UNSTEADY FLOW TO A NONPENETRATING CAVITY WELL IN LEAKY AQUIFERS

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ABSTRACT

This paper presents an analytical solution of unsteady state flow to a nonpenetrating cavity well having the bottom as a segment of a sphere, in the lower layer of an artesian aquifer of infinite depth and extent. The problem is treated as one of flow to a well in a two layered aquifer. The upper layer is less pervious than the lower layer. The well known Laplace transform technique has been used to obtain the solution in terms of rapidly converging error functions. However the method of contour integration is also employed to obtain the solution in terms of a certain integral which is found to be more suitable for numerical computation. The solution for a nonleaky aquifer is obtained as a particular case.

Key words: Hydraulics, Cavity Well, Leaky Aquifers, Segmental bottom.

1. INTRODUCTION

The hydraulics of steady state flow to a nonpenetrating cavity well having a hemispherical bottom is given by Muskat.[1]

Mishra *et al*[2] and Chauhan[3] have analysed on lines similar to that of Muskat.[1] They have modified the hemispherical bottom considered by Muskat into a segment of a sphere as expected to be existing in a cavity well. Sarkar[4] has recently studied the unsteady flow into a nonpenetrating cavity well in a nonleaky aquifer with hemispherical bottom by considering the radius of influence of the circle to be infinite.

In this paper, the solution of unsteady flow into a nonpenetrating cavity well in a leaky aquifer having the bottom as a segment of a sphere in the lower layer of an artesian aquifer of infinite depth and extent has been obtained. The well-known Laplace transform technique has been used to obtain the solution in terms of rapidly converging error functions. However, the method of contour integration is also employed to obtain the

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solution in terms of a certain integral which is found to be more suitable for numerical computation. The solution for a nonleaky aquifer is obtained as a particular case.

NOTATIONS

K Hydraulic conductivity,

S Storativity of the lower pervious bed,

λ Leakage factor,

r_w Radius of the well,

Q Rate of pumping,

T Depth of a segment,

$I_{\frac{1}{2}}(\mu r)$ Bessel function of the first kind of half order,

$K_{\frac{1}{2}}(\mu r)$ Bessel function of the second kind of half order.

2. THEORY

This study is based on the assumption that the change in the rate of leakage is proportional to the drawdown, *i.e.*, the water in the aquifer overlying the semi-pervious layer in steady and the hydraulic gradient through the layer has adjusted itself completely to the new potential distribution due to pumping. When this is not so, unsteady flow exists.

The problem may be treated as one of flow into a well in a two layered aquifer. The less pervious upper layer of hydraulic conductivity K' and thickness b' overlies the lower pervious layer (K, b) in which the well bottom is located. It is assumed that the cavity developed by pumping water is symmetric and is a segment of a sphere of radius r_w, the resistance to the flow within the cavity is zero and that the radius of the influence circle is finite. The storativity in the less permeable upper bed is neglected.

The differential equation and the boundary conditions governing the flow system in spherical polar coordinates are given by

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial s}{\partial r}\right) - \frac{s}{\lambda^2} = \frac{1}{C}\frac{\partial s}{\partial t} \tag{2.1}$$

$$s(r, 0) = 0, \qquad r > 0 \tag{2.2}$$

$$s(R, t) = 0, \qquad t > 0 \tag{2.3}$$

$$\lim_{r \to r_w}\left(\frac{\partial s}{\partial r}\right) = -\frac{Q}{2\pi K T r_w} \tag{2.4}$$

where

$$C = \frac{K}{S}.$$

3. GENERAL SOLUTION

Employing the Laplace Transform with respect to time t and using the initial condition (2.2), one obtains

$$\frac{d^2 \bar{s}}{dr^2} + \frac{2}{r} \frac{d\bar{s}}{dr} - \frac{\bar{s}}{\lambda^2} = \frac{p}{C} \bar{s} \tag{3.1}$$

The solution of (3.1) can be written as

$$\bar{s} = A \sqrt{\frac{\pi}{2\mu r}} \ I_{1/2}(\mu r) + B \sqrt{\frac{\pi}{2\mu r}} \ K_{1/2}(\mu r) \tag{3.2}$$

where

$$\mu^2 = \left(\nu^2 + \frac{1}{\lambda^2} \right), \qquad \nu^2 = p/C.$$

Using the boundary conditions (2.3) and (2.4), results in

$$A I_{1/2}(\mu R) + B K_{1/2}(\mu R) = 0 \tag{3.3}$$

$$\left(\frac{d\bar{s}}{dr} \right) r \to r_w = \mu \sqrt{\frac{\pi}{2\mu r_w}} \ [A I_{3/2}(\mu r_w) - B K_{3/2}(\mu r_w)]. \tag{3.4}$$

Thus from equation (2.4), one gets

$$\mu \sqrt{\frac{\pi}{2\mu r_w}} [A I_{3/2}(\mu r_w) - B K_{3/2}(\mu r_w)] = \frac{-Q}{2\pi \, k T r_w \, p} . \tag{3.5}$$

Evaluating the constants A and B with the help of Eqs. (3.3) and (3.5) one gets

$$A = \frac{Q K_{1/2}(\mu R)}{PEf(R, \, r_w, \, \mu)} \tag{3.6}$$

$$B = \frac{Q I_{1/2}(\mu R)}{PEf(R, \, r_w, \, \mu)} \tag{3.7}$$

where

$$E = (2\pi^3 \, r_w \, \mu)^{1/2} \, KT$$

$$f(R, \, r_w, \, \mu) = I_{3/2}(\mu r_w) \ K_{1/2}(\mu R) + K_{3/2}(\mu r_w) \ I_{1/2}(\mu R)$$

Substituting from (3.6) and (3.7) in (3.2), one gets

$$\bar{s} = \frac{Q \sqrt{\frac{\pi}{2\mu r}} \left[I_{1/2}(\mu R) \ K_{1/2}(\mu r) - I_{1/2}(\mu R) \ K_{1/2}(\mu R) \right]}{PEf(R, r_w, \mu)}. \tag{3.8}$$

The inverse Laplace Transform of (3.8) is not available as such; so using asymptotic expansion of $I_\nu(z)$ and $K_\nu(z)$ as given by Mclachlan[5] one gets

$$[K_{1/2}(\mu r) \ I_{1/2}(\mu R) - I_{1/2}(\mu r) \ K_{1/2}(\mu R)]$$
$$= \frac{1}{2\mu}(rR)^{-1/2} \left[\exp\{\mu(R-r)\} - \exp\{-\mu(R-r)\} \right] \tag{3.9}$$

$$f(R, r_w, \mu) = \frac{1}{2\mu}(r_w R)^{-1/2} \left[\left(1 - \frac{1}{\mu r_w}\right) \exp\{-\mu(R-r_w)\} \right.$$
$$\left. + \left(1 + \frac{1}{\mu r_w}\right) \exp\{\mu(R-r_w)\} \right]. \tag{3.10}$$

Employing (3.9) and (3.10) in (3.8) one obtains

$$\bar{s} = \frac{\frac{Q}{PEr}\sqrt{\frac{\pi r_w}{2\mu}} \left[\exp\{\mu(R-r)\} - \exp\{-\mu(R-r)\}\right]}{\left(1 + \frac{1}{\mu r_w}\right)\exp\{\mu(R-r_w)\} + \left(1 - \frac{1}{\mu r_w}\right)\exp\{-\mu(R-r_w)\}} \tag{3.11}$$

Neglecting the higher order terms and after simplification, one gets

$$\bar{s} = \frac{Qr_w \left[\exp\{-\mu(r-r_w)\} - \exp\{-\mu(2R-r-r_w)\}\right]}{2\pi\,KTrp\,(1+r_w\mu)}. \tag{3.12}$$

Resolving Eq. (3.12) into the following factors, one may write

$$\bar{s} = \frac{Qr_w}{2\pi KTr} \frac{[\exp(-d_1 m) - \exp(-d_2 m)]}{g(a, b, p)} \tag{3.13}$$

where

$$g(a, b, p) = \left[\frac{a^2}{(a^2-b^2)(a+m)} + \frac{a}{2b(a+b)(-b+m)} \right.$$
$$\left. - \frac{a}{2b(a-b)(b+m)} \right]$$
$$a = \frac{\sqrt{C}}{r_w}, \quad \sqrt{p+b^2} = m, \ d_1 = \frac{r-r_w}{\sqrt{C}}$$
$$d_2 = \frac{(2R-r-r_w)}{\sqrt{C}}, \quad b^2 = \frac{C}{\lambda^2}$$

Now employing the shifting property of Laplace Transform and the result contained in Carslaw and Jaeger,[6] one gets

$$
\begin{aligned}
s = \frac{Qr_w}{2\pi KTr} \Bigg[&- \frac{a^2}{(a^2 - b^2)} \exp\{ad_1 + t(a^2 - b^2)\} \\
&\times \operatorname{erfc}\left(\frac{d_1}{2\sqrt{t}} + a\sqrt{t}\right) + \frac{a \exp(-bd_1)}{2(a+b)} \operatorname{erfc}\left(\frac{d_1}{2\sqrt{t}} - b\sqrt{t}\right) \\
&+ \frac{a \exp(bd_1)}{2(a-b)} \operatorname{erfc}\left(\frac{d_1}{2\sqrt{t}} + b\sqrt{t}\right) + \frac{a^2}{(a^2 - b^2)} \\
&\times \exp\{ad_2 + t(a^2 - b^2)\} \operatorname{erfc}\left(\frac{d_2}{2\sqrt{t}} + a\sqrt{t}\right) - \frac{a}{2(a+b)} \\
&\times \exp(-bd_2) \operatorname{erfc}\left(\frac{d_2}{2\sqrt{t}} - b\sqrt{t}\right) - \frac{a}{2(a-b)} \exp(bd_2) \\
&\times \operatorname{erfc}\left(\frac{d_2}{2\sqrt{t}} + b\sqrt{t}\right) \Bigg] .
\end{aligned}
\tag{3.14}
$$

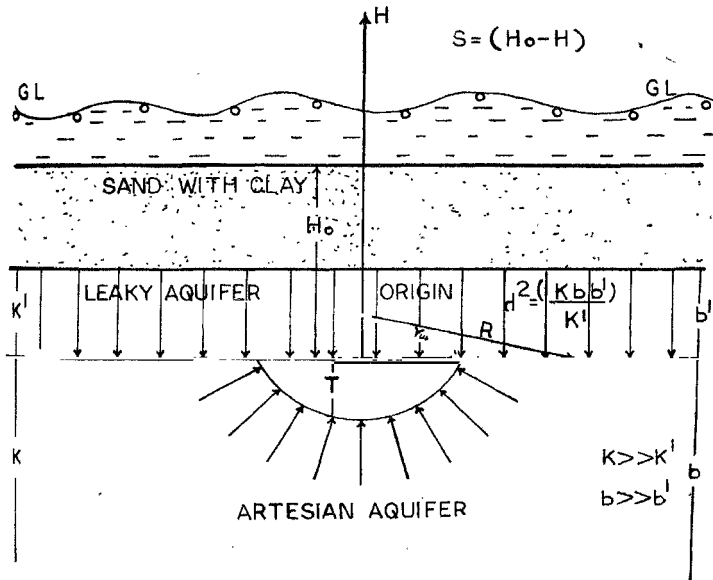FIG. 1. Well having segmental bottom in the artesian aquifer.

where

$$\text{erfc}\,(z) = \frac{2}{\sqrt{\pi}}\int\limits_{z}^{\infty}\exp\,(-\,\beta^2)\,d\beta.$$

Letting $R \to \infty$, $r_w \to r_0$ and $b \to 0$ in Eq. (3·14), one obtains

$$s = \frac{Q}{2\pi Kr}\left[-\exp\,(ad_1 + a^2 t)\,\text{erfc}\,\left(a\,\sqrt{t} + \frac{d_1}{2\,\sqrt{t}}\right)\right.$$
$$\left.+\,\text{erfc}\,\left(\frac{d_1}{2\,\sqrt{t}}\right)\right] \tag{3.15}$$

This is the solution for a nonleaky aquifer, obtained by Sarkar.[4]
However, the inverse transform of s can also be obtained as follows:
From the table of transform[6] one has

$$\frac{\exp\,(-\,d\,\sqrt{p})}{(a + \sqrt{p})} = \int\limits_{0}^{\infty}\exp\,(-\,pt)\left[\frac{\exp\,(-\,d^2/4t)}{\sqrt{\pi t}}\right.$$
$$\left.-\,\exp\,(ad + a^2\,t)\,\text{erfc}\,\left(a\,\sqrt{t} + \frac{d}{2\,\sqrt{t}}\right)\right]dt. \tag{3.16}$$

Substitution of $(p + b^2)$ for the variable p in Eq. (3.16) gives

$$\frac{\exp\,(-\,d\,\sqrt{p + b^2})}{(a + \sqrt{p + b^2})} = \int\limits_{0}^{\infty}\exp\,(-\,pt)\left[\frac{\exp\,(-\,d^2/4t - b^2\,t)}{\sqrt{\pi t}}\right.$$
$$\left.-\,a\exp\,(ad + a^2\,t - b^2\,t)\,\text{erfc}\,\left(a\,\sqrt{t} + \frac{d}{2\sqrt{t}}\right)\right]dt. \tag{3.17}$$

Finally, using the theorem that multiplication by p^{-1} of the Laplace Transform of a function corresponds to integration with respect to the time of the original function, it is found that

$$\frac{\exp\,(-\,d\,\sqrt{p + b^2})}{p\,(a + \sqrt{p + b^2})} = \int\limits_{0}^{\infty}\exp\,(-\,pt)\left[\int\limits_{0}^{t}\frac{\exp\,(-\,d^2/4\tau - b^2\,\tau)}{\sqrt{\pi\tau}}\right.$$
$$\left.-\,a\exp\,(ad + a^2\,\tau)\,\text{erfc}\,\left(\frac{d}{2\,\sqrt{\tau}} + a\,\sqrt{\tau}\right)\,d\tau\right]dt. \tag{3.18}$$

Employing Eq. (3.18) to determine the inverse transform of Eq. (3.12), one gets

$$
s = \frac{Qr_w a}{2\pi KTr} \int_0^t \left[\exp\left(-b^2 t\right) \{ \exp\left(-d_1{}^2/4t\right) \right.
$$
$$
- \exp\left(-d_2{}^2/4t\right)\} - a \exp\left(a^2 t - b^2 t\right)
$$
$$
\times \left\{ \exp\left(ad_2\right) \operatorname{erfc}\left(a\sqrt{t} + \frac{d_2}{2\sqrt{t}} \right) - \exp\left(ad_1\right) \right.
$$
$$
\left. \left. \times \operatorname{erfc}\left(a\sqrt{t} + \frac{d_1}{2\sqrt{t}} \right) \right\} \right] dt. \tag{3.19}
$$

4. EVALUATION OF THE INVERSE LAPLACE TRANSFORM

It is found that results (3.14) and (3.19) are not suitable for numerical computation; hence the inverse transform will be evaluated by integrating along the contour shown in Fig. 2, which consists of a line $R(s) = \gamma$ and a large semi-circle to the left of it, having a pole at $s = 0$ and a branch cut extending from $s = -b^2$ to the real negative axis.
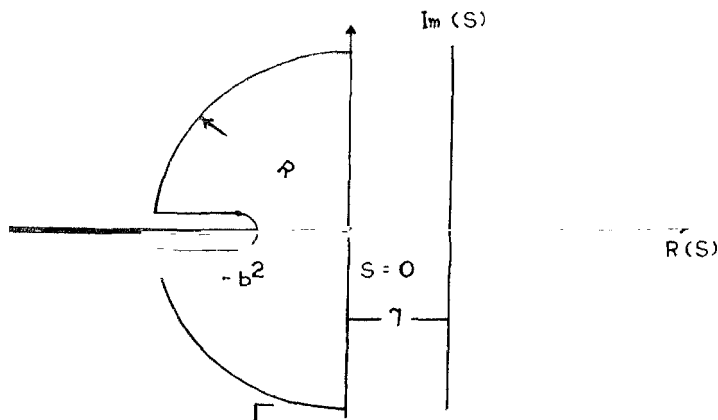

FIG. 2

Employing Cauchy's residue theorem one obtains

$$\frac{1}{2\pi i} \left[\int_{\gamma-iR}^{\gamma+iR} + \int_{\Gamma} + \int_{cut} \right] f(s) \exp(ts)\, ds$$

$$= \Sigma \text{ Residue} \tag{4.1}$$

where Γ is a large semi-circle of radius R, having a branch cut from $-b^2$ to the negative real axis and $f(s)$ is given by

$$f(s) = \frac{\exp\{-d_1(\sqrt{s+b^2})\} - \exp\{-d_2(\sqrt{s+b^2})\}}{s(a+\sqrt{s+b^2})}.$$

In the integral on the cut, substituting $(s+b^2) = \rho\, e^{-\pi i}$ along the lower side of the cut and $(s+b^2) = \rho e^{+\pi i}$ along the upper side of the cut and since the integrand on Γ is of $0\,(R^{-3/2})$ and $R \to \infty$; it follows that the integral along Γ vanishes and finally, we have

$$\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f(s) \exp(ts)\, ds = \frac{\exp(-d_1 b^2) - \exp(-d_2 b^2)}{(a+b)} + I \tag{4.2}$$

where

$$I = \frac{2}{\pi} \int_0^\infty \exp - (a^2 + b^2)\, t \left[a\alpha \left\{ \frac{\sin(d_2\alpha) - \sin(d_1\alpha)}{(a^2+\alpha^2)(b^2+\alpha^2)} \right\} \right.$$
$$\left. + \alpha^2 \left\{ \frac{\cos(d_2\,\alpha) - \cos(d_1\,\alpha)}{(a^2+\alpha^2)(b^2+\alpha^2)} \right\} \right] d\alpha.$$

Using Eq. (4.2), one may finally write

$$s = \frac{Qr_w\,a}{2\pi KTr} \left[\frac{\exp(-d_1\,b^2) - \exp(-d_2\,b^2)}{(a+b)} + I \right]. \tag{4.3}$$

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