RADIAL DEFORMATIONS OF NONHOMOGENEOUS SPHERICALLY ANISOTROPIC ELASTIC MEDIA

SUBHAS CHANDRA ROY

(Department of Mathematics, Jalpaiguri Government Engineering College, Jalpaiguri, West Bengal, India)

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ABSTRACT

This paper deals with the elasticity problem of a spherically anisotropic elastic medium bounded by two concenrtric spherical surfaces subjected to normal pressures. The material of the structure is spherically anisotropic and, in addition, is continuously inhomogeneous with mechanical properties varying exponentially along the radius. An exact solution of the problem in terms of Whittaker functions is presented. The St. Venant's solution in the case of homogeneous material and Lamè's solution in the case of homogeneous isotropic material are derived here from the general solution. The problem of a solid sphere of the same medium under the external pressure is also solved as a particular case of the above problem. Lastly, the displacements and stresses of a composite sphere consisting of a solid spherical body made of homogeneous material and a nonhomogeneous concentric spherical shell covering the inclusion, both of them being spherically anisotropic, are obtained when the sphere is under uniform compression,

Keywords: Radial deformation, Stresses, Spherical shell, Nonhomogeneity, Inclusion.

1. INTRODUCTION

The elastic behavior of a spherically aeolotropic material was first studied by St. Vernant in 1865. He considered the problem of a spherical shell under uniform internal and external pressures and applied his results to some piezometer experiments. A description of his analysis may be found in the treatise by Love [1] or in the book of anisotropic elasticity by Lekhnitskii [2].

Increasing use of composite materials in aerospace applications calls for the study of problems of nonhomogeneous anisotrpic elastic media. Grief and Chou [3] have treated a dynamic problem of a non-homogeneous cylindrically aeolotropic shell. Sengupta and Basu Mallick [4] have investigated the radical deformation of a nonhomogeneous spherically anisotropic elastic shell under uniform internal and external pressures. All of them consider the elastic prameters to be proportional to n-th power of the radius.

An attempt has been made here to find the analytical solution for the radial deformation and corresponding stresses in a spherical shell made of spherically aeolotropic heterogeneous material under the influence of normal pressures on both boundaries. The corresponding results for homogeneous spherically aeolotropic material are derived here as a particular case and these were obtained by St. Venant, as quoted in Lekhnitskii [2]. The expressions calculated by Lamè and given in Love [1] for homogeneous isotropic bodies are found from the general results. The results for a solid sphere of nonhomogeneous spherically anisotropic medium under the external pressure are derived from the general expressions when the radius of the inner surface approaches zero. At the end, radial displacements and stresses for both the portions of nonhomogeneous spherically anisotropic shell having concentric homogeneous spherically anisotropic inclusion are presented here, when the outer surface is loaded with a uniform normal pressure. In all the cases, the nonhomogeneity of the material is characterized by the elastic parameters C_{ij}, *vide* Grief and Chou [3], Sengupta and Basu Mallick [4] as

$$C_{ij} = \lambda_{ij} \exp(-kr), \quad (i, j = 1, 2, 3) \tag{1}$$

a new variation, where λ_{ij} and k are the prescribed parameters of the material concerned.

2. FUNDAMENTAL EQUATIONS

The basic system of field equations in linear isothermal static elasticity theory are:

(*a*) the generalized Hooke's law, (*b*) the linearized strain displacement equations, and (*c*) the stress equations of equilibrium. Here the centre of a spherical shell or sphere is taken as origin and spherical polar co-ordinates (r, θ, ϕ) are used.

For a spherically anisotropic body, the generalized Hooke's law may be written as, *vide* Lekhnitskii]2]

$$\begin{aligned}
\bar{\sigma}_r &= c_{11}\bar{e}_{rr} + c_{12}\bar{e}_{\theta\theta} + c_{12}\bar{e}_{\phi\phi} \\
\bar{\sigma}_\theta &= c_{12}\bar{e}_{rr} + c_{22}\bar{e}_{\theta\theta} + c_{23}\bar{e}_{\phi\phi} \\
\bar{\sigma}_\phi &= c_{12}\bar{e}_{rr} + c_{23}\bar{e}_{\theta\theta} + c_{22}\bar{e}_{\phi\phi} \\
\bar{\tau}_{\theta\phi} &= \tfrac{1}{2}(c_{22} - c_{23})\,\bar{e}_{\theta\phi} \\
\bar{\tau}_{r\theta} &= c_{44}\bar{e}_{r\theta} \\
\bar{\tau}_{r\phi} &= c_{44}\bar{e}_{r\phi}
\end{aligned} \tag{2}$$

where C_{ij} are functions of the elastic moduli, Poison's ratios, and (for a non-homogeneous material) also functions of the spatial position. For the present problem C_{ij} are already mentioned in equation (1).

Now for a purely radial deformation of the body, the displacement components $(\bar{u}, \bar{v}, \bar{w})$ must be of the type $\bar{u} = \bar{u}(r)$, $\bar{v} = 0$ and

$$\bar{w} = 0. \tag{3}$$

Due to this assumption the strain components, in terms of displacements, are

$$\bar{e}_{rr} = \frac{d\bar{u}}{dr}, \quad \bar{e}_{\theta\theta} = \frac{\bar{u}}{r} = \bar{e}_{\phi\phi}$$

$$\bar{e}_{\theta\phi} = \bar{e}_{r\theta} = \bar{e}_{r\phi} = 0. \tag{4}$$

The non-zero stress components in equations (2), in terms of displacement and λ_{ij}, may now be written as

$$\bar{\sigma}_r = \exp(-kr)\left[\lambda_{11}\frac{d\bar{u}}{dr} + 2\lambda_{12}\frac{\bar{u}}{r}\right]$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = \exp(-kr)\left[\lambda_{12}\frac{d\bar{u}}{dr} + (\lambda_{22} + \lambda_{23})\frac{\bar{u}}{r}\right]. \tag{5}$$

In the absence of body forces, two equations of equilibrium are identically satisfied and the non-trivial equation of equilibrium becomes

$$\frac{d}{dr}(\bar{\sigma}_r) + \frac{2}{r}(\bar{\sigma}_r - \bar{\sigma}_\theta) = 0.$$

This equation of equilibrium, with the help of equations [5], stands as

$$r^2\frac{d^2\bar{u}}{dr^2} + (2 - kr)\,r\frac{d\bar{u}}{dr} - 2\left\{\frac{\lambda_{22} + \lambda_{23} + (kr-1)\lambda_{12}}{\lambda_{11}}\right\}\bar{u} = 0. \tag{6}$$

3. Method of Solution

We use transformations

$$x = kr \quad \text{and} \quad \bar{u} = V\exp\frac{x}{2} \tag{7}$$

in the equation (6) and rewrite it accordingly,

$$x^2\frac{d^2v}{dx^2} + 2x\frac{dv}{dx} + \left\{\frac{-2(\lambda_{22} + \lambda_{23} - \lambda_{12})}{\lambda_{11}}\right.$$

$$\left. + \left(1 - \frac{2\lambda_{12}}{\lambda_{11}}\right)x - \frac{x^2}{4}\right\}V = 0. \tag{8}$$

Again for

$$V = x^{-1} U, \tag{9}$$

the equation (8) reduces to

$$x^2 \frac{d^2 U}{dx^2} + \left\{ -\frac{2(\lambda_{22} + \lambda_{23} - \lambda_{12})}{\lambda_{11}} + \left(1 - \frac{2\lambda_{12}}{\lambda_{11}}\right) x - \frac{x^2}{4} \right\} U = 0 \tag{10}$$

Following Whittaker and Watson [5] one can write down the solution of the equation (10) in the form

$$U = AM_{\overset{*}{k}, p}(x) + BM_{\overset{*}{k}, -p}(x) \tag{11}$$

where $M_{\overset{*}{k}, \pm p}(x)$ are Whittaker functions in which

$$2p = \left\{ 1 + \frac{8}{\lambda_{11}} (\lambda_{22} + \lambda_{23} - \lambda_{12}) \right\}^{\frac{1}{2}} > 0, \qquad \text{(noninteger)} \tag{12}$$

and

$$\overset{*}{k} = 1 - \frac{2\lambda_{12}}{\lambda_{11}}. \tag{13}$$

A and B are arbitrary constants.

If $2p$ be an integer or zero, the solution of the equation (10) may be written as

$$U = AW_{\overset{*}{k}, p}(x) + BW_{-\overset{*}{k}, p}(-x) \tag{14}$$

where

$$W_{\overset{*}{k}, p}(x) = \frac{\Gamma(c-1)}{\Gamma(d-c+1)} M_{\overset{*}{k}, p}(x) + \frac{\Gamma(1-c)}{\Gamma(d)} M_{\overset{*}{k}, -p}(x) \tag{15}$$

in which $C = 1 \pm 2p$, $d = \frac{1}{2} - \overset{*}{k} \pm p$ and $\Gamma(t)$ is a gamma function of t.

Finally the radial displacement $\bar{u}(r)$ satisfying the equilibrium equation (6) is obtained with the help of equations (7), (9) and (11) as

$$\bar{u} = \frac{\exp(kr/2)}{kr} [AM_{\overset{*}{k}, p}(kr) + BM_{\overset{*}{k}, -p}(kr)]. \tag{16}$$

Substituting this expression for u in equations (5), the nonvanishing stresses may now be obtained in general form

$$\left.\begin{aligned} \bar{\sigma}_r &= \frac{\exp(-kr/2)}{r} [A\alpha_p(r) + B\alpha_{-p}(r)] \\ \bar{\sigma}_\theta = \bar{\sigma}_\phi &= \frac{\exp(-kr/2)}{r} [A\beta_p(r) + B\beta_{-p}(r)] \end{aligned}\right\} \tag{17}$$

where

$$\left.\begin{aligned} \alpha_{\pm p}(r) &= \left[\frac{kr-2}{2kr}\lambda_{11} + \frac{2\lambda_{12}}{kr}\right] M^{*}_{k,\pm p}(kr) + \lambda_{11} M'^{*}_{k,\pm p}(kr) \\ \beta_{\pm p}(r) &= \left[\frac{kr-2}{2kr}\lambda_{12} + \frac{\lambda_{22}+\lambda_{23}}{kr}\right] M^{*}_{k,\pm p}(kr) + \lambda_{12} M'^{*}_{k,\pm p}(kr) \end{aligned}\right\} \tag{18}$$

The prime indicates the derivative of the function with respect to its argument.

4. The Problem of a Non-homogeneous Spherical Shell

We consider here a spherical shell $a \leq r \leq b$. The structure is made of nonhomogeneous spherically anisotropic material. The shell is under the influence of uniformly distributed internal and external pressures. The boundary conditions are as follows:

$$\left.\begin{aligned} \bar{\sigma}_r &= -p_0, \text{ on the surface } r = a \\ \bar{\sigma}_r &= -p_1, \text{ on the surface } r = b \end{aligned}\right\} \tag{19}$$

On application of these boundary conditions in the first equation of (17), we get

$$A\alpha_p(a) + B\alpha_{-p}(a) + p_0 a \exp(ka/2) = 0$$

$$A\alpha_p(b) + B\alpha_{-p}(b) + p_1 b \exp(kb/2) = 0.$$

Solving the above equations for A and B and inserting their values in equations (16) and (17), one obtains the complete solution for radial displacement and stresses as

$$\bar{u} = \frac{\exp(kr/2)}{Mkr}\left[\{p_1 b\alpha_{-p}(a)\exp(kb/2) - p_0 a\alpha_{-p}(b)\exp(ka/2)\} M_{k^*,p}(kr)\right.$$
$$\left. + \{p_0 a\alpha_p(b)\exp(ka/2) - p_1 b\alpha_p(a)\exp(kb/2)\} M_{k^*,-p}(kr)\right],$$

$$\bar{\sigma}_r = \frac{\exp(-kr/2)}{Mr}\left[\{p_1 b\alpha_{-p}(a)\exp(kb/2) - p_0 a\alpha_{-p}(b)\right.$$
$$\times \exp(ka/2)\}\alpha_p(r) + \{p_0 a\alpha_p(b)\exp(ka/2) - p_1 b\alpha_p(a)$$
$$\times \exp(kb/2)\}\alpha_{-p}(r),$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = \frac{\exp(-kr/2)}{Mr}\left[\{p_1 b\alpha_{-p}(a)\exp(kb/2) - p_0 a\alpha_{-p}(b)\right.$$
$$\times \exp(ka/2)\}\beta_p(r) + \{p_0 a\alpha_p(b)\exp(ka/2) - p_1 b\alpha_p(a)$$
$$\left.\times \exp(kb/2)\}\beta_{-p}(r)\right],$$

where

$$M = a_p(a)\, a_{-p}(b) - a_{-p}(a)\, a_p(b). \tag{20}$$

4.1 *Dilation of the Cavity*

The volume of the cavity of the shell changes under the action of internal and external pressures; the relative change of this volume V_0 say is found to be

$$v = \frac{\nabla V_0}{V_0} = \left(\frac{3\bar{u}}{r}\right)_{r=a} = \frac{3 \exp(ka/2)}{Mka^2} [\{p_1 b a_{-p}(a) \exp(kb/2) - p_0 a a_{-p}(b) \exp(ka/2)\} M_{k^*, p}(ka) + \{p_0 a a_p(b) \exp(ka/2) - p_1 b a_{-p}(a) \exp(kb/2)\} M_{k^*, -p}(ka).] \tag{21}$$

4.2 *Stress Concentration in the Neighbourhood of the Cavity*

It is of interest to find the stress distribution in the vicinity of a spherical cavity. Generally the stress reaches its maximum value in the area which passes through the radius vector near the inner surface. Therefore we have on the surface of the cavity $r = a$

$$[\bar{\sigma}_\theta]_{\max} = [\bar{\sigma}_\phi]_{\max} = \frac{\exp(-ka/2)}{Ma} [\{p_1 b a_{-p}(a) \exp(kb/2) - p_0 a a_{-p}(b) \exp(ka/2)\} \beta_p(a) + \{p_0 a a_p(b) \exp(ka/2) - p_1 b a_p(a) \exp(kb/2)\} \beta_{-p}(a)]. \tag{22}$$

4.3 *Stresses in a Compressed Shell Due to Internal Pressure*

If the spherical shell undergoes compression due to the action of internal pressure only, the external surface being stress-free, the stresses of such a compressed shell are obtained from the equations (20) to be

$$\bar{\sigma}_r = \frac{p_0 a \exp[k(a-r)/2]}{Mr} [a_p(b)\, a_{-p}(r) - a_{-p}(b)\, a_p(r)],$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = \frac{p_0 a \exp[k(a-r)/2]}{Mr} [a_p(b)\, \beta_{-p}(r) - a_{-p}(b)\, \beta_p(r)]. \tag{23}$$

4.4 *St. Venant's Solution*

A spherical shell ($a \leq r \leq b$), made of homogeneous spherically anisotropic material, is considered under the same boundary conditions (19).

Stresses of such a shell may be found from the second and third equations of (20) on making $k \to 0$ and they are

$$\sigma_r = \underset{k\to 0}{\mathrm{lt}}\,(\bar{\sigma}_r) = \frac{r^{p-3/2}}{\left(\frac{a}{b}\right)^p - \left(\frac{b}{a}\right)^p}\left[\left(\frac{p_1 b^{3/2}}{a^p} - \frac{p_0 a^{3/2}}{b^p}\right) + \frac{1}{r^{2p}}\left(p_0 a^{3/2} b^p - p_1 b^{3/2} a^p\right)\right],$$

Similarly,

$$\sigma_\theta = \sigma_\phi = \left[\frac{r^{p-3/2}}{\left(\frac{a}{b}\right)^p - \left(\frac{b}{a}\right)^p}\right]\left[\left(\frac{p_1 b^{3/2}}{a^p} - \frac{p_0 a^{3/2}}{b^p}\right) \times \frac{2(\lambda_{22}+\lambda_{23}) + (2p-1)\lambda_{12}}{4\lambda_{12} + (2p-1)\lambda_{11}} + \frac{1}{r^{2p}}\left(p_0 a^{3/2} b^p - p_1 b^{3/2} a^p\right) \times \frac{2(\lambda_{22}+\lambda_{23}) - (2p+1)\lambda_{12}}{4\lambda_{12} - (2p+1)\lambda_{11}}\right] \tag{24}$$

respectively.

It is to be noted that the following limits are used to compute the stresses in (24)

$$\underset{k\to 0}{\mathrm{Lt}}\,\frac{M^{*}_{k,\pm p}(k\xi)}{M^{*}_{k,\pm p}(kr)} = \left(\frac{\xi}{r}\right)^{\frac{1}{2}\pm p}, \quad \underset{k\to 0}{\mathrm{Lt}}\,\frac{k\xi M'^{*}_{k,\pm p}(k\xi)}{M^{*}_{k,\pm p}(kr)} = \left(\frac{\xi}{r}\right)^{\frac{1}{2}\pm p}, \ (\tfrac{1}{2} \pm p) \tag{24 a}$$

Same sign of p is to be retained for the above limits.

The expressions of stresses in equations (24) for the above shell problem are the same found by St. Venant and given in Lekhnitskii [2].

4.5 *Isotropic Body and Love's Results*

The elastic property of a spherically anisotropic material is describes by five elastic parameters and they reduce to two independent parameter for isotropic material. It is treated as a special case of spherical anisotropy. For thepresent nonhomogeneous problem the relations are

$$C_{11} = C_{22} = (\lambda_0 + 2\mu_0)\exp(-kr), \quad C_{12} = C_{23} = \lambda_0 \exp(-kr)$$
$$C_{44} = \mu_0 \exp(-kr) \tag{25}$$

where λ_0 and μ_0 are Lamè constants. Following equation (1) and equation (25) the above relations in terms of λ_{ij} are

$$\lambda_{11} = \lambda_{22} = \lambda_0 + 2\mu_0, \quad \lambda_{12} = \lambda_{23} = \lambda_0, \quad \lambda_{44} = \mu_0. \tag{26}$$

The results obtained in equations (20)–(23) for spherically anisotropic nonhomogeneous bodies can also be used as the results for isotropic nonhomogeneous bodies if we replace there λ_{ij} by λ_0 and μ_0 as given in equations (26). We further note that the application of relations (26) in equation (12) follows

$$2p = 3. \tag{27}$$

As a test case we make use of equations (26) and (27) in the last two equations of (20) and take the limit as $k \to 0$ (for homogeneous material) and arrive at the following results:

$$\sigma_r = \frac{p_0 a^3 - p_1 b^3}{b^3 - a^3} + \frac{1}{r^3} \frac{(p_1 - p_0)\, a^3 b^3}{b^3 - a^3},$$

$$\sigma_\theta = \sigma_\phi = \frac{p_0 a^3 - p_1 b^3}{b^3 - a^3} - \frac{1}{2} \cdot \frac{1}{r^3} \frac{(p_1 - p_0)\, a^3 b^3}{b^3 - a^3}. \tag{28}$$

These are the stresses in a spherical shell ($a \leq r \leq b$) of homogeneous isotropic elastic material, subjected to internal and external pressures on the boundaries as in equations (19). These expressions were calculated by Lamè and are presented in Love [1].

4.6 *Solid Spherical Body*

A solid spherical body ($0 \leq r \leq b$) of nonhomogeneous spherically aeolotropic material undergoes compression by an uniformly distributed external pressure p_1. The stresses of such a sphere are obtained from the last two equations of (20) when the cavity of radius 'a' diminishes to zero, *i.e.*, as $a \to 0$:

$$\bar{\sigma}_r = -p_1 (b/r)^2 \exp\,[k(b - r)/2] \times \frac{\{(kr - 2)\,\lambda_{11} + 4\lambda_{12}\}\, M^*_{k,p}(kr) + 2kr\,\lambda_{11}\, M'^*_{k,p}(kr)}{\{(kb - 2)\,\lambda_{11} + 4\lambda_{12}\}\, M^*_{k,p}(kb) + 2kb\,\lambda_{11} M'^*_{k,p}(kb)},$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = -p_1 (b/r)^2 \exp\,[k(b - r)/2] \times \frac{\{(kr - 2)\,\lambda_{12} + 2(\lambda_{22} + \lambda_{23})\}\, M^*_{k,p}(kr) + 2kr\,\lambda_{12}\, M'^*_{k,p}(kr)}{\{(kb - 2)\,\lambda_{11} + 4\lambda_{12}\}\, M^*_{k,p}(kb) + 2kb\lambda_{11}\, M'^*_{k,p}(kb)}, \tag{29}$$

since

$$\underset{z\to 0}{\text{Lt}} \frac{M^{*}_{k,p}(z)}{M^{*}_{k,-p}(z)} = 0, \quad \underset{z\to 0}{\text{Lt}} \frac{zM'^{*}_{k,p}(z)}{M^{*}_{k,-p}(z)} = 0$$

and

$$\underset{z\to 0}{\text{Lt}} \frac{zM'^{*}_{k,\pm p}(z)}{M^{*}_{k,\pm p}(z)} = \tfrac{1}{2} \pm p. \tag{29 a}$$

5. The Problem of a Composite Sphere

We now consider a homogeneous solid sphere $(0 \leq r \leq a)$ of spherically anisotropic matrial surrounded by a nonhomogeneous concentric spherical shell $(a \leq r \leq b)$ of spherically anisotropic medium and the whole body is acted upon by a uniform radial pressure on the external bounding surface $r = b$. At the surface of separation $r = a$ the materials are sufficiently rough to ensure the continuity of radial stresses and displacements. The relevant boundary conditions are then

$$\bar{\sigma}_r = -p_1 \text{ on the surface } r = b$$

and

$$u = \bar{u}, \quad \sigma_r = \bar{\sigma}_r \text{ on the surface } r = a \tag{30}$$

In the case of a homogeneous solid sphere $(0 \leq r \leq a)$ $C_{ij} = \lambda_{ij}$ for $k = 0$ in equation (1) and the stress equation of equilibrium corresponding to the equation (6) turns out to be

$$\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) - \frac{2\,(C_{22} + C_{23} - C_{12})}{C_{11}}\, u = 0.$$

The general solution of the above equation may be put as

$$u = \mathrm{C}r^{m-1/2} + \mathrm{D}r^{m-1/2}$$

where

$$m = \{\tfrac{1}{4} + 2\,(C_{22} + C_{23} - C_{12})/C_{11}\}^{1/2} \tag{31}$$

To ensure the finiteness of the stress at every point of the solid sphere, including the neighbourhood of the origin, we are to take the displacement and stresses as (supposing $m > 3/2$)

$$\begin{aligned} u &= Cr^{(m-1)/2} \\ \sigma_r &= Ca_m r^{(m-3)/2} \\ \sigma_\theta &= \sigma_\phi = Cb_m r^{(m-3)/2} \end{aligned} \tag{32}$$

where

$$a_m = 2C_{12} + C_{11}(m - 1/2), \quad b_m = (C_{22} + C_{23}) + C_{12}(m - 1/2). \tag{33}$$

And for the spherical shell ($a \leq r \leq b$) the displacements and stresses are given in equations (16) and (17). Boundary conditions (30) are applied to equations (16), (17) and (32) to get

$$A\alpha_p(b) + B\alpha_{-p}(b) + p_1 b \exp(kb/2) = 0$$

$$\frac{\exp(ka/2)}{ka}[AM^*_{k,p}(ka) + BM^*_{k,-p}(ka)] = Ca^{(m-1/2)}$$

$$\frac{\exp(-ka/2)}{a}[A\alpha_p(a) + B\alpha_{-p}(a)] = Ca_m a^{(m-3/2)}. \tag{34}$$

Solving the above equations for A, B and C and inserting their values in equations (32) and in equations (16), (17) we get the following sets of results:

Displacement and stresses in the sphere ($0 \leq r < a$)

$$u = \frac{p_1 b \exp[k(a+b)/2]}{Nka^{(m+1/2)}}[\alpha_p(a) M^*_{k,-p}(ka) - \alpha_{-p}(a) M^*_{k,p}(ka)] \times r^{(m-1/2)},$$

$$\sigma_r = \frac{p_1 b \exp[k(a+b)/2]}{Nka^{m+1/2}}[\alpha_p(a) M^*_{k,-p}(ka) - \alpha_{-p}(a) M^*_{k,p}(ka)] \times a_m r^{m-3/2},$$

$$\sigma_\theta = \sigma_\phi = \frac{p_1 b \exp[k(a+b)/2]}{Nka^{m+1/2}}[\alpha_p(a) M^*_{k,-p}(ka) - \alpha_{-p}(a) M^*_{k,p}(ka)] \times b_m r^{m-3/2}, \tag{35}$$

Displacement and stresses in the spherical shell ($a \leq r \leq b$)

$$\bar{u} = \frac{p_1 b \exp[k(b+r)/2]}{Nkr}\left[\left\{\frac{a_m \exp(ka)}{ka} M^*_{k,-p}(ka) - \alpha_{-p}(a)\right\} \times M^*_{k,p}(kr) - \left\{\frac{a_m \exp(ka)}{ka} M^*_{k,p}(ka) - \alpha_p(a)\right\} M^*_{k,-p}(kr)\right]$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = \frac{p_1 b \exp[k(b-r)/2]}{Nr}\left[\left\{\frac{a_m \exp(ka)}{ka} M^*_{k,-p}(ka) - \alpha_{-p}(a)\right\} \times \beta_p(r) - \left\{\frac{a_m \exp(ka)}{ka} M^*_{k,p}(ka) - \alpha_p(a)\right\} \beta_{-p}(r)\right],$$

$$\bar{\sigma}_r = \frac{p_1 b \exp[k(b-r)/2]}{Nr}\left[\left\{\frac{a_m \exp(ka)}{ka} M^*_{k,p}(ka) - a_{-p}(a)\right\} a_p(r) - \left\{\frac{a_m \exp(ka)}{ka} M^*_{k,p}(ka) - a_p(a)\right\} a_{-p}(r)\right],$$

where

$$N = a_{-p}(b)\left\{\frac{a_m \exp(ka)}{ka} M^*_{k,p}(ka) - a_p(a)\right\} - a_p(b)\left\{\frac{a_m \exp(ka)}{ka} M^*_{k,-p}(ka) - a_{-p}(a)\right\}. \tag{36}$$

6. Numerical Results

All the results are for structures with finite outer radius b that is twice the inner radius a. It should be noted that the results previously derived are quite general. The problem investigated involve inhomogeneous materials with properties varying exponemtially with the radius according to the equation (1).

We choose the elastic constants $\lambda_{11} = 26\cdot92$, $\lambda_{12} = 13\cdot46$, $\lambda_{22} = 8\cdot47$, $\lambda_{23} = 3\cdot12$, $\lambda_{44} = 6\cdot53$ in terms of a unit 10^{11} dynes per square centimeter and $k = 2/a$ (numerically) for Material I. The present analysis may be useful in studying the stresses for layered media having exponentially increasing or decreasing stiffness. We make use of the values of λ_{ij} in equations (12) and (13) to obtain $p = 1/3$ and $\mathring{k} = 0$ and from equations (18) we get.

r/a	$a_p(r)/\Gamma(4/3)$	$a_{-p}(r)/\Gamma(2/3)$	$\beta_p(r)/\Gamma(4/3)$	$\beta_{-p}(r)/\Gamma(2/3)$
1	54·03	24·89	29·96	12·87
2	161·40	64·33	89·03	35·30

For the first problem the internal surface of the shell structure is always under a uniform normal pressure p_0, whereas its surface is supposed to be stress-free for system I and is subjected to a pressure which is half of the internal pressure for the system II. Our main interest lies in computing the stress concentration near the vicinity of the caivity. The above obtained values for Material I are applied to the last equation of (20) and it yields $M = -540\cdot9\ \Gamma(2/3)\ \Gamma(4/3)$. Ultimately the equation (22) shows $[\bar{\sigma}_\theta]_{max.} = (-.2776\,p_0)$ and $(-.5297\,p_0)$ for the leading systems I and II respectively. Also the third equation of (20) leads to $[\bar{\sigma}_\theta] = (\cdot 0105\,p_0)$ and $(-.2748\,p_0)$ for the loading systems I and II respectively.

In the second problem with inclusion we choose $C_{11} = 6 \cdot 17$, $C_{12} = 2 \cdot 17$, $C_{22} = 5 \cdot 97$, $C_{23} = 2 \cdot 62$ $C_{44} = 1 \cdot 64$ (with the same unit mentioned previously) for Material II for the homogeneous portion $0 \leq r \leq a$. Using these values of C_{ij} in equations (31) and (33) the values of m, a_m, b_m are found to be $m = 1 \cdot 527$, $a = 10 \cdot 67$, $b_m = 10 \cdot 82$. The other portion $a \leq r \leq b$ is filled up with Material I. We calculate the value of the constant N of equation (36) for Material I and find $N = -2054\,\Gamma(2/3)\,\Gamma(4/3)$. Stresses of equations (35) and (36) may now be had from the table below:

TABLE I

	Nonhomogeneous		Homogeneous	
r/a	$-\bar{\sigma}_r/p_1$	$-\bar{\sigma}_\theta/p_1$	$-\sigma_r/p_1$	$-\sigma_\theta/p_1$
1·0	1·635	·960	1·635	1·657
1·4	1·435	·793		
1·6	1·252	·693		
1·8	1·113	·615		
2·0	1	·554		

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