}{2 + q} \right] u \frac{V}{R}. \tag{3.3}$$

By substitution of equations (3.1), (3.3) and boundary conditions (2.4) to (2.6) into the equations (2.1), (2.2) and (2.3), the following set of equations are obtained;

$$\frac{1}{X} \frac{dX}{dx} (AY - x) + A \frac{dY}{dx} + 2A \frac{Y}{x} + \alpha = 0, \tag{3.4}$$

$$\frac{1}{Y} \frac{dY}{dx} (AY - x) + \left[\frac{q - (\alpha + 3)}{2 + q} \right] + \frac{B}{A} \frac{1}{YX} \frac{dZ}{dx} = 0, \tag{3.5}$$

$$\frac{1}{Z} \frac{dZ}{dx} (AY - x) - \frac{\gamma}{X} \frac{dX}{dx} (AY - x) + \delta - \gamma\alpha = 0, \tag{3.6}$$

where

$$x = \frac{r}{R} \; ; \; X = \frac{\rho}{\rho_1} \; ; \; Y = \frac{u}{u_1} \; ; \; Z = \frac{p}{p_1},$$

and

$$A = \frac{2 \left(1 - \frac{1}{M^2} \right)}{\gamma + 1} \; ; \quad B = \frac{[2\gamma M^2 - (\gamma - 1)][(\gamma - 1) M^2 + 2]}{\gamma (\gamma + 1)^2 M^4}.$$

At the shock boundary we have
$$x = 1 \; ; \; X = 1 \; ; \; Y = 1 \; ; \; Z = 1.$$

Starting with these initial values equations (3.4) to (3.6) are integrated for $\gamma = 5/3$; $M^2 = 6$; $\alpha = -1$; $q = 0, 2/3, 2$ corresponding values of δ are $-3, -2, -1$ respectively.

A singular surface occurs at $AY - x = 0$ because on solving the equations (3.4) to (3.6) for $\frac{dX}{dx}$, $\frac{dY}{dx}$ and $\frac{dZ}{dx}$ for computation, two terms $(x - AY)$ and $\left[(x - AY)^2 - B \frac{\gamma Z}{X} \right]$ in denominator of the three differential equations are obtained. The denominator vanishes at $x - AY = 0$ and $\left[(x - AY)^2 - B \frac{\gamma Z}{X} \right] = 0$ within the range of integration from the shock surface towards the inner expanding surface, none of these cases occurs. The integration starts from the shock front, i.e., $x = 1$ and proceed inwards towards the inner boundary, the inner expanding surface or piston. It is found that work of integration fails as the critical surface $AY - x = 0$ is approached. One may identify this surface with that of the inner expanding surface or piston. This singular surface occurs outside the range of integration. The second term given above also does not occur because the flow behind a shock surface must be subsonic. The well-known RKGS programme is employed to solve the system of equations on DEC–system 1090 computer installed at I.I.T., Kanpur. The variations of the field variable are shown in figs. 1 to 3.

4. Isothermal case

In the presence of intense heat exchange the adiabaticity condition is violated, that instead of the condition of adiabaticity behind the shock one may suppose the the temperature gradient is absent. Such flows are called isothermal. In ordinary gas dynamics the problem of self-similar flows behind a spherical shock due to point explosion with zero temperature gradient was first solved by Korobeinikov[8] and was later extended to the piston problem by Melnikova[9].

In this case equation (2.3) is of the form

$$\frac{\partial T}{\partial r} = 0.$$

(4.1)

Using the equation of state $p = \Gamma \rho T$ into the above equation, we get

$$\frac{p}{p_1} = \frac{\rho}{\rho_1},$$

(4.2)

where Γ is the gas constant and T the temperature.
The shock conditions are

$$\rho_1 = \frac{1}{\beta} \rho_0 \; ; \; u_1 = (1 - \beta) V \; ; \; p_1 = (1 - \beta + Q) \rho_0 V^2,$$

(4.3)

where β and Q are parameters. Also it is obvious to note that Mach number is related to Q and is equal to $1/\sqrt{\gamma Q}$.

After using the equations (3.1) to (3.3) and (4.2) into the equations (2.1) and (2.2) and utilising above boundary conditions, the following set of equations are obtained :

$$\frac{1}{X} \frac{dX}{dx} [(1 - \beta) Y - x] + (1 - \beta) \frac{dY}{dx} + 2 (1 - \beta) \frac{Y}{x} + \alpha = 0.$$

(4.4)

$$\frac{1}{Y} \frac{dY}{dx} [(1 - \beta) Y - x] + \left[\frac{q - (\alpha + 3)}{2 + q} \right] + \frac{\beta (1 - \beta + Q) Q}{(1 - \beta)} - \frac{1}{YX} \frac{dX}{dx} = 0$$

(4.5)

The parameter β $(0 < \beta < 1)$ can be determined by iteration using the equation of conservation of mass in Lagrangian form $(\rho_0 r_0^2 dr_0 = \rho r^2 dr)$. Integrating this equation between the limits $r*$ $(0 < r* < R)$ and R and writing the resultant expression in non-dimensional form we get

$$\frac{\beta}{3 - \alpha} = \int_{x*}^{1} x^2 X \, dx,$$

(4.6)

where $x* = r*/R$. The range of β can be found from the fact that the relative velocity of fluid just behind the shock must be subsonic,

5. Conclusions

The value of the Mach number in the above problem is unrestricted and may be large and small. For $\delta = -3$, the velocity first decreases near the shock front and then after some time it shows an upward trend. The density starts to decrease continuously from the shock front until it reaches near the point of explosion and the pressure first decreases near the shock front, increases in the middle of the flow fields for a small distance and again decreases continuously towards the centre of explosion.

For $\delta = -2$ the velocity shows similar behaviour as in the above case with higher values at all the points but both density and pressure slowly start decreasing continuously from the shock front up to near the point of explosion and they are higher than their values for $\delta = -3$.

For $\delta = -1$, the velocity, density and pressure all show an upward trend.

Hence, it is obvious from figs. 1–3 that as δ increases, the flow variables get their higher values throughout their fields at all the points. This theory of self-similar flows behind a shock wave is of considerable physical interest for example, in sonic booms, phenomena associated with laser production of plasmas, high altitude nuclear detonation, supernova explosions and sudden expansion of corona into the interplanetary space. Shock waves are also employed in laboratories to obtain high temperatures.
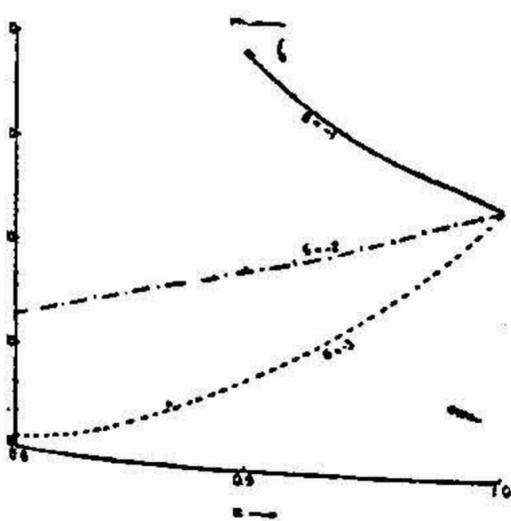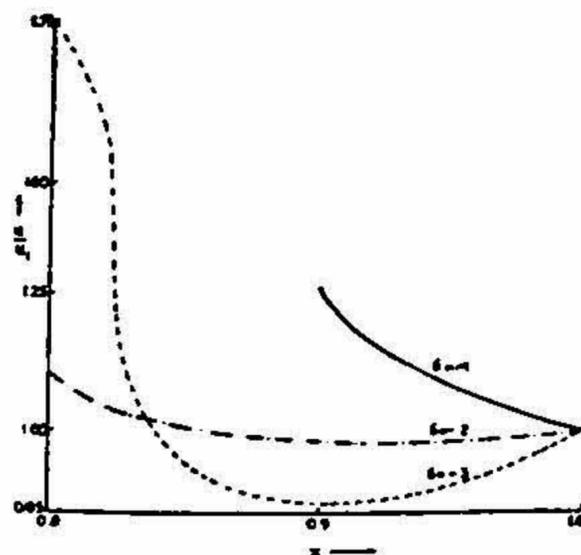

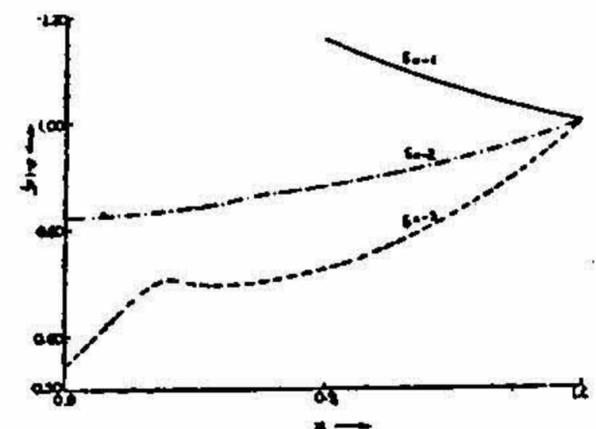FIG. 1. Density distribution.   FIG. 2. Velocity distribution.   FIG. 3. Pressure distribution.

References

1. TAYLOR, G. I.   *Proc. Royal Soc. London*, 1950, 201, 159.

2. TAYLOR, J. L.   *Phil. Mag.*, 1955, 46, 317.

3. DEB RAY, G.   *Proc. Natn. Inst. Sci. India*, 1957, 23A, 420.

4. SINGH, J. B. AND   *Astrophys. Space Sci.*, 1981, 79, 345.
   SRIVASTAVA, S. K.

5. ROGERS, M. H.   *Q. J. Mech. Appl. Math.*, 1958, 11, 411.

6. SINGH, J. B.   *Indian J. Math Soc.*, 1980, 44,

7. SUMMERS, D.   *Astr. Astrophys*, 1975, 45, 151.

8. KOROBEINIKOV, V. P.   *DANSSR*, 1956, 109, 271.

9. MELNIKOVA, N. S.   Unsteady motions of compressible media with blast waves, *Inst.* *Steklov. Math. Inst.* (Edited by L. I. Sedov), American Mathematical Society.