J. Indian Inst. Sci., 64 (B), May 1983, Pp. 97–103

Short Communication

Group theoretic study of certain generalised functions of Jacobi polynomial

ASIT BARAN CHAKRABARTI
Education Dept., Govt. of West Bengal, 2nd Floor, Block E, Writers' Buildings, Calcutta 700 001.

Received on June 5, 1982 ; Revised on January 17, 1983.

Abstract

Making suitable interpretations to both the index (n) and the parameter (β) of Jacobi polynomials $P_n^{(\alpha,\beta)} x$ in order to derive the elements of Lie-algebra, we have constructed a four parameter Lie-group for this polynomials which does not seem to appear before. By means of group-theoretic method a new generating function for Jacobi polynomials is obtained, from which several special generating functions can easily be derived.

Key words : Jacobi polynomials, generating functions.

1. Introduction

The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ satisfies the following ordinary differential equation

$$(1-x^2)\frac{d^2 y}{dx^2} + [\beta - \alpha - (2+\alpha+\beta)x]\frac{dy}{dx} + n(1+\alpha+\beta+n)y = 0. \tag{1.1}$$

The object of the present paper is to derive some generating functions, which are believed to be new, of Jacobi polynomial, by suitably interpreting n and β simultaneously with the help of Weisner's[1] group theoretic method.

It may be of interest to remark that in the course of constructing Lie-algebra for Jacobi polynomial, we have obtained two operators, *viz.*, A_{12} and A_{22} of §2 which simultaneously raise the parameter and the index respectively of the same polynomial. Such type of operators do not seem to appear before. Here we have obtained the following generating functions involving Jacobi polynomial by finding a set of infinitesimal operators A_{ij} $(i, j = 1, 2)$ generating a Lie-algebra.

$$(1-t)^n P_n^{(\alpha,\beta)}\left[\frac{x-t}{1-t}\right] = \sum_{p=0}^{\infty} \frac{1}{p!}(-n-\alpha)_p P_{n-p}^{(\alpha,\beta+p)}(x)\, t^p. \tag{1.2}$$

$$\frac{(1+2t)^\beta}{\{1+(1+x)t\}^{1+\alpha+\beta+n}} P_n^{(\alpha,\beta)}\left[\frac{x+(x+1)t}{1+(x+1)t}\right]$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}(-2)^k (n+1)_k P_{n+k}^{(\alpha,\beta-k)}(x)\, t^k. \tag{1.3}$$

$$\frac{(1+2t)^\beta \left\{t+\frac{1}{\omega}(1+2t)\right\}^n}{\{1+(1+x)t\}^{1+\alpha+\beta+n}} P_n^{(\alpha,\beta)}\left[\frac{t\{x+(x+1)t\}+\frac{1}{\omega}(1+2t)\{1+(x+1)t\}}{\{1+(x+1)t\}\left\{t+\frac{1}{\omega}(1+2t)\right\}}\right]$$

$$= \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \frac{(-2)^k}{k!}\frac{\left(-\frac{1}{\omega}\right)^p}{p!}(n-p+1)_k(-\alpha-n)_p P_{n-p+k}^{(\alpha,\beta+p-k)}(x)\, t^{p+k} \tag{1.4}$$

2. Group theoretic method

The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)}{n!}\,{}_2F_1\left[-n, \begin{matrix} 1+\alpha+\beta+n \\ 1+\alpha \end{matrix} ; \begin{matrix} \frac{1-x}{2} \end{matrix}\right] \tag{2.1}$$

is a solution of the differential equation represented by (1.1).

Now replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, β by $y\frac{\partial}{\partial y}$, n by $z\frac{\partial}{\partial z}$ and $P_n^{(\alpha,\beta)}(x)$ by $u(x,y,z)$ we obtain from (1.1) the following partial differential equation

$$(1-x^2)\frac{\partial^2 u}{\partial x^2} - \{\alpha + x(2+\alpha)\}\frac{\partial u}{\partial x} + y(1-x)\frac{\partial^2 u}{\partial y\,\partial x}$$

$$+ yz\frac{\partial^2 u}{\partial y\,\partial z} + z(2+\alpha)\frac{\partial u}{\partial z} + z^2\frac{\partial^2 u}{\partial z^2} = 0. \tag{2.2}$$

Thus we see that $u_1(x, y, z) = P_n^{(\alpha,\beta)}(x)\, y^\beta z^n$ is a solution of (2.2), since $P_n^{(\alpha,\beta)}(x)$ is a solution of (1.1). Let us now seek two first order partial different'al operators A_{12} and A_{22} such that

$$A_{12}\,[P_n^{(\alpha,\beta)}(x)\, y^\beta z^n] = a_{12}(\beta, n)\, P_{n-1}^{(\alpha,\beta+1)}(x)\, y^{\beta+1} z^{n-1} \tag{2.3}$$

and

$$A_{22}\,[P_n^{(\alpha,\beta)}(x)\, y^\beta z^n] = a_{22}(\beta, n)\, P_{n+1}^{(\alpha,\beta-1)}(x)\, y^{\beta+1} z^{n-1}. \tag{2.4}$$

where $a_{12}(\beta,n)$ and $a_{22}(\beta, n)$ are coefficients involving α, β and n.

Using the relation (2.3) and the relation

$$\frac{d}{dx}\,[P_n^{(\alpha,\beta)}(x)] = \frac{1}{x-1}\,[n\, P_n^{(\alpha,\beta)}(x) - (\alpha + n)\, P_{n-1}^{(\alpha,\beta+1)}(x)] \tag{2.5}$$

we get

$$A_{12} = (x-1)\, y z^{-1} \frac{\partial}{\partial x} - y \frac{\partial}{\partial z}.$$

so that

$$A_{12}\,[P_n^{(\alpha,\beta)}(x)\, y^\beta z^n] = -(n + \alpha)\, P_{n-1}^{(\alpha,\beta+1)}(x)\, y^{\beta+1} z^{n-1}.$$

similarly using (2.4) and the relation

$$\frac{d}{dx}[P_n^{(\alpha,\beta)}(x)] = \frac{1}{1-x^2}\{(1 + \alpha + \beta + n)(x + 1) - 2\beta\}\, P_n^{(\alpha,\beta)}(x) - 2(n+1)\, P_{n+1}^{(\alpha,\beta-1)}(x). \tag{2.6}$$

we get

$$A_{22} = (1 - x^2)\, y^{-1} z \frac{\partial}{\partial x} - z(x-1)\frac{\partial}{\partial y} - (1 + x)\, y^{-1} z^2 \frac{\partial}{\partial z} - (1+\alpha)(1+x)\, y^{-1} z.$$

so that

$$A_{22}\,[P_n^{(\alpha,\beta)}(x)\, y^\beta z^n] = -2(n+1)\, P_{n+1}^{(\alpha,\beta-1)}(x)\, y^{\beta-1} z^{n+1}.$$

To find the group of operators, let us write

$$A_{11} = y\frac{\partial}{\partial y}\;;\; A_{21} = z\frac{\partial}{\partial z}.$$

and we have

$$[A_{11}, A_{12}] = A_{12}$$

$$[A_{21}, A_{12}] = -A_{12}$$

$$[A_{11}, A_{22}] = -A_{22}.$$

$$[A_{11}, A_{21}] = 0$$
$$[A_{21}, A_{22}] = A_{22}$$
$$[A_{22}, A_{12}] = -2\,[2\,A_{21} + (1 + \alpha)]. \tag{2.7}$$

where $[A, B] = AB - BA$ which shows that the operators 1, A_{ij} $(i, j = 1, 2)$, 1 being the identity operator, generate a Lie-algebra.

Now the operator L given by

$$L = (1 - x^2)\frac{\partial^2}{\partial x^2} - \{\alpha + (2 + \alpha)\,x\}\frac{\partial}{\partial x} + y\,(1 - x)\frac{\partial^2}{\partial y \partial x} + yz\frac{\partial^2}{\partial y \partial z} + (2 + \alpha)\,z\frac{\partial}{\partial z} + z^2\frac{\partial^2}{\partial z^2}.$$

can be expressed as follows

$$(x - 1)\,Lu = (A_{22}\,A_{12} - 2A_{21}^2 - 2\alpha A_{21})\,u. \tag{2.8}$$

It can be easily verified that the operators A_{ij} with $i, j = 1, 2$, commute with $(x - 1)\ L$, *i.e.*,

$$[(x - 1)\,L, A_{ij}] = 0. \tag{2.9}$$

The extended form of the groups generated by A_{ij} $(i, j = 1, 2)$ are given by

$$e^{a_{11}A_{11}}\,u\,(x, y, z) = u\,(x, e^{a_{11}}\,y, z)$$
$$e^{a_{21}A_{21}}\,u\,(x, y, z) = u\,(x, y, e^{a_{21}}\,z)$$
$$e^{a_{12}A_{12}}\,u\,(x, y, z) = u\left(\frac{zx - a_{12}y}{z - a_{12}\,y}, y, z - a_{12}\,y\right)$$
$$e^{a_{22}A_{22}}\,u\,(x, y, z) = \left(\frac{y}{y + a_{22}\,(x + 1)\,z}\right)^{\alpha+1} \times u\left(\frac{xy + a_{22}\,(1 + x)\,z}{y + a_{22}\,(1 + x)\,z}, \frac{y\,(y + 2a_{22}z)}{y + a_{22}\,(1 + x)\,z}, \frac{yz}{y + a_{22}\,(1 + x)\,z}\right)$$

where A_{ij}, $i = j = 1, 2$ are constants.

Thus we have

$$e^{a_{22}A_{22}}\,e^{a_{12}A_{12}}\,e^{a_{21}A_{21}}\,e^{a_{11}A_{11}}\,u\,(x, y, z) = \left(\frac{y}{y + a_{22}\,(1 + x)\,z}\right)^{\alpha+1} u\,(\underset{\xi}{\epsilon}, \eta, \rho)$$

where

$$\xi = \frac{z\,\{xy + a_{22}\,(x + 1)\,z\} - a_{12}\,(y + 2a_{22}z)\,\{y + a_{22}\,(x + 1)\,z\}}{\{y + a_{22}\,(x + 1)\,z\}\,\{z - a_{12}\,(y + 2a_{22}\,z)\}}$$
$$\eta = e^{a_{11}}\,\frac{y\,(y + 2a_{22}z)}{y + a_{22}\,(1 + x)\,z}.$$

$$\rho = \frac{y\{z - a_{12}(y + 2a_{22}z)}{y + a_{22}(1 + x)z}.$$

3. Generating functions

From the foregoing discussion we see that $u(x, y, z) = P_n^{(\alpha,\beta)}(x)\, y^\beta z^n$ is a solution of the system

$$Lu = 0\ ; \qquad Lu = 0\ ; \qquad Lu = 0$$

$$(A_{11} - \beta)u = 0 \qquad (A_{21} - n)u = 0 \qquad (A_{11} + A_{21} - \beta - n)u = 0$$

From (2.10) we easily get

$$S((x-1)L)(P_n^{(\alpha,\beta)}(x)\, y^\beta z^n) = ((x-1)L)\, S(P_n^{(\alpha,\beta)}(x)\, y^\beta z^n) = 0$$

where

$$S = e^{a_{22}A_{22}}\, e^{a_{12}A_{12}}\, e^{a_{21}A_{21}}\, e^{a_{11}A_{11}}.$$

Therefore the transformation $S[P_n^{(\alpha,\beta)}(x)\, y^\beta z^n]$ is also annuled by L.

By putting $a_{11} = 0 = a_{21}$ we get

$$e^{a_{22}A_{22}}\, e^{a_{12}A_{12}}\, [P_n^{(\alpha,\beta)}(x)\, y^\beta z^n]$$

$$= \left(\frac{y}{y + a_{22}(1+x)z}\right)^{1+\alpha+\beta+n} (y + 2a_{22}z)^\beta \{z - a_{12}(y + 2a_{22}z)\}^n$$

$$\times P_n^{(\alpha,\beta)}\left[\frac{z\{xy + a_{22}(x+1)z\} - a_{12}(y + 2a_{22}z)\{y + a_{22}(1+x)z\}}{\{y + a_{22}(x+1)z\}\{z - a_{12}(y + 2a_{22}z)\}}\right]. \tag{3.1}$$

But

$$e^{a_{22}A_{22}}\, e^{a_{12}A_{12}}\, [P_n^{(\alpha,\beta)}(x)\, y^\beta z^n]$$

$$= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(a_{22})^k}{k!} \frac{(a_{12})^p}{p!} (-\alpha - n)_p (-2)^k (n - p + 1)_k$$

$$\times P_{n-p+k}^{(\alpha,\beta+p-k)}(x)\, y^{\beta+p-k} z^{n-p+k}. \tag{3.2}$$

Equating the results (3.1) and (3.2) we get

$$\left(\frac{y}{y + a_{22}(x+1)z}\right)^{1+\alpha+\beta+n} (y + 2a_{22}z)^\beta \{z - a_{12}(y + 2a_{22}z)\}^n$$

$$\times P_n^{(\alpha,\beta)}\left[\frac{z\{xy + a_{22}(x+1)z\} - a_{12}(y + 2a_{22}z)\{y + a_{22}z(x+1)\}}{\{y + a_{22}(x+1)z\}\{z - a_{12}(y + 2a_{22}z)\}}\right]$$

$$= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(a_{22})^k}{k\,!} \frac{(a_{12})^p}{p!} (-\alpha - n)_p (-2)^k (n - p + 1)_k$$

$$\times P_{n-p+k}^{(\alpha,\beta+p-k)}(x)\, y^{\beta+p-k} z^{n-p+k}. \tag{3.3}$$

Now we shall consider the following cases :

Case I

Let us put $a_{12} = 1$, $a_{22} = 0$ and writing $y/z = t$ we get

$$(1 - t)^n P_n^{(\alpha,\beta)}\left[\frac{x - t}{1 - t}\right] = \sum_{p=0}^{\infty} \frac{1}{p\,!} (-n - \alpha)_p P_{n-p}^{(\alpha,\beta+p)}(x)\, t^p \tag{3.4}$$

which is (1.2).

Case II

Let $a_{12} = 0$, $a_{22} = 1$ and writing $y/z = t^{-1}$ we get

$$\frac{(1 + 2t)^\beta}{\{1 + (1 + x)\, t\}^{1+\alpha+\beta+n}} P_n^{(\alpha,\beta)}\left[\frac{x + (x + 1)\, t}{1 + (x + 1)\, t}\right]$$

$$= \sum_{k=0}^{\infty} \frac{1}{k\,!} (-2)^k (n + 1)_k P_{n+k}^{(\alpha,\beta-k)}(x)\, t^k. \tag{3.5}$$

which is (1.3).

Case III

If we put $a_{12} = -\dfrac{1}{\omega}$, $a_{22} = 1$ and write $y/z = t^{-1}$ we get

$$\frac{(1 + 2t)^\beta \left\{t + \frac{1}{\omega}(1 + 2t)\right\}^n}{\{1 + (1 + x)\, t\}^{1+\alpha+\beta+n}} P_n^{(\alpha,\beta)} \left[\frac{t\{x + (x + 1)\, t\} + \frac{1}{\omega}\{1 + (x + 1)\, t\}}{\{1 + (x + 1)\, t\}\left\{t + \frac{1}{\omega}(1 + 2t)\right\}}\right]$$

$$= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-2)^k}{k\,!} \frac{\left(-\frac{1}{\omega}\right)^p}{p\,!} (n - p + 1)_k (-\alpha - n)_p P_{n-p+k}^{(\alpha,\beta+p-k)}(x)\, t^{k-p,1} \tag{3.6}$$

which is (1.4).

4. Derivation of Feldhim's formula from (3.5)

Putting $n = 0$, $t = -\frac{u}{2}$ in (3.5) we get

$$(1 - u)^{\beta}\left(1 - \frac{u}{2}(x + 1)\right)^{-1-\alpha-\beta} = \sum_{k=0}^{\infty} P_k^{(\alpha, \beta-k)}(x)\, u^k$$

which is Feldhim's formula[2].

Acknowledgement

I am greatly indebted to Professor Asit Kumar Chongdar for introducing me to this subject and for suggesting this particular area of investigation as well as for his constant encouragement and guidance. I also express my heartfelt thanks to the referee for his valuable suggestions to the improvement of the paper.

References

1. WEISNER, L. Group theoretic origins of certain generating functions, *Pacific J. Math.*, 1955, 5, 1033-39.

2. FELDHIM, E. Relations entre les polynomel de Jacobi, Laguerre of Hermite, *Acta Math.*, 1943, 75, 117-198, 120.