J. Indian Inst. Sci. 63 (B), Feb. 1981, Pp. 25–33.

On a generalised orthogonality relation and its use in the problem of elasticity of a truncated cylindrical wedge

BAIDYANATH PATRA

Department of Mathematics, Bengal Engineering College, Howrah 711 103, West Bengal, India.

Received on June 23, 1980; Revised on January 15, 1981.

Abstract

A generalised orthogonality relation containing Bessel functions has been derived to use in solving the problem of elasticity in an infinite truncated cylindrical wedge when there are non-homogeneous boundary conditions on the lateral sides and homogeneous conditions on the circular surface such that there is symmetry of displacement components about the middle radius.

Key words: Orthogonality relations, Neuber Papkovitch potentials, harmonics.

1. Introduction

Schiff[1] derived generalised orthogonality relation while studying the solution of first fundamental problem of elasticity theory for an infinite hollow cylinder. However, it was Papkovitch's paper[2, 3] which actually gave impetus to study variant problems of elasticity by using generalised orthogonality relations. Grinberg[4], Prokopov[5] and others dealt the same theory in the framework of plane problems of elasticity. While Steklov[6] used Schiff's method in his work, Filon appraised the work incorrectly in his widely known paper[7]. Nariboli[8] also used the method to solve plate problem of elasticity in two dimensions.

Recently, Nuller[9, 10] extended Schiff's method to obtain several general orthogonality relations for solving three-dimensional problems of elasticity in finite cylinders (solid and hollow) with various forms of boundary conditions. Chiu, Weinstein and Zorowski[11] considered such a problem in an infinite elastic cylinder by using double Fourier series with modified Bessel function coefficients. Analytical determination of the unknown constants was an involved process in this case and was not indicated by the authors. Recently, Prokopov[12] derived some generalised orthogonality relations in connection with plane problems of elasticity and pointed out further class of problems which can be solved by them.

In this paper we have derived a generalised orthogonality relation using modified Bessel functions with respect to the order in its first section. In the next section we have used the same relation in determining the rotation components of an infinite truncated cylindrical wedge.

2. A generalised orthogonality relation

Let

$$\epsilon_{2\nu}(kr) = r\, I'_{2\nu}(kr) - \frac{4\nu^2}{k^2 a}\,\frac{I_{2\nu}(ka)}{I'_{2\nu}(ka)}\, I_{2\nu}(kr) \tag{2.1}$$

be a function of ν such that

$$\epsilon_{2\nu}(ka) = 0. \tag{2.2}$$

Let

$$\nu = \nu_1,\ \nu_2,\ \nu_3,\ \ldots,\ \nu_n,\ \ldots \tag{2.3}$$

form a set of infinite number of distinct roots of (2.2).

$I_{2\nu}(kr)$ is modified Bessel function of order 2ν and k, r are real positive constants. The differential equation satisfied by $\epsilon_{2\nu}(kr)$ is then

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\epsilon_{2\nu}(kr)\right] - \left[k^2 + \frac{4\nu^2}{r^2}\right]\epsilon_{2\nu}(kr) = 2k\, I_{2\nu}(kr) \tag{2.4}$$

Let us consider the integral

$$\int_0^a \frac{d}{dr}\left[r\frac{d}{dr}\epsilon_{2\nu_i}(kr)\right] I_{2\nu_j}(kr)\, dr,$$

where $i, j = 1, 2, 3, \ldots$

Integration by parts yields

$$\int_0^a \frac{d}{dr}\left[r\frac{d}{dr}\epsilon_{2\nu_i}(kr)\right] I_{2\nu_j}(kr)\, dr$$

$$= \left[r\, I_{2\nu_j}(kr)\frac{d}{dr}\epsilon_{2\nu_i}(kr) - r\,\epsilon_{2\nu_i}(kr)\frac{d}{dr} I_{2\nu_j}(kr)\right]_0^a$$

$$+ \int_0^a \frac{d}{dr}\left[r\frac{d}{dr} I_{2\nu_j}(kr)\right]\epsilon_{2\nu_i}(kr)\, dr. \tag{2.5}$$

Using (2.2) and the properties of modified Bessel function (2.5) becomes

$$\int_0^a \frac{d}{dr}\left[r\frac{d}{dr}\epsilon_{2\nu_i}(kr)\right] I_{2\nu_j}(kr)\, dr$$

$$= \int_0^a \frac{d}{dr}\left[r \frac{d}{dr} I_{2\nu_i}(kr)\right] c_{\nu_i}(kr)\, dr + a^2 k\, I_{\nu_i}(ka)\, I_{2\nu_j}(ka),$$

$$\operatorname{Re} \nu > 0. \tag{2.6}$$

By virtue of (2.6), expression (2.4) implies that

$$(4\nu_i^2 - 4\nu_j^2) \int_0^a c_{2\nu_i}(kr)\, I_{2\nu_j}(kr) \frac{dr}{r} + 2k \int_0^a I_{2\nu_i}(kr)\, I_{2\nu_j}(kr)\, dr$$

$$= a^2 k\, I_{2\nu_i}(ka)\, I_{2\nu_j}(ka), \quad \operatorname{Re} \nu > 0. \tag{2.7}$$

Interchanging the subscripts i and j we obtain

$$(4\nu_j^2 - 4\nu_i^2) \int_0^a c_{2\nu_j}(kr)\, I_{2\nu_i}(kr) \frac{dr}{r} + 2k \int_0^a I_{2\nu_i}(kr)\, I_{2\nu_j}(kr)\, dr$$

$$= a^2 k\, I_{2\nu_i}(ka)\, I_{2\nu_j}(ka), \quad \operatorname{Re} \nu > 0. \tag{2.8}$$

Subtracting (2.8) from (2.7) we get

$$4(\nu_i^2 - \nu_j^2) \int_0^a [c_{2\nu_i}(kr)\, I_{2\nu_j}(kr) + c_{2\nu_j}(kr)\, I_{2\nu_i}(kr)] \frac{dr}{r} = 0, \quad \operatorname{Re} \nu > 0.$$

If ν_i and ν_j are two distinct roots of (2.2) then we obtain the desired generalised orthogonality relation as

$$\int_0^a [c_{2\nu_i}(kr)\, I_{2\nu_j}(kr) + c_{2\nu_j}(kr)\, I_{2\nu_i}(kr)] \frac{dr}{r} = 0, \quad \operatorname{Re} \nu > 0. \tag{2.9}$$

3. The problem

We consider the elastic region to be an infinite truncated cylindrical wedge bounded by the planes $\theta = \pm\, \alpha$ and curvilinear boundary $r = a$ as shown in the figure. We seek to determine the cylindrical components of rotation ω_r, ω_θ and ω_z where

$$2\omega_r = \frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z},$$

$$2\omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r},$$

$$2\omega_z = \frac{1}{r} \frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r} \frac{\partial u_r}{\partial \theta}, \tag{3.1}$$

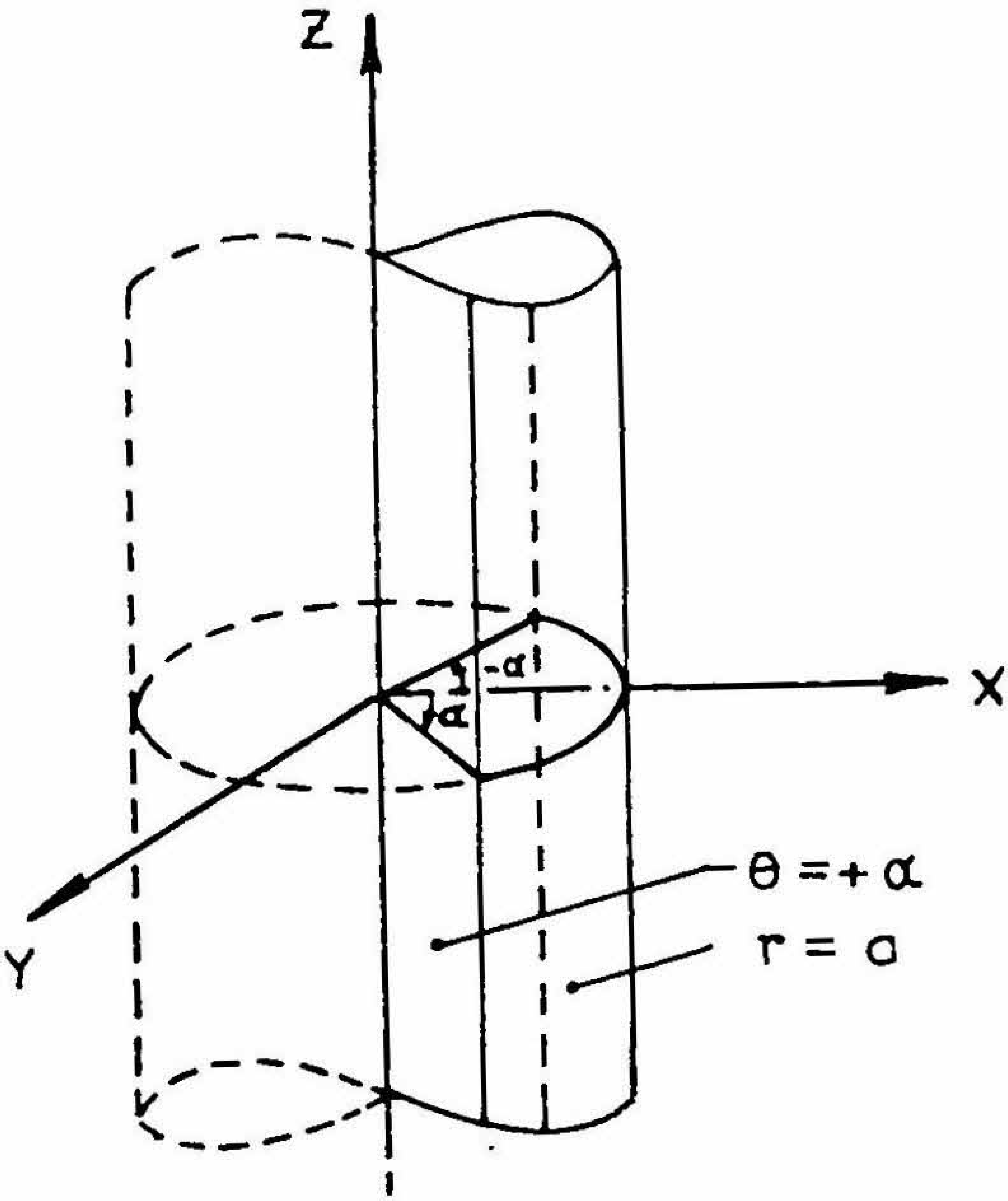

Fig. 1. Elastic region.

u_r, u_θ, u_z being the cylindrical components of displacement vector inside the region when their surface values are prescribed as

$$\omega_r = 0, \text{ on } r = a \tag{3.2}$$

$$\omega_\theta = 0, \text{ on } r = a \tag{3.3}$$

$$\omega_z = 0, \text{ on } r = a \tag{3.4}$$

$$\omega_r = \pm f_1(r, z), \text{ on } \theta = \pm \alpha \tag{3.5}$$

$$\omega_z = \pm f_2(r, z), \text{ on } \theta = \pm \alpha \tag{3.6}$$

$$\omega_\theta = f_3(r, z), \text{ on } \theta = \pm \alpha. \tag{3.7}$$

They represent boundary conditions for non-existence of twist on circular face and symmetric resultant twist in the form of couples in opposite directions on plane wedge-faces. These form boundary conditions which are solvable in a closed form. But the method is applicable for other boundary conditions to give approximate solution in the form of Fredholm integral equation of second kind which is preferable to double infinite series solution which is generally obtained in such cases (*cf.* Chiu, Weinstein and Zorowski[11]). Further, by choice of functions $f_i\,(r, z)$, $i = 1, 2, 3$ we can give it a physical meaning of twist on the plane side faces including the directions at right angles to the faces. Introducing Neuber-Papkovitch potentials the solutions of equilibrium equation in terms of displacements are given by

$$2\mu\, u_r = -\frac{\partial F}{\partial r} + 4(1-\sigma)\;[\varphi_1 \,.\, \cos\theta + \varphi_2 \sin\theta]$$

$$2\mu\, u_\theta = -\frac{1}{r}\frac{\partial F}{\partial\theta} + 4(1-\sigma)\,[-\varphi_1 \,.\, \sin\theta + \varphi_2 \cos\theta]$$

$$2\mu\, u_z = -\frac{\partial F}{\partial z} + 4(1-\sigma)\,\varphi_3 \tag{3.8}$$

where

$$F = r\cos\theta \,.\, \phi_1 + r\sin\theta \,.\, \varphi_2 + z\varphi_3 + \varphi_0. \tag{3.9}$$

and φ_i, $i = 0, 1, 2, 3$ are harmonic functions in cartesian co-ordinates.

We assume that the displacement components u_r, u_z are symmetric with respect to $\theta = 0$ and therefore u_θ is antisymmetric about that plane. Then the rotation components ω_r, ω_z are antisymmetric and ω_θ is symmetric with respect to $\theta = 0$. Further we assume that the displacements and hence the rotation components are periodic with respect to z with period $2l$. Let the prescribed surface values $f_1(r, z)$ and $f_2(r, z)$ of ω_r and ω_z respectively be given in the form

$$f_1(r, z) = \sum_{n=1}^{\infty} g_k(r)\cos kz, \tag{3.10}$$

$$f_2(r, z) = \sum_{n=1}^{\infty} h_k(r)\sin kz, \tag{3.11}$$

where $k = \dfrac{n\pi}{l}$. Accordingly, we set the potentials φ_i, $i = 0, 1, 2, 3$ as

$$\varphi_0 = \sum_{n=1}^{\infty}\sum_{\nu} E(\nu)\, I_\nu(kr)\cos\nu\theta \sin k_z, \tag{3.12}$$

$$\varphi_1 = \sum_{n=1}^{\infty}\sum_{\nu} A(\nu)\, I_\nu\,(kr)\cos\nu\theta\sin kz, \tag{3.13}$$

$$\varphi_2 = \sum_{n=1}^{\infty}\sum_{\nu} B(\nu)\, I_\nu\;(kr)\sin\nu\theta\sin kz, \tag{3.14}$$

$$\varphi_3 = 0, \tag{3.15}$$

under the restriction that $\operatorname{Re}\nu > 0$.

Since the loads are periodic in the axial direction and since any one of the above four potentials φ_i, $i = 0, 1, 2, 3$ can always be chosen arbitrarily, we choose φ_3 as zero in (3.15). Then the displacement components in (3.8) and (3.9) become

$$2\mu u_r = \sum_{n=1}^{\infty} \sin kz \sum_{\nu} \Big[-r \frac{d}{dr} \{\alpha(\nu) I_{2\nu-1}(kr) + \beta(\nu) I_{2\nu+1}(kr)\}$$

$$- E(2\nu) \frac{d}{dr} I_{2\nu}(kr) + (3 - 4\sigma)\{\alpha(\nu) I_{2\nu-1}(kr) + \beta(\nu) I_{2\nu+1}(kr)\}\Big] \cos 2\nu\theta, \tag{3.16}$$

$$2\mu u_\theta = \sum_{n=1}^{\infty} \sin kz \sum_{\nu} \Big[(2\nu + 4\sigma - 4)\, \alpha(\nu) I_{2\nu-1}(kr)$$

$$+ (2\nu - 4\sigma + 4)\, \beta(\nu) I_{2\nu+1}(kr) + \frac{2\nu}{r} E(2\nu) I_{2\nu}(kr)\Big] \sin 2\nu\theta, \tag{3.17}$$

$$2\mu u_z = - \sum_{n=1}^{\infty} k \cos kz \sum_{\nu} [r\{\alpha(\nu) I_{2\nu-1}(kr) + \beta(\nu) I_{2\nu+1}(kr)\}$$

$$+ E(2\nu) I_{2\nu}(kr)] \cos 2\nu\theta, \tag{3.18}$$

where

$$\alpha(\nu) = \tfrac{1}{2}[A(\nu - 1) - B(\nu - 1)], \qquad \beta(\nu) = \tfrac{1}{2}[A(\nu) + B(\nu)].$$

The rotation components are therefore given by

$$\mu\omega_r = (1 - \sigma) \sum_{n=1}^{\infty} k \cos kz \sum_{\nu} [\alpha(\nu) I_{2\nu-1}(kr) - \beta(\nu) I_{2\nu+1}(kr)] \sin 2\nu\theta, \tag{3.19}$$

$$\mu\omega_\theta = (1 - \sigma) \sum_{n=1}^{\infty} k \cos kz \sum_{\nu} [\alpha(\nu) I_{2\nu-1}(kr) + \beta(\nu) I_{2\nu+1}(kr)] \cos 2\nu\theta \tag{3.20}$$

and

$$\mu\omega_z = (1 - \sigma) \sum_{n=1}^{\infty} \sin kz \sum_{\nu} \frac{1}{r} \Big[(2\nu - 1)\, \alpha(\nu) I_{2\nu-1}(kr) + (2\nu + 1)$$

$$\times \beta(\nu) I_{2\nu+1}(kr) - r \frac{d}{dr} \{\alpha(\nu) I_{2\nu-1}(kr) - \beta(\nu) I_{2\nu+1}(kr)\}\Big] \sin 2\nu\theta.$$

The last expression for ω_z after some simplification becomes

$$\mu\omega_z = (1 - \sigma) \sum_{n=1}^{\infty} \sin kz \sum_{\nu} [-\alpha(\nu) + \beta(\nu)] I_{2\nu}(kr) \sin 2\nu\theta. \tag{3.21}$$

Using the boundary conditions (3.2) and (3.3) in (3.19) and (3.20) respectively, we get for each particular value of k, the equations

$$\alpha(\nu) I_{2\nu-1}(ka) + \beta(\nu) I_{2\nu+1}(ka) = 0,$$
$$\alpha(\nu) I_{2\nu-1}(ka) - \beta(\nu) I_{2\nu+1}(ka) = 0. \tag{3.22}$$

From (3.22) we get the characteristic equation for the determination of ν as

$$I_{2\nu-1}(ka) \,.\, I_{2\nu+1}(ka) = 0$$

or

$$I_{2\nu}'^{2}(ka) - \frac{4\nu^2}{k^2 a^2} I_{2\nu}^2(ka) = 0. \tag{3.23}$$

Comparing (3.23) with (2.2) we see that the infinite set of roots of (3.23) are same as those of (2.2) for which $0 < \text{Re}\,\nu < \frac{1}{2}$. Hence, if we choose $\nu = \nu_i$, $i = 1, 2, \ldots$, where ν_i are those roots of (2.2) for which $0 < \text{Re}\,\nu < \frac{1}{2}$ in the summation in (3.19) and (3.20), the boundary conditions (3.2) and (3.3) will be identically satisfied. Further by virtue of (3.22), the unknown constants $\alpha(\nu)$ and $\beta(\nu)$ take the following form:

$$\alpha(\nu) = L(\nu) I_{2\nu+1}(ka),$$
$$\beta(\nu) = L(\nu) I_{2\nu-1}(ka).$$

So expressions for components of rotation satisfying (3.2) and (3.3) become

$$\mu\omega_r = (1-\sigma) \sum_{n=1}^{\infty} k \cos kz \sum_{\nu_n} L(\nu) [I_{2\nu+1}(ka) I_{2\nu-1}(kr) - I_{2\nu-1}(ka) I_{2\nu+1}(kr)] \times \sin 2\nu\theta, \tag{3.24}$$

$$\mu\omega_\theta = (1-\sigma) \sum_{n=1}^{\infty} k \cos kz \sum_{\nu_n} L(\nu) [I_{2\nu+1}(ka) I_{2\nu-1}(kr) + I_{2\nu-1}(ka) I_{2\nu+1}(kr)] \times \cos 2\nu\theta \tag{3.25}$$

and

$$\mu\omega_z = (1-\sigma) \sum_{n=1}^{\infty} \sin kz \sum_{\nu_n} L(\nu) \frac{2\nu}{ak} I_{2\nu}(ka) I_{2\nu}(kr) \sin 2\nu\theta. \tag{3.26}$$

Now using boundary conditions (3.5) and (3.6), the expressions (3.24) and (3.26) by virtue of (3.10) and (3.11) respectively give

$$\sum_{n=1}^{\infty} k \cos kz \sum_{\nu_n} L(\nu) [I_{2\nu+1}(ka) I_{2\nu-1}(kr) - I_{2\nu-1}(ka) I_{2\nu+1}(kr)] \sin 2\nu\alpha$$
$$= \frac{\mu}{1-\sigma} \sum_{n=1}^{\infty} g_k(r) \cos kz. \tag{3.27}$$

and

$$\sum_{n=1}^{\infty} \sin kz \sum_{\nu_n} \frac{2\nu}{ak} L(\nu)\, I_{2\nu}(ka)\, I_{2\nu}(kr) \sin 2\nu\alpha$$

$$= \frac{\mu}{1-\sigma} \sum_{n=1}^{\infty} h_k(r) \sin kz. \tag{3.28}$$

Equating like harmonics from the two sides of (3.27) we get

$$k \sum_{\nu_n} L(\nu)\,[I_{2\nu+1}(ka)\, I_{2\nu-1}(kr) - I_{2\nu-1}(ka)\, I_{2\nu+1}(kr)] \sin 2\nu\alpha$$

$$= \frac{\mu}{1-\sigma}\, g_k(r).$$

Introducing $\epsilon_{2\nu}(kr)$ defined in (2.1), a little simplification of the above relation leads to

$$\sum_{\nu_n} \frac{2\nu \sin 2\nu\alpha}{kar} L(\nu)\; I_{2\nu}(ka)\; \epsilon_{2\nu}(kr) = -\frac{\mu}{(1-\sigma)k}\, g_k(r). \tag{3.29}$$

Again equating like harmonics from the two sides of (3.28) we get

$$\sum_{\nu_n} \frac{2\nu \sin 2\nu\alpha}{kar} L(\nu)\, I_{2\nu}(ka)\, I_{2\nu}(kr) = \frac{\mu}{1-\sigma}\, \frac{h_k(r)}{r}. \tag{3.30}$$

Now, making use of the general orthogonality relation derived in (2.9) we obtain the unknown coefficient $L(\nu_j)$ from (3.29) and (3.30) as,

$$\frac{4\nu_j}{ka} L(\nu_j) \sin 2\nu_j\alpha\, I_{2\nu_j}(ka)$$

$$= \left[\int_0^a \epsilon_{2\nu_j}(kr)\, I_{2\nu_j}(kr)\, \frac{dr}{r}\right]^{-1} \left[-\frac{\mu}{(1-\sigma)k} \int_0^a g_k(r)\, I_{2\nu_j}(kr)\, dr\right.$$

$$\left. + \frac{\mu}{1-\sigma} \int_0^a h_k(r)\, \epsilon_{2\nu_j}(kr)\, \frac{dr}{r}\right]. \tag{3.31}$$

Thus $L(\nu_j)$ will be obtained from (3.31) after performing the integrations. Hence the solution of the problem is reduced to a quadrature. This value of $L(\nu_j)$ determines the values ω_r only which satisfies the conditions (3.2) and (3.5). Proceeding similarly with other sets and other orthogonality relations we can determine ω_θ, ω_z satisfying (3.3) and (3.7), (3.4) and (3.6) respectively.

Acknowledgement

The author is grateful to Prof. S. C. Dasgupta for suggesting the problem and for guidance at various stages of the work.

References

1. SCHIFF, P. A. — Sur l'équilibre d'un cylindre élastique, *J. Math. Pure Appl. Ser. III*, 1883, 9.

2. PAPKOVITCH, P. F. — One form of the solution of the plane problem of elasticity theory for a rectangular strip, *Dokl. Akad. Nauk. SSSR*, 1940, 27 (4).

3. PAPKOVITCH, P. F. — Two problems in the theory of bending of thin elastic plates, *P.M.M.*, 1941, 5 (3).

4. GRINBERG, G. A. — On the Papkovitch method for solving the plane problems of elasticity theory for a rectangular domain and the problem of bending of a rectangular thin plate with two fixed edges and some generalisation of this method, *P.M.M.*, 1953, 17 (2).

5. PROKOPOV, V. K. — On the relation of the generalised orthogonality of P. F. Papkovitch for rectangular plate, *P.M.M.*, 1964, 18 (2).

6. STEKLOV, V. A. — On the equilibrium of elastic solids of revolution, *Soobshcheniia Khar' kovsk. Mat. Obshchestva. Series* 2, 1893, 3.

7. FILON, L. N. G. — On the elastic equilibrium of circular cylinders under certain practical systems of load, *Trans. Roy. Phil. Soc. (London) Series A*, 1902, 193.

8. NARIBOLI, G. S. — Eigenfunctions for the strip problem, *Mathematica*, 1965, 12 (23), Part I.

9. NULLER, B. M. — On the homogeneous solution of elasticity theory and the orthogonality relation of P. A. Schiff, *Inzh. Zhurnal Mekh. Tverd. Tela*, 1968, No. 2.

10. NULLER, B. M. — On the generalised orthogonality relation of P. A. Schiff, *P.M.M.*, 1969, 33 (2).

11. CHIU, Y. P., WEINSTEIN, A. S. AND ZOROWSKI, C. F. — Elastic behaviour of a cylinder subjected to a biaxially symmetric normal surface loading, *Q. J. Mech. Appl. Maths.*, 1964, 17, Part 2, 199.

12. PROKOPOV, V. K. — O Sootnosheniakh Obobshennoi Ortogonalnosti, imeushikh prilozhenia V teorii uprugosti, *Proceedings of symposium on continuum mechanics and related problems of analysis* (*Tbilisi*, 23–29. *IX*. 1971), 1.