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Two results on table matrix *L*-systems

NALINAKSHI NIRMAL
Department of Mathematics, Madras Christian College, Madras 600 059, India.

AND

KAMALA KRITHIVASAN
Computer Centre, Indian Institute of Technology, Madras 600 036, India.

Abstract

Array control on the rules of the tables of Part TMLS are defined as controlled associated derivation. Two main results are proved.

Key words: Partial table matrix *L*-systems, controlled associated derivation, context-free matrix languages.

1. Introduction

L-systems have stimulated a substantial amount of research much of which can be found in literature[1,2]. In the recent past, several attempts were made by many investigators to incorporate the developmental type of generation used in *L*-systems to higher dimensions[3,4,5]. Motivated by the idea of generating array languages which expand linearly and where the inner elements also grow apart from the growth along the edges, we define table matrix *L*-systems[6]. In this model, in each column (row) of a rectangular array, the growth or derivation is controlled by tables but in the entire rectangular array, the growth or derivation takes place in such a way that the rectangular format is maintained. The productions are given in the form of row (column) tables. The right side of every rule in a row (column) table has the same number of rows (columns). Every table consists of at least one rule for every element of the alphabet. The axiom is a rectangular array whose elements are elements of the alphabet Σ. The rules from the row (column) tables are applied to the elements of the axiom row by row (column by column). By such application if the derived word is rectangular, then the rules from the row (column) tables can be applied again, otherwise the derivation comes to an end. If the completeness condition is removed then we call the system as partial table matrix *L*-system.

In this paper we define an array control called the controlled associated derivation on the rules of the tables of Part TMLS. The first result that we prove is that the family of Part TMLL with det_1 (Part TMLL) as controlled associated derivation will be equal to the family of det_1 (Part CTMLL). The authors propose a linguistic model called the abstract family of matrices for the generation of rectangular array of terminal.[6], by the substitution of regular sets into well-known families of formal languages. This model generates interesting class of pictures like token I, T of different sizes and proportions.

In this paper, the second result that we prove is that the family of CFML[6] is a proper subset of the family of CTMLL.

2. Definitions and main results

In this section, we first review some basic definitions including the definitions of table matrix *L*-systems, Part TMLS and Part CTMLS.

Definition 2.1 : A table matrix *L*-system with row (column) tables $TMLS_r$ $(TMLS_c)$ is a 3-tuple $G = (\Sigma, \mathcal{P}, \omega)$ where

(i) Σ is a finite non-empty set, the alphabet of G.

(ii) $\mathcal{P}$ is a finite non-empty set of row (column) tables which we denote as $\mathcal{P}_r$ $(\mathcal{P}_c)$, *i.e.*, $\mathcal{P} = \{P_1, P_2, \ldots, P_f\}$ for some $f \geq 1$. Each element of P is a finite subset of $\Sigma \times \Sigma^{**}$ consisting of rules of the form

$$\{a_1 \rightarrow \alpha_1, a_2 \rightarrow \alpha_2, \ldots, a_k \rightarrow \alpha_k), \text{ where } \alpha_1, \alpha_2, \ldots, \alpha_k$$

have the same number of rows (columns). $\mathcal{P}$ satisfies the following completeness condition

$$(\forall P)_P (\forall_a)_\Sigma (\exists_\alpha)_\Sigma{}^{**} (< a, \alpha > \in P)$$

(iii) $\omega \in \Sigma^{++}$ is the start matrix or axiom of G.

The derivations are defined as follows:

$$\text{If } \omega = \begin{matrix} a_{11} \cdots\cdots a_{1s} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ a_{t1} \cdots\cdots a_{ts} \end{matrix} \quad \text{and if } \mathcal{P} = \mathcal{P}_r = \{P_1, P_2, \ldots, P_s\}$$

$(\mathcal{P} = \mathcal{P}_c = \{P_1, P_2, \ldots, P_t\})$, then we apply to ω, the rules from the tables of $\mathcal{P}_r (\mathcal{P}_c)$ row by row (column by column), *i.e.*, we choose a P_{i_1} in $\mathcal{P}_r$ (P_{i_1} in $\mathcal{P}_c$) and apply the rules to the first row (column) of ω. Next we choose a P_{i_2} in $\mathcal{P}_r$ (P_{i_2} in $\mathcal{P}_c$) and apply the rules to the second row (column). Proceeding in this manner we apply the rules from a table to a row (column) of ω. By such applications if the resultant array is rectangular then the rules from the tables $\mathcal{P}_r$ ($\mathcal{P}_c$) can be applied again, otherwise the derivation comes to an end.

Definition 2.2: Let $G = (\Sigma, \mathcal{P}, \omega)$ be a TMLS. Let M_1, $M_2 \in \Sigma^{++}$

$$\text{and let } M_1 = \begin{matrix} a_{11} & \cdots & a_{1k} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{l1} & \cdots & a_{lk} \end{matrix}, \quad M_2 = \begin{matrix} \alpha_{11} & \cdots & \alpha_{1k} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \alpha_{l1} & \cdots & \alpha_{lk} \end{matrix}$$

where $a_{ij} \in \Sigma$ and $\alpha_{ij} \in \Sigma^{+*}$. We say that M_1 *directly derives* M_2 in $G(M_1 \underset{G}{\Rightarrow} M_2)$ if M_2 is the resultant rectangular array obtained by applying rules in $\mathcal{P}$ to M_1 row by row (column by column). $\Rightarrow^*$ is the reflexive transitive closure of $\Rightarrow$.

Let $G = (\Sigma, \mathcal{P}, \omega)$ be a TMLS. The language generated by G is defined as

$$L(G) = \{x \in \Sigma^{**} / \omega \overset{*}{\underset{G}{\Rightarrow}} x\}.$$

Let Σ be a finite alphabet and $L \subseteq \Sigma^{**}$. L is called a table matrix L language (TMLL) if and only if, there exists a TMLS G such that $L = L(G)$.

We denote the TMLS with row (column) tables by TMLS$_r$ (TMLS$_c$) and the languages by TMLL$_r$ (TMLL$_c$). If the meaning is clear, we omit c or r and just write as TMLS and TMLL.

Definition 2.3: Let $G = (\Sigma, \mathcal{P}, \omega)$ be a TMLS. G is said to be (i) *deterministic* iff for each P in $\mathcal{P}$ and each a in Σ there is exactly one rule $a \to \alpha$ in P,

(ii) *propagating* if for each P in $\mathcal{P}$ we have $P \subset \Sigma \times \Sigma^{++}$.

It is seen that if the completeness condition is not imposed in definition 2.1, then some interesting classes of pictures like Kirsch's triangles will be generated[4].

Definition 2.4: A partial table matrix L-system row (column) (Part TMLS$_r$) (Part TMLS$_c$) is a 3-tuple $G = (\Sigma, \mathcal{P}, \omega)$ where

(i) Σ is a finite non-empty set, the alphabet of G.

(ii) $\mathcal{P}$ is a finite non-empty set of row (column) tables which we denote as $\mathcal{P}_r$ ($\mathcal{P}_c$), *i.e.*, $\mathcal{P} = \{P_1, P_2, \ldots, P_f\}$ for some $f \geq 1$, each element P_i is a finite subset of $\Sigma \times \Sigma^{**}$, consisting of rules of the form $\{a_1 \to \alpha_1, a_2 \to \alpha_2, \ldots, a_k \to \alpha_k\}$, where $\alpha_1, \ldots, \alpha_k$ have the same number of rows (columns).

(iii) $\omega \in \Sigma^{++}$ is the start matrix or axiom of G.

The derivations are defined as follows:

$$\text{If } \omega = \begin{matrix} a_{11} \cdots\cdots\cdots a_{1s} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ a_{t1} \cdots\cdots\cdots a_{ts}. \end{matrix} \quad \text{and if } \mathcal{P} = \mathcal{P}_r = \{P_1, P_2, \ldots, P_k\}$$

($\mathcal{P} = \mathcal{P}_c = \{P_1, P_2, \ldots, P_k\}$), then we apply to ω, the rules from the tables of $\mathcal{P}_r$ ($\mathcal{P}_c$) row by row (column by column) *i.e.*, if we choose a P_i in $\mathcal{P}_r$ (P_i in $\mathcal{P}_c$) which has rules with $a_{11}, a_{12}, \ldots, a_{1s}$ ($a_{11}, a_{21}, \ldots, a_{t1}$) on the left side and apply to the first row (column) of ω. Next we choose a P_j in $\mathcal{P}_r$ (P_j in $\mathcal{P}_c$) which has rules with $a_{21}, a_{22}, \ldots, a_{2s}$ ($a_{12}, a_{22}, \ldots, a_{t2}$) on the left side and apply to the second row (second column) of ω.

Proceeding in this way we choose a P_k in $\mathcal{P}_r$ (P_k in $\mathcal{P}_c$) which has rules with $a_{t1}, a_{t2}, \ldots, a_{ts}$ ($a_{1s}, a_{2s}, \ldots, a_{ts}$) on the left side and apply to the tth row (sth column) of ω. The rules are applied in such a manner that a rectangular array results. Rules may again be applied to the resultant rectangular array in a similar manner. If a rectangular array cannot be obtained by applying the rules row by row (column by column) then the derivation comes to an end. $M_1 \underset{G}{\Rightarrow} M_2$, the language generated by a Part TMLS can be defined similar to that of TMLS.

Definition 2.5: Let $G = (\Sigma, \mathcal{P}, \omega)$ be a Part TMLS, G is said to be (*i*) *deterministic of the first kind* if there is only one rule in $\mathcal{P}$ for any $a \in \Sigma$ and denoted by $\det_1$ (Part TMLS). (ii) *deterministic of the second kind* if for every P in $\mathcal{P}$ there is atmost one rule $a \to \alpha$ for any $a \in \Sigma$ and denoted by $\det_2$ (Part TMLS), (iii) propagating if for each P in $\mathcal{P}$ we have $P \subset \Sigma \times \Sigma^{++}$.

Definition 2.6: The partial coding table matrix L-system row (column) (Part CTMLS_r (Part CTMLS_c)) is a 5-tuple $G = (V, \mathcal{P}, \omega, \Sigma, h)$ where
(i) $(V, \mathcal{P}, \omega)$ is a Part TMLS.
(ii) Σ is a non-empty finite set called the target alphabet.
(iii) h is the partial coding from V into Σ.

Starting from ω, arrays are derived by Part TMLS $(V, \mathcal{P}, \omega)$ and then coding h is applied to these arrays, where

$$h\begin{bmatrix} a_{11} \cdots\cdots\cdots a_{1n} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ a_{m1} \cdots\cdots\cdots a_{mn} \end{bmatrix} = \begin{matrix} h(a_{11}) \cdots\cdots h(a_{1n}) \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ h(a_{m1}) \cdots\cdots h(a_{mn}) \end{matrix}$$

$$h\begin{bmatrix} a_{11} & \cdots\cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots\cdots & a_{mn} \end{bmatrix}$$ is undefined if there is at least one a_{ij} for which $h(a_{ij})$ is not defined.

Let $G = (V, \mathcal{P}, \omega, \Sigma, h)$ be a Part CTMLS. The language generated by G is defined as

$$L(G) = \{M \in \Sigma^{**} / \omega \Rightarrow^* M', \ h(M') = M\}. \quad L \subseteq \Sigma^{**}$$

is called a Part CTMLL if and only if there exists a Part CTMLS G such that $L = L(G)$.

Definition 2.7: Let $G = (V, \mathcal{P}, \omega, \Sigma, h)$ be a Part CTMLS. G is said to be (i) *deterministic of the first kind* if the Part TMLS $(V, \mathcal{P}, \omega)$ is deterministic of the first kind, (ii) *deterministic of the second kind* if the Part TMLS $(V, \mathcal{P}, \omega)$ is deterministic of the second kind. (iii) *propagating* if the Part TMLS $(V, \mathcal{P}, \omega)$ is propagating.

Now let us define an associated derivation of a Part TMLS.

Definition 2.8: Let $G = (\Sigma, \mathcal{P}_r, \omega)$ be a Part TMLS, $\mathcal{P}_r = \{P_1, P_2, \ldots, P_k\}$, each P_i consists of a set of rules of the form $a_i \rightarrow \alpha_i$.

If $M_1 = \begin{matrix} a_{11} \cdots\cdots a_{1n} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ a_{m1} \cdots\cdots a_{mn} \end{matrix}$ and $M_1 \underset{G}{\Rightarrow} M_2$, $M_2 = \begin{matrix} \alpha_{11} \cdots\cdots \alpha_{1n} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \alpha_{m1} \cdots\cdots \alpha_{mn} \end{matrix}$

where $\{a_{i1} \rightarrow \alpha_{i1}, \ldots\ldots, a_{in} \rightarrow \alpha_{in}\}$ is a table of $\mathcal{P}_r$, $\alpha_{i1} \ldots\ldots, \alpha_{in}$ having the same number of rows. We attach a label to each rule in a table, *i.e.*, lab $(a \rightarrow \alpha) = t_p$. If t_{ij} is the label of the production $a_i \rightarrow \alpha_{ij}$, then the

array $\begin{matrix} t_{11} \cdots\cdots t_{1n} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ t_{m1} \cdots\cdots t_{mn} \end{matrix} = \beta$ is associated with the derivation

$M_1 \underset{G}{\Rightarrow} M_2$. We call β the array associated with the derivation $M_1 \underset{G}{\Rightarrow} M_2$.

If $\omega = M_0 \Rightarrow M_1 \Rightarrow M_2 \Rightarrow \ldots\ldots \Rightarrow M_n$ is a derivation in Part TMLS, G and β_i is the array associated with the derivation $M_{i-1} \Rightarrow M_i$, then we call $\{\beta_1, \beta_2, \ldots\ldots\ldots, \beta_n\}$ the sequence of arrays associated with the derivation $M_0 \Rightarrow * M_n$ in the Part TMLS, G. We note that the size of the arrays M_{i-1} and β_i are the same, $\beta_1, \ldots\ldots, \beta_n$ are the arrays over the alphabet Σ', which is the set of labels of rules in $\mathscr{P}_r$, *i.e.*, $\Sigma' = \text{lab}\ (\mathscr{P}_r)$.

Now in any Part TMLS G, instead of allowing all possible derivations, we allow only derivations where the sequence of arrays $\{\beta_1 \ldots\ldots, \beta_n\}$ associated with the derivation $M_0 \Rightarrow *M_n$, is a sequence of steps in a derivation $\beta_1 \Rightarrow \beta_2 \Rightarrow \ldots\ldots \Rightarrow \beta_n$ in some system, then we obtain a Part TMLS with controlled associated derivation.

As would be naturally expected, the generative power of Part TMLS increases. If $\beta_1 \Rightarrow *\beta_n$ is a derivation in a system G' (G' may be a Part TMLS itself) then $L(G')$ is called the controlled associated derivation language (CADL).

Definition 2.9: If $G = (\Sigma, \mathscr{P}, \omega)$ is a Part TMLS and C is the controlled associated derivation language, then

$$L(G, C) = \{M \in \Sigma^{++} / \omega \underset{\beta_1}{\Rightarrow} M_1 \underset{\beta_2}{\Rightarrow} \ldots\ldots \underset{\beta_n}{\Rightarrow} M_n\},$$

$\beta_1, \beta_2, \ldots\ldots\ldots, \beta_n$ in C and $\beta_1 \Rightarrow \beta_2 \Rightarrow \ldots\ldots \Rightarrow \beta_n$ is a derivation in the grammar generating C.

Now we state the main results without proofs as the proofs can be found in Nirmal[7].

Theorem 2.1: If G is a Part TMLS, and C is a deterministic Part TMLL, of the first kind, then $L(G, C)$ is a deterministic Part CTMLL, of the first kind.

Theorem 2.2: Corresponding to every det_1 (Part CTMLL) L, a Part TMLS G' and a det_1 (Part TMLL) C can be found such that $L = L(G', C)$.

Definition 2.10: Let $\mathscr{L}$ (Part TMLS, $\mathscr{F}$) denote the family of languages of the form $L(G, C)$ where G is a part TMLS and $C \in \mathscr{F}$.

Theorem 2.3 If $C = \mathscr{F}\ \text{det}_1$ (Part TMLL) then,

$$\mathscr{L}(\text{Part TMLS}, C) = \mathscr{F}\ \text{det}_1\ (\text{Part CTMLL}).$$

Now let us compare $\mathscr{F}$ Part CTMLL with the family of CFML[8]. CFMG will generate pictures like token I, T of all sizes and proportions. But token I and T of fixed proportions are generated by a Part CTMLS, but not by a CFMG. Hence, we have the second main results.

Theorem 2.4; $\mathscr{F}\text{CFML} \subset \mathscr{F}$ Part CTMLL.
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