J Indian Inst. Sci., July–August 1994, 74, 569–582.

The science in computation: An engineer's defence

G. PRATHAP
National Aerospace Laboratories, Bangalore 560 017, and
Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore 560 012, India.

Abstract

We now recognise three segments of the scientific enterprise: The theoretical stage, the empirical stage and the computational stage. The scientific nature of the theoretical and empirical components of the procedure of investigation into the problems of the natural or hard sciences has been accepted for a long time. It is still not clear if computational modelling and simulation of scientific and engineering problems can have a richness that allows it to be legitimately designated as science. In this paper, an argument has been advanced to show that the body of knowledge and the modes of enquiry that emerge from a computational discipline like the finite element method also possesses properties that permit it to meet the stipulations of the conceptual legislation demanded by Popper's critical rationalism for admission as a science.

Key words: Computational structural mechanics, finite element method, scientific computing, scientific method, critical rationalism, falsification.

1. Introduction

"The purpose of the lecture series is to stimulate the student's critical abilities in order to enable him to appreciate the excellence of a work, and to induce him to an attempt to produce good things himself, [with the hope] that each of the lecturers will speak from his own experience on the work of his art or profession, and that he will demonstrate its value by elucidating its nature, formulating its purpose, and explaining its techniques."

Extract from letter from Robert Maynard Hutchins, then the Chancellor of the University of Chicago to Professor S. Chandrasekhar inviting him to give the lecture on "The Scientist" as part of The Works of the Mind lecture series.

I think, when Professor C.N.R. Rao invited us to deliver these lectures, it was precisely this motivation he had in mind; that is, to demonstrate in each case, the value of the work of his *art* or *profession*. Note the emphasis placed on *art* and *profession* and not on *science*. It is in demonstrating its value, its purpose, its

Text of lecture delivered on November 19, 1993, at the Annual Faculty Meeting of the Jawaharlal Nehru Centre for Advanced Scientific Research.

technique and its universality that the elements of science in an art or profession are discovered. This morning, I shall attempt to give such a rationalisation to some of the work that my colleagues and I have been doing in the field of computational structural mechanics.

My profession is engineering, my art is structural mechanics and the specialised area on which I've focussed attention in recent years is called computational structural mechanics. If there is a ring of apologia about my lecture, it is understandable because as an engineer and as a computationalist, I'm doubly defensive about my credentials as a scientist. Let me briefly enunciate the source of my misgivings.

2. Science *vs* engineering: The nature of technological knowledge

The clearest and most concise statement on the nature of technological knowledge that I've read in recent times is one made by Professor R. Narasimha at his lecture during the National Debate on Science and Technology organised by the JNCASR earlier this year and which has appeared in full in *Current Science*[1]. I quote:

> " . . . It is the codification of knowledge that becomes the directive of innovation.
>
> There is a widespread tendency to think of technology as indistinguishable from science, or as being at best applied science. However, while there is nothing more practical than a good theory, and while scientific knowledge is most valuable for and increasingly more often can lead to technology development, technology is not mere applied science; it is enormously richer, and indeed autonomous in many ways, for technologists will and must develop new artifacts for human use even when all the 'understanding' that is motivation for science is not available.
>
> Science has to do with understanding Nature, and with explaining and predicting phenomena with the fewest possible independent hypothesis. [It] has to do with intellectual economy.
>
> The objective of technology . . . is to make new products, or artifacts . . . to meet human needs . . . in the most economic way possible."

Thus, while science has to do with economy and unity of understanding, engineering or technology has to do with economy of utility. It is not always manifest that the practitioners of the latter art do also conduct themselves in a manner that is in conformity with economy of understanding. Thus, the question arises, How does technological knowledge become science? Before we answer this question, it is worthwhile to explore the kind or creatures who inhabit these various worlds.

3. The computationalist

There are three kinds of animals in the scientific enterprise; and all these and more in the engineering zoo, reflecting the vastly greater richness of engineering and technology—the list shown in Table I is perhaps more representative than exhaustive.

In recent years, a new branch of scientific and engineering activity called computational modelling and simulation has emerged. Today, there are increasingly large

Table I

Scientists	*Engineers*
1. Theorists	1. Toolmakers, inventors, innovators
2. Experimentalists	2 Scientists
	a) Theorists
	b) Experimentalists
	c) *Computationalists*
3. *Computationalists*	3. Designers
	4 Analysts
	5. Manufacturers, fabricators
	6. Planners, managers
	7. Diagnosticists, troubleshooters
Computationalists	
Egyptians, Babylonians	
Archimedes	
Kepler	
Halley	

numbers of people, who work entirely with computational modelling—often totally ignoring theoretical and experimental work. Computationalists, as I shall call the group of workers who deal exclusively with such problems, are not really a new breed emerging suddenly in the post-computer age. The Egyptians who devised computational schemes to monitor the flooding of the Nile and maintain their water works; the ancient astronomers who invented algorithmic procedures to predict astronomical phenomena; Archimedes' work on integratory schemes for computing areas and volumes which foreshadowed the emergence of the integral calculus by several centuries; the twenty or more years that Kepler spent in immensely laborious calculations using Tycho Brahe's data are classical examples of science beginning as computation. Thus, much before there was algebra or analysis, computation was used to solve problems of a scientific or technical nature.

It goes without debate that theorists and experimentalists do science, their roles have been traditionally defined. But what about those who perform tasks that are entirely computational in nature. Is their work scientific? Is computation unavoidable poisonous drudgery needed only to fill in the gaps on a scientific map or can computation be revelatory as well? Can modelling and simulation itself have a richness in its activity that allows it to be held to scrutiny by methods which are used to verify or falsify the scientific quality of any area or branch of knowledge? This is the question I will address now using as an example some of my recent work in the area of the finite element method in computational structural mechanics.

4. Structural mechanics and the engineer

Let me briefly explain how and where structural mechanics plays a crucial role in the life of the engineer. The most poetic description of the role of an engineer in society was made, perhaps very predictably, by Shelley. In his *A Defence of Poetry,* Shelly wrote thus about the role of technology:

> "Undoubtedly the promoters of utility, in this limited sense, have their appointed office in society. They follow the footsteps of poets, and copy the sketches of their creations into the book of common life. They make space and give time."

The structural engineer's appointed office in society is to enclose space for activity and living, and sometimes does so giving time as well—the ship builder, the railway engineer and the aerospace engineer enable travel in enclosed spaces that provide safety with speed in travel. He did this first, by imitating the structural forms already present in Nature. Shelly was therefore only partially right—engineers also copy the sketches of their creations from the book of common life. From Art imitating Life, one is led to codification of the accumulated wisdom as science—the laws of mechanics, elasticity, theories of the various structural elements like beams, plates and shells, etc.

From Archimedes' use of the Principle of Virtual Work to derive the law of the lever, through Galileo and Hooke to Euler, Lagrange, Love, Kirchhoff, Rayleigh, etc., we see the theoretical and mathematical foundations being laid, and then copiously used by engineers to fabricate structural forms for civil and military functions. Solid and structural mechanics is therefore the scientific basis for the design, testing, evaluation and certification of structural forms made from material bodies to ensure proper function, safety, reliability and efficiency.

Today, analytical methods of solution, which are too restricted in application, have been replaced by computational schemes ideal for implementation on the digital computer. By far the most popular method in computational structural mechanics is that called the finite element method.

5. The finite element method

The early history of the finite element method shows the convergence of methods and practices from three distinct branches of knowledge. From civil engineering, it borrowed the concept of matrix method of analyses. From aerospace engineering came the finite element discretisation procedures, as we know it today. Variational calculus then provided the formal mathematical rationalisations in terms of energy theorems, least action principles, virtual work methods, etc.

The finite element world is now a billion dollar industry, both in terms of software sold and in terms of analyses costs using such installed software. It is one area of engineering knowledge that has become an undisputed commercial success.

This is not to say that the growth of the method was free of difficulties. There were enormous hurdles to cross as well. The best known is a problem known as *locking*. We shall take it up for closer examination now.

6. Finite element discretisation and the locking phenomenon

At the heart of the finite element method is a process of discretisation providing algorithms that capture the physics of the structural behaviour of the constituent parts of the structure in terms of matrices of discrete numbers relating forces to displacement at the nodes. All structural regions are replaced by a set of sub-domains called finite elements. Each element has to model the elasto-mechanical behaviour of the region it replaces. In effect, the differential equations of the infinitesimal calculus governing the region are now substituted with discrete matrix relationships. These matrices are eminently amenable for manipulation on a digital computer through clever book-keeping and database operations. Thus, very large problems which are otherwise intractable by analytical techniques can be solved.

Over nearly four decades, a systematic process by which continuum structural behaviour is replaced by a discretised description has evolved. At first, efforts to do this were founded on engineering intuition and heuristic judgement. It was extremely successful in most cases although there were instances where mere technique could not resolve difficulties. A scientific basis was required to reconcile this.

To furnish a scientifically acceptable basis to the procedure it is necessary to establish that paradigmatic principles can be identified. Our studies[2] show that four fundamental criteria inform this procedure—we shall call them the c-concepts, namely, *Continuity, Completeness, Consistency* and *Correctness*. This conceptual framework is then used to guide the construction of stiffness matrices and to assess the performance of these matrices.

It is not surprising that of particular concern to us in the finite element method is the quality of approximation that can be achieved. We desire a quick convergence to the correct solution as the number of elements increase (or the element sizes decrease). This will depend on the quality of the element stiffness matrix which is a function of element shape, size and aspect ratio and of the order of the chosen displacement field representation by what are called the shape functions (also called basis functions, trial and test functions, etc.). In simple displacement elements, these shape functions are usually taken in the form of polynomials. The conventional wisdom around the time we began our work around 1978 specified two well-known conditions that these polynomial functions must satisfy to ensure convergence: these are the *completeness* and *continuity* conditions.

Continuity

The displacement functions chosen must allow strains at element interfaces to be finite—this means that there must be a certain degree of continuity of displacement between adjacent elements. It is not always easy to ensure this. Within each element, one can argue that the continuous representation of the displacement fields will ensure compatibility, but this may be violated along element edges unless special care is taken. Where strains are defined by first derivatives of the displacement fields, a simple continuity of the displacement fields across element edges suffices—this is called C^0-continuity. There are problems, as in the classical Kirchhoff–Love theories of plates and shells, where strains are based on second derivatives of displacement fields—in this case, continuity of first derivatives of displacement fields across element edges are demanded; this is known as C^1-continuity.

We shall, however, find a large class of problems (Timoshenko beams, Mindlin plates and shells, plane stress/strain flexure, incompressible elasticity) where this simplistic view of continuity does not assure reasonable (*i.e.*, practical) rates of convergence—in fact, formulations that take liberties, *e.g.*, the non-conforming or incompatible approaches, significantly improve convergence. The reasons for this will be found in the consistency requirements later.

Completeness

Displacement functions must be so chosen that no straining within an element takes place when nodal displacements equivalent to a rigid body motion of the whole element are applied. This is called the *strain-free rigid body motion condition.* In addition, it is necessary that each element must be able to reproduce a state of constant strain, *i.e.*, if nodal displacements applied to the element are compatible with a constant strain state, this must be reflected in the strains computed within the element. There are simple rules that allow these conditions to be met and these are called the *completeness* requirements. If polynomial trial functions are used, then a simple assurance that the polynomial functions contain the constant and linear terms, etc. (*e.g.*, $1, x$ in a one-dimensional C^0problem; $1, x,y$ in a two-dimensional C^0problem) will meet this requirement.

Unfortunately, elements derived rigorously from these basic paradigms can behave in unreasonably erratic ways in many important situations; errors as large as *100%* are often reported! These difficulties are most pronounced in the lowest order finite elements. The problems encountered were called *locking, parasitic shear,* etc. By locking, we mean that finite element solutions vanish quickly to zero (errors saturating quickly to nearly *100%* !) as some structural parameters become very large.

Some of the problems may have gone unrecorded with no adequate framework or terminology to classify them. As a very good example, for a very long time, it was believed that curved beam and shell elements performed poorly because they could not meet the *strain-free rigid body motion* condition. However, more recently, the correct error-inducing mechanism has been discovered and these problems have come to be called *membrane locking.*

Initially, this discouraged the use of low-order displacement elements and attention was turned to higher order elements and to the assumed strain/stress elements. Around the same time, many 'tricks' were tried out on the displacement formulations, and some of these resulted in acceptably accurate elements. These tricks included techniques such as reduced integration, addition of non-conforming modes, energy balancing, B-bar methods, etc. Some of these violated the well-known norms for finite element formulation but were accepted because the elements thus formulated were more accurate than the rigorously formulated ones. It was clear at this stage that the paradigms known so far were neither sufficient nor always necessary. It was imperative that some new paradigms be found to make the study of such elements more scientific. We address one aspect of this problem in this lecture.

To put the locking phenomenon in a proper perspective, it is first necessary to recognise that errors, whether in displacements, stresses or energies, due to finite element discretisation must converge rapidly, at least in a $O(h^2)$ manner or better, where h is the 'diameter' of the element. If a large structure (domain) of dimension L is sub-divided into elements (sub-domains) of dimension l, one expects errors of the order of $(l/L)^2$. Thus, with ten elements in a one-dimensional structure, errors must not be more than a few per cent. However, in problems where locking is noticed, errors are much larger—the discretisations fail in a dramatic fashion, and this cannot be resolved by the classical (pre-1977) understanding of the finite element method. We shall explain the issues involved using the example of the linear Timoshenko beam element.

Most published literature, including all textbooks, associate locking with the rank or non-singularity of the stiffness matrix linked to the penalty term (*e.g.*, the shear stiffness matrix in a Timoshenko beam element which becomes very large as the beam becomes very thin). However, on reflection, it is obvious that these are symptoms of the problem and not the cause. The high rank and non-singularity is the outcome of certain assumptions made (or not made, *i.e.*, leaving certain unanticipated requirements unsatisfied) during the discretisation process. It is therefore necessary to trace this to the origin. An explanation offered by Prathap and co-workers[2] is promising—they have argued that it is necessary in such problems to discretise the penalty-linked strain fields in a consistent way so that only physically meaningful constraints appear. In this lecture, we show how what originated as an exercise in computation led to the formulation of a paradigmatic principle like consistency.

7. Analysis of the Timoshenko beam element

The Timoshenko beam theory[3] offers a general formulation of beam flexure. The total strain energy functional is now constructed from the two independent functions for transverse deflection (wx) and section rotation $\theta(x)$, and it will now account for the bending (flexural) energy and an energy of shear deformation.

$$\pi = \int_0^L (1/2\ EI\ \theta_{,x}^2 + 1/2\ \alpha(\theta - w_{,x})^2 - qw)\ dx \tag{1}$$

where the curvature $x = \theta_{,x}$, the shear strain $\gamma = \theta - w_{,x}$ and $\alpha = kGA$ is the shear rigidity. E and G are the Young's and shear moduli and k, the shear correction factor used in Timoshenko's theory. I and A are the moment of inertia and the area of cross-section.

The Timoshenko beam theory will asymptotically recover the elementary beam theory as the beam becomes very thin, or as the shear rigidity becomes very large, *i.e.*, $\alpha \to \infty$. This requires that the Kirchhoff constraint $\theta - w_{,x} \to 0$ must emerge in the limit. For a very large α, these equations lead directly to the simple fourth-order differential equation for w of elementary beam theory. Thus, this is secured very easily in the infinitesimal theory but it is this very same point that poses difficulties when a simple finite element approximation is made.

A two-noded beam element based on this theory will need only C^0 continuity and can be based on simple linear interpolations[4]. It was therefore very attractive for general-purpose applications. However, the element was beset with problems, as we shall presently see. We can show that locking in C^0 displacement-type finite element formulations is due to a lack of consistent definition of the strain fields that are constrained in the penalty regime when the discretisation is made.

7.1. *The conventional formulation of the linear beam element*

The strain energy of a Timoshenko beam element of length $2l$ can be written as the sum of its bending and shear components as

$$\int (1/2\ EI\ x^T x\ +\ 1/2\ kGA\ \gamma^T \gamma) dx \tag{2}$$

where

$$x = \theta_{,x} \tag{3a}$$

$$\gamma = \theta - w_{,x}. \tag{3b}$$

In the conventional procedure, linear interpolations are chosen for the displacement field variables[4]. This ensures that the element is capable of strain-free rigid body motion and can recover a constant state of strain (*completeness* requirement) and that the displacements are continuous within the element and across the element boundaries (*continuity* requirement). We can compute the bending and shear strains directly from these interpolations using the strain gradient operators given in eqns (3a) and (3b). These are then introduced into the strain energy computation in eqn (2), and the element stiffness matrix is calculated in an analytically or numerically exact (a 2-point Gauss Legendre integration rule) way.

For the beam element of length $2l$ the stiffness matrix can be split into two parts, a bending related part and a shear related part[4]. It turns out that the rank of the shear stiffness matrix is two. It is also useful to introduce a note about the singularity aspect. This element stiffness matrix can be used to model a cantilever beam by assembling into a global stiffness system with the rigid body motions suppressed. This can be done by deleting the first two rows and columns of the stiffness matrices. It can be seen that the shear related part of the assembled global stiffness matrix is non-singular.

Following Hughes *et al*[4] we shall model a thin cantilever beam under a tip load using this element. We choose E=1000, G=37500000, t=1, L=4; using a fictitiously large value of G to simulate the thin beam condition. Table II gives the normalized tip displacements for this case as obtained by Hughes *et al*[4] (where an error was present) and as later corrected by Prathap and Bhashyam[5]. We can see a trend emerge as the number of elements are increased. The tip deflections obtained for the thin beam, which are several orders of magnitude lower than the correct answer, are directly related to the square of the number of elements used for the idealization. In other words, the discretisation process has introduced an error so large that the resulting answer has a stiffness related to the inverse of N^2, where N is the number of elements used in the computation. This is clearly unrelated to the physics of the Timoshenko beam and also not the usual sort of discretisation errors encountered in the finite element method. It is this very phenomenon that is known as shear locking.

Table II

Normalised tip deflections for a thin cantilever beam

No. of elements	*Hughes et al*[4]	*Prathap & Bhashyam*[5]
1	0.00002	0.00002
2	0.00008	0.00008
4	0.00032	0.00032
8	0.000128	0.00128
16	0.000512	0.00512

The error in each element must be related to the element length, and therefore when a beam of overall length L is divided into N elements of equal length $2l$, the additional stiffening introduced in each element due to shear locking is seen to be proportional to l^2. In fact, numerical experiments[5] showed the locking stiffness progresses without limit as the element depth t decreases. Thus, we now have to look for a mechanism that can explain how this spurious stiffness of $(l/t)^2$ can be accounted for by considering the mathematics of the discretisation process.

The magic formula proposed by Hughes *et al*[4] to overcome the locking seen for the linear beam element is the reduced integration method. The bending component of the strain energy of a Timoshenko beam element of length $2l$ shown in eqn (2) is integrated with a one-point Gaussian rule as this is the minimum order of integration required for exact evaluation of this strain energy. However, a mathematically exact evaluation of the shear strain energy will demand a two-point Gaussian integration rule. It is this rule that resulted in the shear stiffness matrix of rank two that locked. Hughes *et al*[4] experimented with a one-point integration of the shear strain energy component and the shear-related stiffness matrix changed, the rank now having reduced to one. The performance of this element was extremely good, showing no signs of locking at all.

If we repeat the exercise of using a single element to model a cantilever beam, the shear-related part of the assembled global system matrix will be singular. The conventional wisdom was to relate this singularity to the improved performance seen above. The argument proceeded thus. The functional of eqn (2) becomes constrained when $kGAl^2 >> EI$. This leads to finite element equations (after assembly) of the form

$$(K_1 + \alpha K_2)\, a + f = 0 \tag{4}$$

where a is the displacement vector and f, the load vector. K_1 is the unconstrained part of the stiffness matrix (in this instance, that derived from the bending energy) and K_2 the constrained part (here derived from the shear energy). The penalty parameter α (here, we know this is $kGAl^2/EI$) increases as the beam becomes thinner, and it is argued that eqn (4) degenerates to

$$K_2 a = -f/\alpha \rightarrow 0. \tag{5}$$

This degeneration can take place only if K_2 is non-singular. It is possible to establish then that locking will set in, *i.e.*, $a \rightarrow 0$ as α becomes very large. In a conventional displacement-type formulation of constrained media elasticity (as in the exactly integrated case), this singularity does not arise naturally. The reduced integration strategy is therefore viewed as an artifice that can bring about the required singularity so that in the penalty limit, the eqn (4) does not degenerate as seen above.

There are several weaknesses in this heuristically appealing argument. It is not certain that no violation of the variational theorems has taken place in this 'trick' of introducing singularity into the constrained matrix. The argument also does not assert that there is a unique way in which singularity must be achieved. Thirdly, there is no possibility of constructing a numerical experiment that can 'falsify' (verify) this paradigm and lead at the same time to a measure of error in terms of the penalty parameter $kGAl^2/EI$. With the field-consistency paradigm it was possible to do this[2].

Our findings indicate that K_2 (after assembly) is non-singular only in the lowest order representations. For higher order elements, therefore, even with exact integration, there will be true constraints which reflect some degree of singularity. An argument in terms of rank becomes more useful here; however, it is not always easy to establish for a high-order element what the correct rank of the penalty-linked stiffness matrix should be. Our paradigmatic requirement that there should only be true constraints and no spurious constraints, which we will derive below, will automatically ensure that the correct rank is maintained.

There are other closely related arguments which have found their way to the textbooks but which are no more scientifically valid than the singularity argument. One relates to the rank of the shear stiffness matrix—we can understand why it is insisted that the rank must not be too high. Reduced integration helps to reduce this rank condition. Another very closely related paradigm concerns the number of constraints contained in the stiffness matrix, the so-called constraint-counting procedure. Reduced integration lowers the constraint count as one can show quite easily that the number of constraints activated are linked to the number of integration points used to integrate the constrained strain energy. Another argument that was current some time ago was that of relating locking to the spectral condition number: exactly integrated stiffness matrices always had a higher spectral condition number and this was linked to the locking effect. Note that these are all heuristic arguments, reflecting the symptoms of the problem (locking is seen where there is a non-singular constrained matrix, or where the rank is too high, etc.) and not really the cause of the problem. We shall now look forward to a paradigm that can trace the problem to the root and then can argue forward to what can be called a falsifiable error estimate.

7.2 *The field-consistency paradigm*

It is clear from the formulation of the linear Timoshenko beam element using exact integration (we shall call it the *field-inconsistent* element) that the *completeness* and *continuity* paradigms, which had been for a long time considered to be necessary and sufficient conditions for describing displacement interpolations, are really not enough in some problems. We shall propose a requirement for a *consistent* interpolation of the constrained strain fields as the necessary paradigm to make our understanding of the phenomena complete.

If we start with linear trial functions for w and θ, we can associate two generalized displacement constants with each of the interpolations in the following manner

$$w = a_0 + a_1\,(x/l); \tag{6a}$$

$$\theta = b_0 + b_1\,(x/l). \tag{6b}$$

We can relate such constants to the field variables obtaining in this element and in discretised sense; thus, $a_1/l = w,_x$ at $x = 0$, $b_0 = \theta$ and $b_1/l = \theta,_x$ at $x = 0$. This denotation would become useful when we try to explain how the discretisation process can alter the infinitesimal description of the problem if the strain fields are not consistently defined.

If the strain fields are now derived from the displacement fields given in eqns (3), we get

$$\chi = (b_1/l); \tag{7a}$$

$$\gamma = (b_0 - a_1/l) + b_1\,(x/l). \tag{7b}$$

An exact evaluation of the strain energies for an element of length $2l$ will now yield the bending and shear strain energy as

$$U_B = 1/2\,(EI)\,(2l)\,\{(b_1/l)\}^2 \tag{8a}$$

$$U_S = 1/2\,(kGA)\,(2l)\,\{(b_0 - a_1/l)^2 + 1/3\;b_1^2\}. \tag{8b}$$

It is possible to see from this that in the constraining physical limit of a very thin beam modelled by elements of length $2l$ and depth t, the shear strain energy in eqn (8b) must vanish. An examination of the conditions produced by this requirement shows that the following constraints would emerge in such a limit

$$b_\sigma - a_1/l \to 0; \tag{9a}$$

$$b_1 \to 0. \tag{9b}$$

In the new terminology, constraint (9a) is field consistent as it contains constants from both the contributing displacement interpolations relevant to the descriptions of the shear strain field. These constraints can then accommodate the true Kirchhoff constraints in a physically meaningful way, *i.e.*, in an infinitesimal sense, this is equal to the condition $(\theta - w,_x) \to 0$ at the element centroid. In direct contrast, constraint (9b) contains only a term from the section rotation θ. A constraint imposed on this will lead to an undesired restriction on θ. In an infinitesimal sense, this is equal to the condition $\theta,_x \to 0$ at the element centroid (*i.e.*, no bending is allowed to develop in the element region). This is the 'spurious constraint' that leads to shear locking and also violent disturbances in the shear force prediction over the element[2,5].

7.3. *A 'falsifiable' error model for the field-consistency paradigm*

We must now determine that this field-consistency paradigm has a scientific quality. To do this, we borrow an idea from the philosophy of science, the falsifiability theorem of Karl Popper[6]. We know that the discretised finite element model will contain an error which can be recognised when digital computations made with these elements are compared with analytical solutions where available. The *consistency* requirement has been offered as the missing paradigm for the error-free formulation of the constrained media problems. Therefore, to establish the scientific validity of this coneptual scheme, it is necessary to first devise a procedure that will trace the errors due to an inconsistent representation of the constrained strain field and obtain precise *a priori* measures for these. We must then show by actual numerical experiments with the original elements that the errors are as projected by these *a priori* error models. Only such an exercise will complete the scientific validation of the consistency paradigm. Fortunately, a procedure we shall call the *functional reconstitution* technique makes it possible to do this verification[2,5]. It is however beyond the scope of this lecture to go into the details of this procedure here.

7.4. *Numerical experiments to verify error prediction*

Our functional reconstitution procedure (note that this is an auxiliary procedure, distinct from the direct finite element procedure that yields the stiffness matrix) now provides an instrument for the critical self-examination of the consistency paradigm. It indicates that an exactly integrated or field-inconsistent finite element model tends to behave as a shear flexible beam with a much stiffened flexural rigidity I'. This can be related to the original rigidity I of the system by comparing the expressions derived from the functional reconstitution exercise[2,5] as

$$I'/I = 1 + kGAl^2/3EI. \tag{10}$$

We can show through a numerical experiment that this estimate for the error, which has been established entirely *a priori*, starting from the consistency paradigm and introducing the functional reconstitution technique, anticipates very accurately, the behaviour of a field-inconsistent linearly interpolated shear flexible element in an actual digital computation[2,5]. This has shown us that the consistency paradigm can be scientifically verified. Traditional procedures such as counting constraint indices, or computing the rank or condition number of the stiffness matrices could offer only a heuristic picture of how and why locking sets in.

7.5. *Concluding remarks on the Timoshenko beam element*

These exercises show us why it is important to maintain consistency of the basis functions chosen for terms in a functional which are linked to penalty multipliers. The same conditions are true for the various finite element formulations where locking, poor convergence and stress oscillations are known to appear. It is also clear why the imposition of the consistency condition into the formulation allows the correct rank or singularity of the penalty-linked stiffness matrix to be maintained so that the system is free of locking or sub-optimal convergence. Again, it is worthwhile to observe that non-singularity of the penalty-linked matrix occurs only when the approximate fields are of very low order. In higher order inconsistent formulations, solutions are obtained which are sub-optimal to solutions that are possible if the formulation is carried out with the consistency condition imposed *a priori*. Devices such as reduced integration permit the consistency requirement to be introduced when penalty-linked matrix is computed so that the correct rank which ensures the imposition of the true constraints only is maintained. It is easy to predict all this by examining the constrained strain-field terms from the consistency point of view rather than performing a post-mortem examination of the penalty-linked stiffness matrix from rank or singularity considerations as is mostly advocated in the literature.

8. Conclusions

What I've sought to demonstrate in this lecture is that a critical rationalisation[6] can be made of the procedures adopted in the finite element method (FEM). The FEM is a procedure for solving field equations appearing in engineering analysis. It can be viewed as a purely computational device to eliminate the drudgery from the painful analysis and algebra otherwise required to solve such boundary and initial value problems. However, when difficulties were noticed when the procedures were applied to certain problems (*e.g.*, the locking phenomenon), it became clear that the conceptual framework which existed then (*continuity* and *completeness*) was insufficient to account for this phenomena. When we sought to enlarge this conceptual framework, the consistency paradigm was one that emerged as a satisfactory explanatory scheme. In this lecture, I've briefly outlined how it was used to provide a critical rationalisation of the difficulties encountered. Some of the questions we asked or were asked as we went about this task were:

1. Are there laws (theories, hypotheses, paradigms) governing FEM methodology (or computational methodology in general) as there are in descriptions of natural phenomena, space and time?

2. Can singular statements be derived from these that can be falsified by numerical experimentation?

3. Is error analysis (from the point of view of FEM or any other computational methodology) predictive?

I end my lecture on optimistic note that my answers to all these questions are in the affirmative. We could trace the evolution of paradigms in FEM practice from myth and superstition to the consistency paradigm as a falsifiable basis[2]. I hope this lecture has captured this spirit—that concepts like *completeness, continuity, consistency* and *correctness* (not dealt with here; see Prathap[2] for details) are relevant not only in a paradigmatic description of natural phenomena but also in areas of derived (secondary?) knowledge like computational mechanics.

Acknowledgements

The author is deeply indebted to Professor R. Narasimha, FRS, formerly Director of the National Aerospace Laboratories for having been friend, philosopher and guide for nearly a decade while this work was being carried out. His inspiration was directly responsible for the critical rationalization that has been brought to bear on this present statement on the progress of the finite element method.

The concepts and theories reviewed here were developed partly under the support of Grants-in-Aid Schemes of the Aeronautics Research and Development Board (ARDB) of the Ministry of Defence and this financial support is gratefully acknowledged.

References

1. NARASIMHA, R. — Technology policy in a liberalising economy, *Curr. Sci.*, 1993, **64**, 494–502.

2. PRATHAP, G. — *The finite element method in structural mechanics*, 1993, Kluwer.

3. TIMOSHENKO, S.P — On the transverse vibrations of bars of uniform cross-section, *Phil. Mag.*, 1992, **43**, 125–131.

4. HUGHES, T.J.R., TAYLOR, R.L. AND KANOKNUKULCHAL, W. — A simple and efficient finite element for plate bending, *Int. J. Numer. Meth. Engng*, 1977, **11**, 1529–1543.

5. PRATHAP, G. AND BHASHYAM, G.R — Reduced integration and the shear flexible beam element, *Int. J. Numer. Meth. Engng*, 1982, **18**, 195–210.

6. POPPER, K.R. — *Conjectures and refutations—the growth of scientific knowledge*, 1972, Routledge and Kegan Paul, London.