NOTE ON THE LARGE DEFLECTION OF AN ORTHOTROPIC CIRCULAR PLATE WITH CLAMPED EDGE UNDER SYMMETRICAL LOAD

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1. INTRODUCTION

For large deflection of plates usually a non-linear equation is involved which cannot be exactly solved. But Berger [1] has shown that if in driving the differential equation from strain energy, the energy due to the second strain invariant in the middle plane of the plate is neglected, a simple fourth order differential equation coupled with a non-linear second order equation is obtained. He has also solved such equations for the problem of circular plates under various boundary conditions subjected to normal uniform load throughout the plate. Since then numerous problems have been solved with remarkable ease and accuracy by different authors. Iwinski and Nowinski [2] generalised the procedure of Berger to orthotropic plates and examined the deflections of circular and rectangular plates under uniform load and different boundary conditions.

In this note the author has attempted to solve the problem of the large deflections of orthotropic circular plates under symmetrical load. The hecorresponding problem on isotropic plate is due to Banerjee, B. [3].

NOTATIONS

$f(r)$ = symmetrical load function at a distance from the centre,

u = radial displacement,

w = deflection normal to the middle plane of the plate,

a = radius of the plate,

h = thickness of the plate,

D_r = average flexual rigidity of the plate,

$k^2 = \sigma_t / \sigma_r$,

σ_t, σ_r = Poisson's ratio corresponding to radial and cross-radial directions.

ANALYSIS

Considering the circular symmetry of the plate of thickness h, deflection w, normal to the middle plane, and u, radial displacement in the middle plane, one can write the fundamental equations with the load function $f(r)$ as

$$\frac{d^4 w}{dr^4} + \frac{2}{r}\frac{d^3 w}{dr^3} - \frac{k^2}{r^2}\left(\frac{d^2 w}{dr^2} - \frac{1}{r}\frac{dw}{dr}\right)$$

$$- \frac{12}{h^2}\frac{1}{r}\frac{d}{dr}\left(e_1^* r \frac{dw}{dr}\right) = f(r) \tag{1}$$

$$\frac{de_1^*}{dr} + \frac{1-k}{r} e_1^* = 0, \tag{2}$$

where

$$e_1^* = \frac{du}{dr} + k\frac{u}{r} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \tag{3}$$

Integrating (2) we have

$$e_1^* = Cr^{k-1}. \tag{4}$$

Let us assume the solution of (1) as

$$w = \sum_{s=1}^{\infty} A_s \{r^{\frac{1-k}{2}} J_{\frac{k-1}{k+1}}(p_s r^{\frac{1+k}{2}}) - a^{\frac{1-k}{2}} J_{\frac{k-1}{k+1}}(p_s a^{\frac{1+k}{2}})\} \tag{5}$$

where $p_s a^{\frac{1+k}{2}}$ is the s-th root of $J_{\frac{2k}{k+1}}(p_s a^{\frac{1+k}{2}}) = 0$. It is clear that the above form of w satifies the following boundary conditions for a clamped plate:

$$w = \frac{dw}{dr} = 0, \quad \text{at} \quad r = a \tag{6}$$

Using (5) in equation (1) one gets

$$\sum A_s p_s^{\frac{5k-1}{k+1}} (p_s^2 + \alpha_1^2)\left(\frac{k+1}{2}\right)^4 r^{\frac{3k}{2}-\frac{3}{2}} J_{\frac{k-1}{k+1}}(p_s r^{\frac{1+k}{2}})$$
$$= f(r) \tag{7}$$

where

$$\alpha_1^2 = \frac{12C}{h^2}\left(\frac{2}{1+k}\right)^2 \tag{8}$$

Replacing $r^{\frac{1+k}{2}}$ by t and multiplying both sides of (7) by $t\, J_{\frac{k-1}{k+1}}(p_s t)$, we use Dini-expansion and finally integrating the tesulting equation over the area of the plate, equation (7) reduces to

$$\int_0^{a^{\frac{1+k}{2}}} \sum A_s p_s^{\frac{5k-1}{k+1}} (p_s^2 + \alpha_1^2) \left(\frac{k+1}{2}\right)^4 t\, J^2_{\frac{k-1}{k+1}}(p_s t)\, dt$$

$$= \int_0^{a^{\frac{1+k}{2}}} t^{\frac{2(2-k)}{1+k}} f(t^{\frac{2}{1+k}})\, J_{\frac{k-1}{k+1}}(p_s t)\, dt \tag{9}$$

Thus A_s is given by

$$A_s = \frac{2^5 (1+k)^{-4} (p_s^2 + \alpha_1^2)^{-1}}{p_s^{\frac{5k-1}{k+1}}\, a^{1+k}\, J^2_{\frac{k-1}{k+1}}\left(p_s a^{\frac{1+k}{2}}\right)} \int_0^{a^{\frac{1+k}{2}}} t^{\frac{2(2-k)}{1+k}} f(t^{\frac{2}{1+k}})\, J_{\frac{k-1}{k+1}}(p_s t)\, dt. \tag{10}$$

For an illustration we take the load function over a concentric circular area of radius b, $(b < a)$, as

$$f(r) = Q_0 (b^2 - r^2)^{1/2} \qquad \text{where} \quad 0 < r < b$$
$$= 0, \qquad \text{where} \quad b < r < a$$

and Q_0 = a constant.

For such a load function equation (10) reduces to

$$A_s = \frac{2Q_0 \left(\frac{2}{k+1}\right)^4 b^{\frac{7-k}{2}}\, \psi(p_s b)}{p_s^{\frac{5k-1}{k+1}}\, a^{1+k} (p_s^2 + \alpha_1^2)\, J^2_{\frac{k-1}{1+k}}\left(p_s a^{\frac{1+k}{2}}\right)} \tag{11}$$

where

$$\psi(p_s b) = \frac{1}{2} \sum_{n=0}^{\alpha} (-1)^n \frac{\{p_s b^{\frac{1+k}{2}}/2\}^{\frac{k-1}{k+1}+2n}\, \beta(\alpha_1', \alpha_2)}{n!\, \Gamma\left(\frac{2k}{k+1} + n\right)} \tag{12}$$

Γ, β = Gamma and Beta functions, respectively,

$$a_1' = 1 + \frac{1+k}{2} n, \qquad a_2 = \frac{3}{2}. \tag{13}$$

Thus w is obtained in the following form

$$w = \sum_{s=1}^{\infty} \frac{2Q_0 b^{\frac{7-k}{2}} \left(\frac{2}{1+k}\right)^4 \psi(p_s b^{\frac{1+k}{2}})}{p_s^{\frac{5k-1}{k+1}} (p_s^2 + a_1^2) a^{1+k} J^2_{\frac{k-1}{k+1}} (p_s a^{\frac{1+k}{2}})} \times [r^{\frac{1+k}{2}} J_{\frac{k-1}{k+1}} (p_s r^{\frac{1+k}{2}}) - a^{\frac{1-k}{2}} J_{\frac{k-1}{k+1}} (p_s a^{\frac{1+k}{2}})] \tag{14}$$

As $r - 0$ the central deflection w_0 is obtained as

$$w_0 = \sum_{s=1}^{\infty} \frac{2Q_0 b^{\frac{7-k}{2}} \psi(p_s b^{\frac{1+k}{2}}) \left(\frac{2}{1+k}\right)^4 \left[\frac{p_s^{\frac{k-1}{k+1}}}{2^{\frac{k-1}{k+1}} \Gamma\left(\frac{2k}{k+1}\right)} - a^{\frac{1-k}{2}} J_{\frac{k-1}{k+1}} (p_s^{\frac{1+k}{2}}\right]}{p_s^{\frac{5k-1}{k+1}} a^{1+k} (p_s^2 + a_1^2) J^2_{\frac{k-1}{k+1}} (p_s a^{\frac{1+k}{2}})}$$

To determine the radial displacement u, we integrate equation (3) with the help of (and (5) we get

$$ur^k = C\frac{r^{2k}}{2k} - \frac{1}{2}\sum_{s=1}^{\infty} A_s^2 p_s^2 \left(\frac{1+k}{4}\right) r^{1+k} \left[\left(1 - \frac{4k^2}{p_s^2 (1+k)^2 r^{1+k}}\right) J^2_{\frac{2k}{1+k}} (p_s r^{\frac{1+k}{2}}) + J'^2_{\frac{2k}{1+k}} (p_s r)^{\frac{1+k}{2}}\right] - \frac{1}{2} \sum_{\substack{s=1 \\ s \neq m}}^{\infty} \sum_{m=1}^{\infty} A_s A_m p_s p_m^{\frac{1+k}{2}}$$

$$\times \frac{[p_s J_{\frac{3k+1}{k+1}} (p_s r^{\frac{1+k}{2}}) J_{\frac{2k}{1+1}} (p_m r^{\frac{1+k}{2}}) - p_m J_{\frac{2k}{1+k}} (p_s r^{\frac{1+k}{2}}) J_{\frac{3k+1}{k+1}} (p_m r^{\frac{1+k}{2}})]}{\{2/(1+k)\} (p_s^2 - p_m^2)} + \theta,$$

where θ is the constant of integration. Using $r \to a$, $u \to 0$ θ is given by

$$\theta = \frac{1}{2} \sum_{s=1}^{\infty} A_s^2 p_s^2 \left(\frac{k+1}{4}\right) a^{1+k} J^2_{\frac{k-1}{k+1}} (p_s a^{\frac{1+k}{2}}) - C\frac{a^{2k}}{2k}. \tag{17}$$

To evaluate C and hence α_1, further condition $u \to 0$ as $r \to 0$ can be used when α_1 is given by

$$\alpha_1^2 a^{2k} h^2 (1+k)/(12k) = \sum_{s=1}^{\infty} A_s^2 p_s^2 a^{1+k} J^2_{\frac{k-1}{k+1}} \left(p_s a^{\frac{1+k}{2}}\right). \tag{18}$$

As $k \to 1$, eqns. (14), (18) and (15) will reduce to

$$w = \sum_{s=1}^{\infty} \frac{2Q_0 b^3 P(p_s b)}{a^2 p_s^2 (p_s^2 + \alpha_1^2) J_0^2 (p_s a)} (J_0(p_s r) - J_0(p_s a)],$$

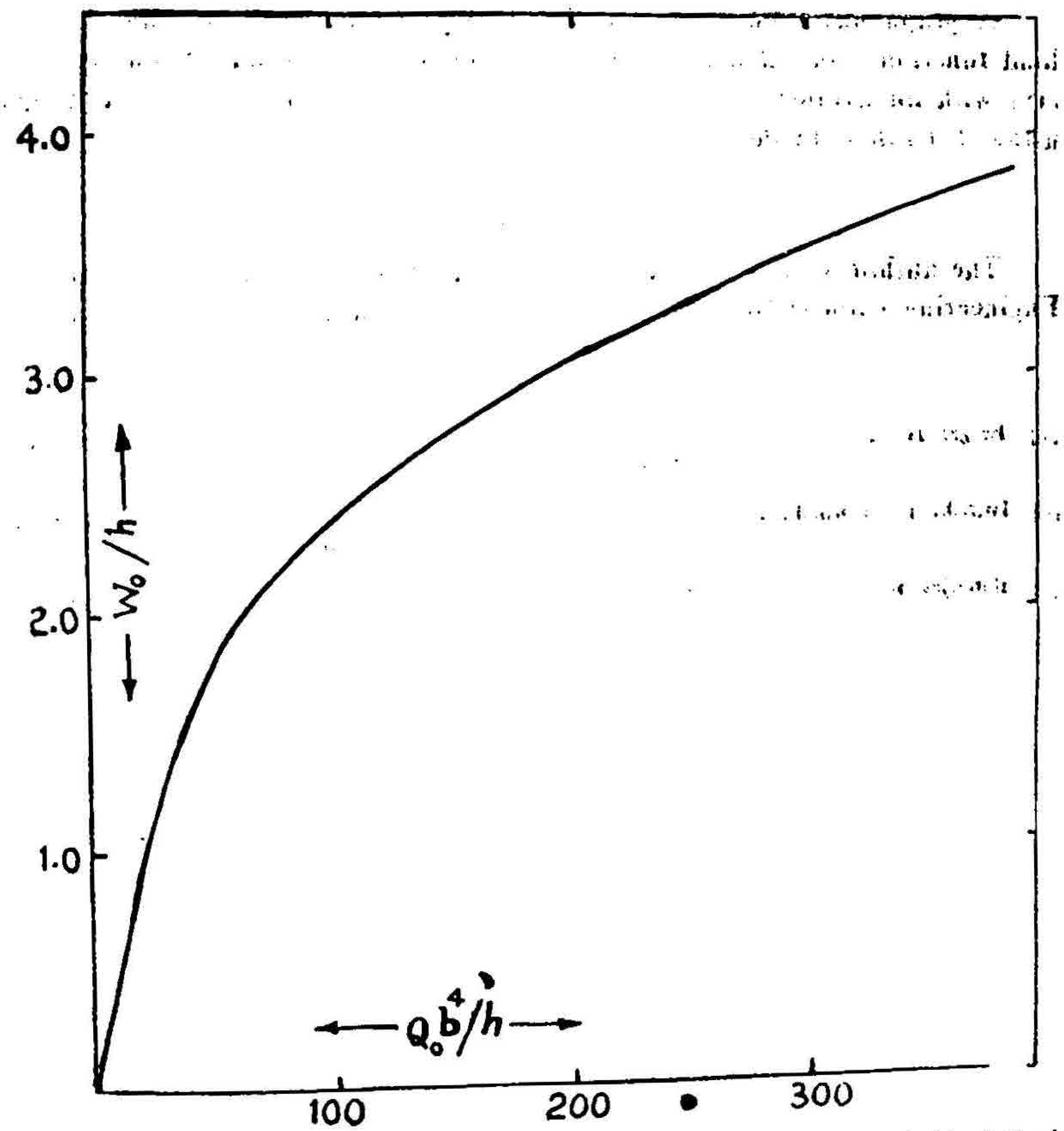

Fig. 1. Centre deflection of a clamped orthotrophic circular plates under symmetrical load. $K = \frac{1}{3}$

$$(\alpha^2 a^2 h^2/6) = \sum_{s=1}^{\infty} A_s^2 p_s^2 J_0(p_s a),$$

$$w_0 = \frac{2b^3 Q_0}{a^2} \sum_{s=1}^{\infty} \frac{p(p_s b)\,[1 - J_0(p_s a)]}{p_s^2 (p_s^2 + \alpha^2) J_0^2(p_s a)},$$

respectively, and these are the results obtained by Banerjee, B. [3] in his corresponding isotropic plate problem.

NUMERICAL RESULTS

A graph has been plotted showing the central deflection against the load function. In calculating the deflection one has to start from equation (18) with an assumed value and then using equations (18) first and then using (15) will yield the results.

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