1. Appendix

A.1. Wave Equation in an Inhomogeneous Medium

Using $D = \epsilon_0 \epsilon(\rho) E$ and $B = \mu_0 \mu H$ in Maxwell's equation for a source-free, isotropic lossless medium and assuming harmonic time dependence for the field vectors, we get

$$\nabla \times E = -j\omega\mu_0 H$$

$$\nabla \times H = -j\omega\epsilon_0 \epsilon(\rho) E$$

which, on elimination of H, yields

$$\nabla(\nabla \cdot E) - \nabla^2 E = k_0^2 \epsilon(\rho) E. \tag{A.1.1}$$

Since

$$\nabla \cdot D = 0,$$

$$\nabla \cdot [\epsilon_0 \epsilon(\rho) E] = \epsilon_0 [\nabla \cdot \epsilon(\rho) E]$$

which reduces to

$$\epsilon_0 [\epsilon(\rho) \nabla \cdot E + \nabla \epsilon(\rho) E] = 0. \tag{A.1.2}$$

The following wave equation for an inhomogeneous medium is obtained with the aid of the two equations (A.1.1) and (A.1.2)

$$\nabla^2 E + \nabla \left[E \cdot \frac{\nabla \epsilon(\rho)}{\epsilon(\rho)}\right] + k_0^2 \epsilon(\rho) E = 0 \tag{A.1.3}$$

A.2. Condition for the Variation of $\epsilon(\rho)$ over One Wavelength

The order of magnitude of the terms occurring in equation (A.1.3) was estimated and the first and the last term which are of equal order are found to influence greatly the magnitude of equation (A.1.3). The following analysis is applied only to estimate the order of magnitude.

Using the approximation

$$k_0^2 \epsilon(\rho) E \simeq \left(\frac{2\pi}{\lambda}\right)^2 E \tag{A.2.1}$$

and replacing the operator ∇ in the second term of the equation (A.1.3) by a derivative with respect to some direction S in space, its order of magnitude can be written as

$$\nabla\left[E \cdot \frac{\nabla \epsilon(\rho)}{\epsilon(\rho)}\right] \simeq \frac{\partial}{\partial S}\left[E \cdot \frac{\nabla \epsilon(\rho)}{\epsilon(\rho)}\right]$$

$$\simeq \left(\frac{2\pi}{\lambda}\right) E \cdot \frac{\nabla \epsilon(\rho)}{\epsilon(\rho)} + E \frac{\partial}{\partial S}\left[\frac{\nabla \epsilon(\rho)}{\epsilon(\rho)}\right] \tag{A.2.2}$$

We are interested in the case when the second term in equation (A.1.3) is much smaller than the first term. Indicating the order of magnitude by placing the expressions in brackets and comparing the second and the third terms in (A.1.3),

$$R_x = \left[\nabla\left\{E \cdot \frac{\nabla \epsilon(\rho)}{\epsilon(\rho)}\right\}\right] \Big/ [k_0^2\, \epsilon(\rho) E]$$

$$\simeq \left\{\frac{2\pi}{\lambda} \frac{\Delta(\rho)}{\epsilon(\rho)}\right\} \Big/ \left(\frac{2\pi}{\lambda}\right)^2$$

which yields

$$R_x \simeq \frac{\lambda}{2\pi} \frac{[\epsilon_1(\rho) - \epsilon_{11}(\rho)]}{\Delta S \cdot \epsilon(\rho)} \tag{A.2.3}$$

where the order of magnitude of the gradient of $\epsilon(\rho)$ is indicated by the ratio of the difference $[\epsilon_1(\rho) - \epsilon_{11}(\rho)]$ of the dielectric constant of two closely spaced points to the distance ΔS between them. Considering $\Delta S = \lambda$, equation (A.2.3) reduces to

$$R_x \simeq \frac{1}{2\pi} \frac{\epsilon_1(\rho) - \epsilon_{11}(\rho)}{\epsilon(\rho)} \tag{A.2.4}$$

If $R_x \ll 1$, *i.e.*, the relative change in $\epsilon(\rho)$ over the distance of one wavelength is very small compared to unity, then the second term in equation (A.1.3) can be ignored and hence equation (2) can be simplified to equation (2 *a*).

The maximum values of d and for different dielectric profiles have been determined in the following way.

If

$$\epsilon(\rho) = \epsilon_1 [1 - (d\rho')^2]$$

where

$$\rho' = \rho - r_2,$$

$$\epsilon(\rho) = \epsilon_1 [- 2d^2 (\rho - r_2)]$$

and

$$R_x = \frac{\lambda}{2\pi} \frac{\epsilon'(\rho)}{\epsilon(\rho)} \ll 1$$

or

$$\frac{\epsilon'(\rho)}{\epsilon(\rho)} \ll k_0 \sqrt{\epsilon(\rho)}$$

which leads to

$$d \ll k_0 \sqrt{\epsilon_1} [1 - (d\rho')^2]^{3/2}/2 (d\rho')$$

Similarly we obtain for the other profiles

(ii) $d \ll k_1 [1 - (d\rho')^4]^{3/2}/4 (d\rho')^3$

(iii) $d \ll k_1 [1 - (d\rho')^2 - (d\rho')^4]^{3/2}/[2 (d\rho') + 4 (d\rho')^3]$

(iv) $d \ll k_1 [\text{sech} (d\rho')]^{1/2}/\tanh (d\rho')$

(v) $d \ll k_1 \text{sech} (d\rho')/2 \tanh (d\rho')$

(vi) $d \ll k_1 \exp (- d\rho')/2.$ (A.2.5)

The maximum values (d_{max}) taken for different profiles are given in Table A 1.

TABLE A 1.

Values of d_{max}

$d\rho'$ \ Type of profile	(i)	(ii)	(iii)	(iv)	(v)	(vi)
0·1	4·0	200·0	4·0	8·5	4·3	0·42
0·2	2·0	27·0	1·9	4·4	2·1	0·4
0·3	1·25	8·0	1·0	2·9	1·6	0·38
0·4	0·75	3·0	0·6	2·2	1·0	0·36
0·5	0·55	1·5	0·3	1·75	0·8	0·34

The values of d and ρ have been chosen appropriately from the values of d_{max} and $d\rho'$ for numerical calculations reported in the text.

A.3 SOLUTION OF EQUATION

The solution of equation

$$x\frac{d^2 y}{dx^2} + \frac{dy}{dx} + \left(\frac{a_1}{b}\right)^2 \times [(1 - r_2^2 b^2) + 2r_2 bx - x^2] y = 0 \quad (A.3.1)$$

where

$b = k_1 d/a_1$ and y is related to E_z as

$E_z = y_e^{-\gamma z}$, is found by assuming

$y = x^r (c_0 + c_1 x + c_2 x^2 + \ldots)$ and determining the coefficients $c_0, c_1, c_2 \ldots$. The first solution is,

$$y_1 = x^r (c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots) \quad (A.3.2)$$

where

$$c_n = \frac{-1}{(r+n)^2}\left(\frac{a_1}{b}\right)^2 [(1 - r_2^2 b^2) c_{n-2} + 2b_2 bc_{n-3} - c_{n-4}]$$

The second solution is found by substituting

$$y_1 = x^r \left[1 + \frac{b_1 x}{(r+1)^3} + \frac{b_2 x^2}{(r+2)^3} + \ldots\right]$$

where

$$b_1 = -\left(\frac{a_1}{b}\right)^2 (1 - r_2^2 b^2) c_0$$

$$b_2 = -\left(\frac{a_1}{b}\right)^2 [(1 - t_2^2 b^2) c_1 + 2r_2 bc_0)]$$

and so on, in equation (A.3.1), carrying out the necessary differentiation and after some manipulation,

$$y_2 = y_1 \log x + \sum_{n=0}^{\infty} d_n x_n \quad (A.3.3)$$

where

$$d_n = -\left[\frac{2}{n} c_n + \left(\frac{a_1}{bn}\right)^2 \xi (1 - r_2^2 b^2) d_{n-2} + 23_2 b\, d_{n-3} - d_{n-4}\}\right]$$

with

$$d_0 = 0, \quad d_1 = 0, \quad d_2 = - c_2$$

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = \left(\frac{a_1}{2b}\right)^2 (1 - r_2^2 b^2)$$

and so on.

The above results are applied to the different cases as follows.

(1) *Profile* :

$$\epsilon_2(\rho) = \epsilon_1 [1 - d^2 (\rho - r_2)^2]$$

wave equation

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(\frac{a_1}{b}\right)^2 [1 - (x - r_2 b)^2] y = 0$$

where

$$x = b\rho, \; b = \frac{k_1 d}{a_1}$$

Solution

$$y = (c_1 + c_2 \log x) \sum_{n=0}^{\infty} c_n x^n + c_2 \sum_{n=0}^{\infty} d_n x^n \tag{A}$$

where

$$c_n = - \left(\frac{a_1}{b^n}\right)^2 [(1 - r_2^2 b^2) c_{n-2} + 2r_2 b c_{n-3} - c_{n-4}]$$

$$d_n = - \left\{\frac{2c_n}{n} + \left(\frac{a_1}{b_n}\right)^2 [(1 - r_2^2 b^2) d_{n-2} + 2r_2 b d_{n-3} - d_{r-4}]\right\}$$

(ii) *Profile*

$$\epsilon_2(\rho) = \epsilon_1 [1 - d^4 (\rho - r_2)^4]$$

Wave equation :

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(\frac{a_1}{b}\right)^2 [1 - (x - r_2 b)^4] y = 0$$

where

$$x = b\rho \quad \text{and} \quad b = \left(\frac{k_1}{a_1}\right)^{1/2} d$$

Solution : as given by (A)

where

$$c_n = -\left(\frac{a_1}{bn}\right)^2 [(1 - r_2^4 b^4) c_{n-2} + 4r_2^3 b^3 c_{n-3}$$

$$- 6r_2^2 b^2 c_{n-4} + 4r_2 b c_{n-5} - c_{n-6}]$$

$$d_n = -\left\{\frac{2c_n}{n} + \left(\frac{a_1}{bn}\right)^2 [(1 - r_2^4 b^4) dn - 2\right.$$

$$\left. + 4r_2^3 b^3 d_{n-3} r_2^2 b^2 d_{n-4} + 4r_2 b d_{n-5} - d_{n-6}]\right\}$$

(iii) Profile :

$$\epsilon_2(\rho) = \epsilon_1 [1 - d^2(\rho - r_2)^2 - d^4(\rho - r_2)^4].$$

Wave Equation:

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(\frac{a_1}{b}\right)^2 \left[1 - (x - r_2 b)^2 - \left(\frac{a_1}{k_1}\right)^2 (x - r_{2b})^4\right] y = 0$$

where

$$x = b\rho, \quad b = \frac{k_1 d}{a_1}.$$

Solution : As given by (A), where

$$c_n = -\left(\frac{a_1}{bn}\right)^2 [(1 - r_2^4 b^4 P - r_2^2 b^2) c_{n-2} + (2r_{2b} + 4r_2^3 b^3 P) c_{n-3}$$

$$- (1 + 6r_2^2 b^2 P) c_{n-4} + 4r_2 bP c_{n-5} - P c_{n-6}]$$

$$d_n = -\left\{\frac{2c_n}{n} + \left(\frac{a_1}{bn}\right)^2 [(1 - -r_2^4 b^4 P - r_2^2 b^2) d_{n-2}\right.$$

$$+ (2r_{2b} + 4r_2^3 b^3 P) dn - 3 - (1 + 6r_2^2 b^2 P) d_{n-4}$$

$$\left. + 4r_2 bP d_{n-5} - P d_{n-6}]\right\}$$

(iv) *Profile*

$$\epsilon_2(\rho) = \epsilon_1 \operatorname{sech} d(\rho - r_2)$$

Wave Equation:

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + a_1^2\left[1 + \sum_{n=1}^{\infty} pn\,(x - r_2)^{2n}\right] y = 0$$

where

$$x = 1, \quad p_n = \left(\frac{k_1}{a_1}\right)^2 d^{2n}\, b_n$$

and

$$b_n = (-1)^n \frac{2^{2n+2}}{\pi^{2n+1}} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{(2k-1)^{2n+1}}\,.$$

Solution : As given by (A), where

$$C_n = -\left(\frac{a_1}{n}\right)^2 \sum_{k=0}^{n-2} p_k c_{n-k-2}$$

$$d_n = -\left[\frac{2c_n}{n} + \left(\frac{a_1}{n}\right)^2 \sum_{k=0}^{n-2} p_k d_{n-k-2}\right]$$

$$p_k = (-1)^k \sum_{n=m}^{\infty} \frac{(2n)\,!}{k\,!\,(2n-k)\,!}\, p_n d_{n-k-2}$$

where

$m = (k+1)|2$, for k, odd

$m = k|2$, for k, even

(v) *Profile* :

$$\epsilon_2(\rho) = \epsilon_1 \operatorname{sech}^2 d(\rho - r_2)$$

Wave Equation :

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + a_1^2\left[1 + \sum_{n=1}^{\infty} q_n\,(x - r_2)^{2n}\right] y = 0$$

where

$$x = \rho \quad \text{and} \quad q_n' = \left(\frac{k_1}{a_1}\right)^2 d^{2n} \sum_{k=0}^{n} b_k b_{n-k}$$

Solution : as given by (A), where

$$c_n = -\left(\frac{a}{n}\right)^2 \sum_{k=0}^{n-2} q_k c_{n-k-2}$$

$$d_n = -\left[\frac{2c_n}{n} + \left(\frac{a_1}{n}\right)^2 \sum_{k=0}^{n-2} q_k d_{n-k-2}\right]$$

$$q_k = (-1)^k \sum_{n=m}^{\infty} \frac{(2n)!}{k!\,(2n-k)!} \times q_n' \cdot r_2^{2n-k}$$

where

$$m = (k+1)/2 \text{ for } k, \text{ odd}$$

and

$$m = k/2 \quad \text{for} \quad k, \text{ even.}$$

(vi) *Profile* :

$$\epsilon_2(\rho) = \epsilon_1 \exp[-d(\rho - r_2)].$$

Wave Equation :

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + a_1^2\left[1 + \sum_{n=1}^{\infty} t_n'(x - r_2)^n\right] y = 0$$

where

$$x = \rho \quad \text{and} \quad t_n' = \frac{(-1)^n}{n!} d^n \left(\frac{k_1}{a_1}\right)^2.$$

Solution : As given by (A), where

$$c_n = -\left(\frac{a_1}{n}\right)^2 \sum_{k=0}^{n-3} t_k c_{n-k-2}$$

$$d_n = -\left[\frac{2c_n}{n} + \left(\frac{a_1}{n}\right)^2 \sum_{k=0}^{n-3} t_k d_{n-k-2}\right]$$

$$t_k = \sum_{n=k}^{\infty} (-1)^n t_n' \frac{n!}{(n-k)!\,k!} r_2^{n-k}.$$

In all the above six cases, $c_0 = 1$, $c_1 = 0$, $d_0 = 0 = d_1$.