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Short Communication

Bounds on the flow rate for pipe flow

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Abstract

The upper and lower bounds on the flow rate of viscous incompresssible fluid through a straight pipe of arbitrary cross-section filled with porous material are derived in a simple manner.

Key words : Bounds, flow rate, porous material, pipe flow.

1. Introduction

The bounds on the flow rate for steady Poiseuille flow through a straight pipe of arbitrary cross-section filled with porous material are obtained by the application of Gauss divergence theorem and standard inequalities and they are found to be in agreement with [1]. Bounds on the flux are given for the following types of cross-section of the pipe :

(*a*) a curvilinear triangle bounded by the arc of a cardioid, arc of a parabola and the axis of the cardioid ; and

(*b*) a curvilinear quadrilateral bounded by the arcs of four confocal parabolas.

An attempt has been made to provide bounds for the flux through a pipe having an annulus as cross-section. It is important to note that we are able to cope with multiply connected flow sections.

2. Derivation of bounds

2.1 *Pipe of arbitrary cross-section*

The equation for the axial velocity for Poiseuille flow of a viscous incompressible liquid in a long straight pipe of arbitrary cross-section filled with porous material is

$$\partial^2 \omega/\partial x^2 + \partial^2 \omega/\partial y^2 = \omega/k + p'/\mu \text{ on } S \tag{2.1}$$

$$\omega = 0 \text{ on } \partial S \tag{2.2}$$

where ∂S denotes the boundary of the cross-section S, $p' < 0$ the constant axial pressure gradient along the axis OZ of the pipe and k is the permeability of the porous medium. The flow rate

$$Q = \int_S \omega \, dS \tag{2.3}$$

In view of (2.1), the divergence theorem leads to

$$Q = -(\mu/p') \int_S (|\nabla \omega|^2 + \omega^2/k) \, dS \tag{2.4}$$

Let $\bar{V}$ be a vector field satisfying

$$\nabla \cdot \bar{V} = \omega/k + p'/\mu \tag{2.5}$$

The divergence theorem and (2.5) imply

$$Q = -(\mu/p') \int_S (\bar{V} \cdot \nabla\omega + \omega^2/k) \, dS. \tag{2.6}$$

Since

$$\bar{V} \cdot \nabla \omega \leqslant \tfrac{1}{2} (|\bar{V}|^2 + |\nabla\omega|^2) \tag{2.7}$$

from (2.6) and (2.7) we have

$$Q \leq -(\mu/p') \int_S (|\bar{V}|^2 + \omega^2/k) \, dS \tag{2.8}$$

Let $u = 0$ on ∂S and $\int_S (|\nabla u|^2 + u^2/k) \, dS \neq 0$. The divergence theorem and (2.1) give

$$-(p'/\mu) \int_S u \, dS = \int_S (\nabla u \cdot \nabla \omega + u\omega/k) \, dS. \tag{2.9}$$

Squaring both sides and applying the Cauchy-Schwartz inequality

$$\left(\int_S (\bar{a} \cdot \bar{b} + \bar{c} \cdot \bar{d}) \, dS\right)^2 \leq \left(\int_S (\bar{a}^2 + \bar{c}^2) \, dS\right) \left(\int_S \bar{b}^2 + \bar{d}^2) \, dS\right).$$

We have

$$-(p'/\mu) \left(\int_S u \, dS\right)^2 / \int_S (|\nabla u|^2 + u^2/k) \, dS \leq Q. \tag{2.10}$$

Instead of (2.10) we consider the bound

$$Q \geq 2 \int_S u\, dS + (\mu/p') \int_S (|\nabla u|^2 + u^2/k)\, dS. \tag{2.11}$$

Evidently the scaled maximum of (2.11) is (2.10). Thus from (2.8) and (2.11) we have

$$2 \int_S u\, dS + (\mu/p') \int_S (|\Delta u|^2 + u^2/k)\, dS \leq Q$$

$$\leq - (\mu/p') \int_S (|\bar{V}|^2 + \omega^2/k)\, dS. \tag{2.12}$$

The bounds given by (2.12) are seen to be in agreement with [1].

2.2. *Pipe of cross-section (a)*

Consider the transformation

$$z = c\,(1 + \exp(\phi))^2 \tag{2.13}$$

where $z = (x + iy) = \gamma \exp(i\theta)$, $\phi = \xi + i\eta$. Then $\xi = 0$ is the cardioid $\gamma = 2c(1 + \cos\theta)$, $\xi = -\infty$ is the point $(c, 0)$, $\eta = 0$ is the part of the real axis extending from $(c, 0)$ to ∞, $\eta = \pi/2$ is the upper-half of the parabola $2c/\gamma = (1 + \cos\theta)$. Using (2.13), eqn. (2.1) transforms to

$$\partial^2 \omega/\partial\xi^2 + \partial^2 \omega/\partial\eta^2 = \exp(2\xi)(\lambda + \beta\omega)(1 + \exp(2\xi) + 2\cos\eta \exp(\xi)) \tag{2.14}$$

where $\lambda = 4c^2 p'/\mu$, $\beta = 4c^2/k$. The boundary conditions are

$$\left.\begin{aligned} \omega = 0 \text{ when } \xi &= 0, -\infty \\ \eta &= 0, \pi/2 \end{aligned}\right\}. \tag{2.15}$$

Proceeding as in section (2.1), the bounds on the flux Q_1 are given by

$$8c^2 \int_S e^{2\xi} (1 + e^{2\xi} + 2e^{\xi} \cos\eta)\, u\, dS$$

$$+ (\mu/p') \int_S (|\nabla u|^2 + \beta u^2 e^{2\xi} (1 + e^{2\xi} + 2e^{\xi}\cos\eta))\, dS$$

$$\leq Q_1 \leq - (\mu/p') \int_S (|\bar{V}|^2 + \beta e^{2\xi} (1 - e^{2\xi} + 2e^{\xi} \cos\eta)\, \omega^2)\, dS \tag{2.16}$$

2.3 *Pipe of cross-section (b)*

Introduce the transformation

$$z = \phi^2. \tag{2.17}$$

Then $\xi = \xi_1$, $\xi = \xi_2$, $\eta = n_1$, $\eta = \eta_2$ are four confocal parabolas. Using (2.17), (2.1) transforms to

$$\partial^2 \omega/\partial\xi^2 + \partial^2 \omega/\partial\eta^2 = (\xi^2 + \eta^2)(\lambda + \beta\omega) \tag{2.18}$$

where $\lambda = 4p'/\mu$, $\beta = 4/k$. The bounds on the flux Q_2 are obtained as

$$8 \int_S (\xi^2 + \eta^2)\, u\, dS + (\mu/p') \int_S (|\nabla u|^2 + \beta(\xi^2 + \eta^2) u^2)\, dS$$
$$\leq Q_2 \leq -(\mu/p') \int_S (|\bar{V}|^2 + \beta(\xi^2 + \eta^2)\omega^2)\, dS. \tag{2.19}$$

As $k \to \infty$ (*i.e.*), $\beta \to 0$ the bounds on the flux for pipe flow in the absence of porous material are deduced.

3. Calculation of upper bound

Annular domain : Let t be the thickness which is taken to be uniform ; s is the arc length along the mid-curve C and a is the distance measured along the normal to C. Let ω_s and ω_a be the components of ω in the s and a directions. In terms of the coordinates (s, a)

$$\nabla \cdot \omega = \frac{1}{(1 - aK)} [\partial \omega_s/\partial s + \partial((1 - aK)\,\omega_a)/\partial a] \tag{3.1}$$

where K is the curvature of C. We seek ω such that $\omega_s = 0$, $\omega_a = \omega(s, a)$. Consequently, from (2.5) and (3.1) we have

$$\partial((1 - aK)\,\omega)/\partial a = p'(1 - aK)/\mu + v(1 - aK)/k \tag{3.2}$$

which on integration leads to

$$\omega(s, a) = -p'(1 - aK)/2\mu K + f(s)/(1 - aK)$$
$$+ (a^3(4 - 3aK)/12 + t^2(1 - aK)^2/8K)/k(1 - aK) \tag{3.3}$$

where we have taken $v = (a^2 - t^2/4)$ and $f(s)$ is an undetermined function. It is clear that $f(s)$ should be chosen so as to minimize the integral $I(f) = \int_S (|\omega|^2 + v^2/k)\, dS$. In terms of the coordinates (s, a), this is written as

$$I(f) = \int_0^l \int_{-t/2}^{t/2} (|\omega|^2 + v^2/k)(1 - aK)\, da\, ds. \tag{3.4}$$

The suitable choice for $f(s)$ is

$$f(s) = \frac{1}{2\log \dfrac{1 - tK/2}{(1 + tK/2)}} \left\{ t^3/4k - p't/\mu - (1/6k)\left[t^3/12 + t/K^2 \right.\right.$$
$$\left.\left. + 1/K^3 \log\left(\frac{1 - tK/2}{1 + tK/2}\right)\right]\right\}. \tag{3.5}$$

Now that $f(s)$ has been found, from (3.3) and (3.5) we have

$$\omega(s, a) = (p'a/\mu - at^2/4k + a^3/3k) \quad (K = 0,\ -t/2 \leq a \leq t2)$$
$$= -p'(1 - aK)/2\mu K + [a^3(4 - 3aK)/12$$
$$+ t^2(1 - aK)^2/8K]/k(1 - aK)$$

$$+ \frac{1}{2(1-\alpha K)} \frac{1}{\log\left(\frac{1 - tK/2}{1 + tK/2}\right)} \left[- p' \, t/\mu - t/6\, kK^2 + 17\, t^3/72k \right.$$
$$\left. - \frac{1}{6\, kK^3} \log\left(\frac{1 - tK/2}{1 + tK/2}\right)\right] (K \neq 0, \; - t/2 \leq \alpha \leq t/2).$$

Thus from (3.4), (2.8) and (3.5) we get the upper bound on Q. Note that the bound is valid for the domain doubly connected.

4. Calculation of lower bound

Annular section : Take $u = (\alpha^2 - t^2/4)$ which satisfies the boundary condition. Now we see that

$$\left.\begin{aligned} &\int_S u\, dS = - t^3\, l/6, \quad \int_S |\nabla u|^2\, dS = t^3 l/3, \\ &\int_S u^2/k\, dS = t^5\, l/30\, k \end{aligned}\right\} \tag{4.1}$$

From (2.11) and (4.1) it follows that

$$Q \geq - (p'/12\mu)\,(t^3 l/(1 + t^2/10\, k)).$$

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Reference

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