GRAVITATIONAL INSTABILITY OF AN INFINITELY EXTENDING LAYER OF FINITE THICKNESS SURROUNDED BY NON-CONDUCTING MATERIAL IN THE PRESENCE OF MAGNETIC FIELD AND ROTATION

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ABSTRACT

The present note deals with the study of gravitational instability of an infinite layer of finite thickness surrounded by a non-conducting material in the presence of magnetic field directed parallel to the interfaces and rotating uniformly about an axis perpendicular to the interfaces. The perturbations considered here are more general than studied by Ognesyan in the case of non-rotating layer and by Chakraborty in the case of rotating layer, as in these two investigations the perturbations are taken symmetrical to the mid-plane of the conducting layer. In the absence of the surrounding non-conducting material the system is always overstable. In the presence of the surrounding material the system remains overstable in the presence of the magnetic field if the surrounding material is lighter than the conducting material but when the surrounding material is heavier than the conducting material there exists a critical wave number k^* for a given l, the wave number transverse to the magnetic field and the axis of rotation, such that when $k < k^*$ the system is unstable. The rotating system in the absence of magnetic field is always unstable irrespective of the relative magnitudes of the densities of the conducting and non-conducting materials.

INTRODUCTION

In the present note we shall study the gravitational instability of a rotating fluid layer of infinite extension but of finite thickness, and with constant density in the presence of an external uniform magnetic field. This ideally conducting fluid with density ρ is surrounded on both sides by non-conducting fluid with density ρ_0. Recently Ognesyan[1,2] has studied the gravitational instability of fluid layer of finite thickness in the presence of a magnetic field and Chakraborty[3] has extended this investigation to include the Coriolis forces arising due to the uniform rotation of the fluid layer. In both these investigations, the perturbations are taken to be symmetrical about the middle-plane of the layer and they have confined their discussion to the case when the surrounding non-conducting medium is absent. In the present note we have studied the problem of general perturbations and have taken account of the non-conducting material surrounding the conducting layer. We note that the system is overstable for all wave-lengths when $\rho_0 = 0$, even in the presence of rotation and magnetic field. Thus in contra-distinction to the case of symmetrical perturbations, there is no critical wavelength separating the domains of stability and instability in this case.

In the presence of surrounding non-conducting material, our conclusions are as follows: (i) in the absence of rotation and magnetic field the system is overstable or unstable according as $\rho_0 <$ or $> \rho$, (ii) in the presence of magnetic field, when rotation is absent, the system is overstable or unstable according as $\rho_0 <$ or $> \rho$ and (iii) in the presence of rotation, when the magnetic field is absent, the system is unstable.

For the sake of comparison, we have discussed the instability of the layer under symmetrical perturbations in the presence of surroundiug non-conducting material in 5.

2. LINEARIZED EQUATIONS AND THEIR SOLUTION

We consider a homageneous distribution of gravitating ideally conducting fluid mass with constant density ρ in the form of a plane layer of thickness $2h$. The xoy plane is taken to coincide with the unperturbed middle-level of the layer and positive z-axis in the upward direction normal to the unperturbed fluid surfaces. This layer is surrounded by a non-conducting uniform matetial of density ρ_0.

In the equilibrium state the conducting fluid is taken immersed in an external uniform magnetic field $\mathbf{H}_0$ directed in the x-direction. The entire material is assumed to be rotating uniformly abount the z-axis with angular velocity Ω

The linearized hydro-magnetic equations for the conducting layer determining the perturbations are:

Equation of continuity:

$$\operatorname{div} \mathbf{v} = 0, \tag{2.1}$$

Equations of momentum:

$$\rho\left[(\partial \mathbf{v}/\partial t) + 2\vec{\Omega} \times \mathbf{v}\right] = -\nabla p + \mu\,(\operatorname{curl} \mathbf{h}) \times \mathbf{H}_0 - \rho \nabla U, \tag{2.2}$$

Maxwell's Equations :

$$\operatorname{div} \mathbf{h} = 0, \tag{2.3}$$

$$\partial \mathbf{h}/\partial t = \operatorname{curl}(\mathbf{v} \times \mathbf{H}_0), \tag{2.4}$$

and

Poisson's Equation :

$$\nabla^2 U = 0, \tag{2.5}$$

where p, v, h, U denote the perturbations in pressure, velocity, magnetic field and gravitational potential.

Using the same notation as for the conducting medium the corresponding linearized equations for the non-conducting fluid are :

Equation of continuity :

$$\operatorname{div} \mathbf{v} = 0, \tag{2.6}$$

Equations of momentum :

$$\rho_0 [(\partial \mathbf{v}/\partial t) + 2\vec{\Omega} \times \mathbf{v}] = -\nabla p - \rho_0 \nabla U, \tag{2.7}$$

Maxwell's Equations :

$$\operatorname{div} \mathbf{h} = 0, \tag{2.8}$$

$$\operatorname{curl} \mathbf{h} = 0, \tag{2.9}$$

Poisson's Equation :

$$\nabla^2 U = 0. \tag{2.10}$$

In order to study the stability of the system, we assume that all the perturbations vary as $\exp[i(\sigma t + kx + ly)]$.

From [2.3] and [2.4] we have

$$\mathbf{h} = (H_0 k/\sigma)\,\mathbf{v}. \tag{2.11}$$

By taking the curl of the momentum equation [2.2] and substituting the value of h from [2.11], we have

$$\left.\begin{aligned} \frac{d^2 v_x}{dz^2} &= m^2 v_x \\ v_y &= v_x \left(\frac{Al - 2\Omega k}{2\Omega l + Ak}\right) \\ \text{and} \quad v_z &= \frac{-iA(k^2 + l^2)}{m^2(2\Omega l + Ak)} \cdot \frac{dv_x}{dz}, \end{aligned}\right\} \tag{2.12}$$

where
$$m^2 = \frac{A^2(k^2+l^2)}{A^2+4\Omega^2}$$
and
$$A = i\left(\sigma - \frac{\mu k^2 H_0^2}{\rho\sigma}\right). \qquad [2.13]$$

From [2.12], we have

$$v_x = (C_1 e^{mz} + C_2 e^{-mz}),$$
$$v_y = \left(\frac{Al - 2\Omega k}{2\Omega l + Ak}\right)(C_1 e^{mz} + C_2 e^{-mz}),$$
$$v_z = \frac{-iA(k^2+l^2)}{m(2\Omega l + Ak)}(C_1 e^{mz} - C_2 e^{-mz}), \qquad [2.14]$$

where C_1 and C_2 are arbitrary constants.

From [2.5]

$$U = g_1 \exp[k^2+l^2)^{1/2} z] + g_2 \exp[-(k^2+l^2)^{1/2} z], \qquad [2.15]$$

where g_1 and g_2 are arbitrary constants.

Then from [2.2] we get the amplitude for the pressure as

$$p = \frac{i\rho}{k}\left[i\sigma - 2\Omega\frac{(Al - 2\Omega k)}{(2\Omega l + Ak)}\right] v_x + \rho U. \qquad [2.16]$$

For the non-conducting material, from [2.7], we get

$$v_x = C' e^{m'z} + C'' e^{-m'z}$$
$$v_y = \frac{(i\sigma l - 2\Omega k)}{(i\sigma k + 2\Omega l)}(C' e^{m'z} + C'' e^{-m'z})$$
and
$$v_z = \frac{\sigma(k^2+l^2)}{m'(i\sigma k + 2\Omega l)}(C' e^{m'z} - C'' e^{-m'z}), \qquad [2.17]$$

where
$$m'^2 = \frac{\sigma^2(k^2+l^2)}{(\sigma^2 - 4\Omega^2)} \qquad [2.18]$$

and C', C'' are arbitrary constants.

From [2.8] and [2.9] we have

$$\mathbf{h} = \text{grad}\, \phi\,,$$

where
$$\phi = L \exp[(k^2 + l^2)^{1/2} z] + L' \exp[-(k^2 + l^2)^{1/2} z] \qquad [2.19]$$

and from Poisson's equation [2.10]

$$U = g' \exp[(k^2 + l^2)^{1/2} z] + g'' \exp[-(k^2 + l^2)^{1/2} z]\,, \qquad [2.20]$$

where L, L', g', g'' are arbitrary constants.

The perturbation in pressure is given by

$$p = (i\,\rho_0/k)\,[i\,\sigma\, v_x - 2\,\Omega\, v_y] - \rho_0\, U. \qquad [2.21]$$

3. Boundary Conditions and Dispersion Relation

The perturbed interfaces between the conducting and the non-conducting fluids are given by

$$z_1 = h + (\delta z)_1 \exp[i(\sigma t + kx + ly)]$$

and
$$z_2 = -h + (\delta z)_2 \exp[i(\sigma t + kx + ly)]\,,$$

where $(\delta z)_{1,2}$ are the amplitudes of the displacements at the interfaces. We note that this type of perturbation is more natural than the one considered in references [1, 2].

The perturbation $\hat{n}_1$ in the unit normal n_0 to the boundary is given by

$$\hat{n}_1 = (\delta z\, ik,\ \delta z\, il,\ 0) \exp[i(\sigma t + kx + ly)].$$

The gravitational potential satisfies the following boundary conditions:

At the interfaces

(i) gravitational potential is continuous, *i.e.*

$$[\mathrm{V}] = 0 \qquad [3.1]$$

and

(ii) the normal component of the gradient of the gravitational potential is continuons, *i.e.*

$$\hat{n}[\nabla V] = 0. \qquad [3.2]$$

The boundary conditions satisfied by the velocity and electromagnetic fields, in reference [4] are:

$$u = \hat{n}.\mathbf{v}, \quad \hat{n}.[\mathbf{B}] = 0, \qquad [3.3], [3.4]$$

$$\hat{n} \times [\mathbf{B}] = \mu_0 \mathbf{j}^*, \quad \mathbf{j}^* \times \bar{\mathbf{B}} - \hat{n}[p] = 0. \qquad [3.5], [3.6]$$

In the above, $\hat{n}$ is the unit normal vector to the surface directed in the conducting fluid, p and $\mathbf{v}$ are the pressure and velocity at an interface, the square brackets denote the jump in the enclosed quantity upon crossing the interface from the non-conducting to the conducting fluid and $\bar{\mathbf{B}}$ denotes the airthmetic mean of the magnetic inductions on the two sides of an interface.

The boundary conditions satisfied by the perturbations are obtained from [3.1] – [3.6] by linearizing these equations. The perturbed boundary conditions have been satisfied at the unperturbed interfaces, $z = \pm h$, after taking into account the contribution of the perturbations of these interfaces to the perturbations evaluated in 2.

The condition [3.4] gives the perturbation in the magnetic field in the outside medium to be zero. Solving for $\mathbf{j}^*$ from [3.5] and substituting in [3.6], we find that the x and y components of [3.6] are identically satisfied.

Finally [3.1], [3.2], [3.3] and the z-component of [3.6] give four boundary conditions to be satisfied at the upper and the lower interfaces Thus we get eight homogenous equation for the eight constants g', g'', C', C'', C_1, C_2, g_1, g_2 and on eliminating these constants we get the dispersion relation:

$$\left[\frac{4\pi G(\rho-\rho_0)^2}{\sigma(k^2+l^2)^{1/2}} \cdot \frac{1}{[1+\cot h(k^2+l^2)^{1/2}h]} + \frac{\rho_0(\sigma^2-4\Omega^2)^{1/2}}{(k^2+l^2)^{1/2}} + \frac{4\pi G\rho h(\rho_0-\rho)}{\sigma}\right]$$

$$\times \frac{(A^2+4\Omega^2)^{1/2}(k^2+l^2)^{1/2}}{(2\Omega l+Ak)}$$

$$+\left[\frac{\rho}{k}\left\{\sigma+\frac{2\Omega i(Al-2\Omega k)}{(2\Omega l+Ak)}\right\} - \frac{k\mu H_0^2}{\sigma}\right]$$

$$\times \tan h\left\{\frac{A(k^2+l^2)^{1/2}h}{(A^2+4\Omega^2)^{1/2}}\right\} = 0. \qquad [3.7]$$

After evaluating the constants in terms of $v_z(0)$, the amplitude of the

z – component of the velocity at the mid-plane of the layer, we find the following expressions for the amplitudes of the perturbations:

Conducting medium	Non-conducting medium
$v_x = v_z(0)\dfrac{im(2\Omega l + Ak)}{A(k^2+l^2)} \times \sinh mz$	$v_x = 2v_z(0)\cosh mh\, e^{m'h} \times \dfrac{m'(i\sigma k + 2\Omega l)}{\sigma(k^2+l^2)} \sinh m'z$
$v_y = v_z(0)\dfrac{im(Al - 2\Omega k)}{A(k^2+l^2)} \times \sinh mz$	$v_y = 2v_z(0)\cosh mh\, e^{m'h} \times \dfrac{m'(i\sigma l - 2\Omega k)}{\sigma(k^2+l^2)} \sinh m'z$
$v_z = v_z(0)\cosh mz$	$v_z = 2v_z(0)\cosh mh\, e^{m'h} \times \cosh m'z$
$U = v_z(0)\dfrac{i2\pi G(\rho-\rho_0)}{\sigma(k^2+l^2)^{1/2}}e^{-\alpha} [\cosh mh \exp\{(k^2+l^2)^{1/2}z\} - \sinh mh \exp\{-(k^2+l^2)^{1/2}z\}]$	$V = v_z(0)\dfrac{i2\pi G(\rho-\rho_0)}{\sigma(k^2+l^2)^{1/2}}e^{-\alpha} \times [(\cosh mh - e^{2\alpha}\sinh mh)\exp\{(k^2 \times l^2)^{1/2}z\} + (e^{2\alpha}\cosh mh - \sinh mh) \times \exp\{-(k^2+l^2)^{1/2}z\}]$
$p = -v_z(0)\dfrac{m(2\Omega l + Ak)}{A(k^2+l^2)}\cdot\dfrac{\rho}{k} \times \left[i\sigma - \dfrac{2\Omega(Al - 2\Omega k)}{(2\Omega l + Ak)}\right] - \rho U$	$p = -2v_z(0)\dfrac{m'(i\sigma k + 2\Omega l)}{i\sigma(k^2+l^2)}\cdot\dfrac{\rho_0}{k} \times \cosh mh\, e^{m'h}\left[i\sigma - \dfrac{2\Omega(i\sigma l - 2\Omega k)}{(i\sigma k + 2\Omega l)}\right] \times \sinh m'z - \rho_0 V$
$\mathbf{h} = \dfrac{H_0 k}{\sigma}\mathbf{v}$.	$\mathbf{h} = 0.$

4. Discussion of the Dispersion Relation

We shall discuss the following special cases:

Case I: $\Omega = 0, \; \rho_0 = 0, \; H_0 = 0.$

In this case the dispersion relation [3.7] reduces to

$$4\pi G\rho\left[1-\frac{1}{\alpha(1+\coth\alpha)}\right]\alpha\coth\alpha=\sigma^2, \qquad [4.1]$$

where

$$\alpha=h(k^2+l^2)^{1/2}.$$

The function $\alpha[1-\{1/\alpha(1+\coth\alpha)\}]$ monotonically increases from 0 to ∞ as α increases from 0 to ∞. Hence the frequency σ is real for all values of the wave number k and l. Thus in the absence of rotation, magnetic field and surrounding matter, the system is overstable for all wave lengths. In the case of symmetric perturbations studied by Ognesyan[1], there exists a critical wave number below which the system is unstable.

Case II: $\rho_0=0,\ \Omega=0,\ H_0\neq 0.$

Here the dispersion relation reduces to

$$4\pi G\rho\alpha\coth\alpha\left[1-\frac{1}{\alpha(1+\coth\alpha)}\right]+\frac{k^2\mu H_0^2}{\rho}=\sigma^2. \qquad [4.2]$$

We note that the presence of the magnetic field introduces anisotropy and the wave number in the direction of the external magnetic field appears separately in addition to the combination $(k^2+l^2)^{1/2}$. In this case also σ is real for all wave numbers k and l, so that the system is again overstable for all the wave numbers as in case I. Hence the magnetic field does not affect the stability of the system and unlike the case discussed in reference [2] there is no critical wave number for the system to be stable or unstable.

Case III: $\rho_0=0,\ \Omega\neq 0,\ H_0=0.$

Here the dispersion relation reduces to:

$$F(\alpha)=\frac{X}{1+X^2}\tan\alpha X, \qquad [4.3]$$

where

$$F(\alpha)=\frac{\pi G\rho\alpha}{\Omega^2}\left[1-\frac{1}{\alpha(1+\coth\alpha)}\right]$$

and

$$X=\frac{\sigma}{(4\Omega^2-\sigma^2)^{1/2}}$$

so that

$$\sigma^2=4\Omega^2\frac{X^2}{1+X^2}. \qquad [4.4]$$

$F(\alpha)$ is a monotronically increasing, positive function for all positive values of α.

We shall now determine the total number of roots of the dispersion relation in a circle $|X| = R_n$, in the complex X-plane where R_n lies between the consecutive zeros of $\sin \alpha X$ and $\cos \alpha X$, say when $(n\pi/\alpha) < R_n < (n+\frac{1}{2})(\pi/\alpha)$.

Let $$f(X) = F(\alpha) \cos \alpha X (1 + X^2)$$

and $$g(X) = -X \sin \alpha X.$$

For large R_n, $|\cos \alpha X|$ and $|\sin \alpha X|$ are of the same order of magnitude.

Hence $$\operatorname*{Lt}_{R_n \to \infty} \frac{|f(X)|}{|g(X)|} \to \infty$$

and by Rouche's Theorem, the dispersion relation [4.3] has the same number $(2n+2)$ of roots within the circle $|X| = R_n$ as the equation

$$F(\alpha)(1 + X^2) \cos \alpha X = 0. \qquad [4.5]$$

Writing the dispersion relation as

$$F(\alpha) \cot \alpha X = \frac{X}{1 + X^2}, \qquad [4.6]$$

we draw the graphs of the right-hand side and the left-hand side of [4.6] for real positive values of X. $[X/(1+X^2)]$ starts from origin, has the maximum at $X = 1$, and tends to zero as $X \to \infty$. $F(\alpha) \cot \alpha X$ will have n vertical asymptotes corresponding to the roots of $\sin \alpha X = 0$ within the circle $|X| = R_n$. Hence there are n number of intersections between the curves representing the two sides of [4 6], thus giving n roots of this equation. Similarly, for X real and negative we get n roots. Thus equation [4.6] admits $2n$ real roots within the circle $|X| = R_n$.

To determine the number of imaginary roots of [4.6], we set $X = iY$, where Y is real, so that it reduces to

$$F(\alpha)(Y^2 - 1) = Y \tanh \alpha Y. \qquad [4.7]$$

Again drawing the graphs of the right-hand side and the left-hand side of [4.7], we note that the graph of left-hand side is a parabola with vertex at $[0, -F(\alpha)]$ and latus-rectum $1/F(\alpha)$ cutting the Y-axis at $(\pm 1, 0)$, $Y \tanh \alpha Y$ is positive and even function of Y for all real values of Y. It is zero at the origin and tends to $+\infty$ asymptotically as $|Y|$. These two curves intersect only at two points for which $Y^2 > 1$. Hence [4.7] has two real roots having modulii greater than unity. Thus out of the $(2n+2)$ roots of [4.3] within $|X| = R_n$, $2n$ are real and other two pure imaginary. From the relation [4.4], we find that for all of these $(2n+2)$ roots for X, σ^2 is always positive, showing thereby that for all wave numbers k and l, the system is overstable.

Case IV: $\rho_0 = 0,\ H_0 \neq 0,\ \Omega \neq 0.$

Here the dispersion relation reduces to :

$$F(\alpha)\left[F(\alpha)\cot\alpha X - \frac{X}{1+X^2}\right] = \frac{k^2 V^2}{\Omega^2}\cdot\frac{\tan\alpha X}{1+X^2}, \qquad [4.8]$$

where

$$V^2 = \frac{\mu H_0^2}{\rho}$$

and

$$X = \frac{(\sigma - k^2 V^2)}{(4\Omega^2\sigma^2 - \sigma^4 + 2\sigma^2 k^2 V^2 - k^4 V^4)^{1/2}}.$$

so that

$$\frac{\sigma^2}{2\Omega^2} = \frac{k^2 V^2}{2\Omega^2} + \frac{X^2}{1+X^2} \pm \left[\frac{X^4}{(1+X^2)^2} + \frac{k^2 V^2}{\Omega^2}\cdot\frac{X^2}{1+X^2}\right]^{1/2}. \qquad [4.9]$$

Let

$$f(X) = F^2(\alpha)\,(1+X^2)\cos^2\alpha X$$

and

$$g(X) = -\left[\frac{k^2 V^2}{4\Omega^2}\sin^2\alpha X + F(\alpha) X \sin\alpha X\cos\alpha X\right].$$

As in the previous case

$$\underset{R_n\to\infty}{\text{Lt.}} \frac{|F(X)|}{|g(X)|} \to \infty,$$

where

$$\frac{n\pi}{\alpha} < R_n < \left(n+\tfrac{1}{2}\right)\frac{\pi}{\alpha}.$$

Hence [4.8] will have the same number of roots within the circle $|X| = R_n$ as the equation

$$F^2(\alpha)\,(1+X^2)\cos^2\alpha X = 0.$$

This equation has $(4n+2)$ roots. Thus the dispersion relation admits $(4n+2)$ roots for X.

We draw the graphs of the right-hand side and the left-hand side of [4.8] against the positive values of X. $[F(\alpha)\cot\alpha X - X/(1+X^2)]$ has n vertical asymptotes corresponding to n roots of $\sin\alpha X$, and it vanishes at the points X, which are the roots of the dispersion relation in case III. The function $(k^2 V^2/4\Omega^2)\tan\alpha X/(1+X^2)$ has n vertical asymptotes corresponding to the roots of $\cos \alpha X$. Hence there are $2n$ number of intersections between the curves representing the two sides of [4.8] within $|X| = R_n$. Similarly for negative values of X these curves have $2n$ intersections giving $4n$ real roots for the equation [4.8].

Let $X = iY$, where Y is real, then [4.8] reduces to

$$F^2(\alpha)(Y^2 - 1) = \left[\frac{k^2 V^2}{4\Omega^2} \tanh^2 \alpha Y + F(\alpha) Y \tanh \alpha Y\right]. \qquad [4.10]$$

The function $F^2(\alpha)(Y^2 - 1)$ represents a parabola with vertex at $[0, -F^2(\alpha)]$ and latus-rectum $1/F^2(\alpha)$. It cuts the Y-axis at $(\pm 1, 0)$. The function on the right-hand side is even and positive and attains zero value at the origin and $\rightarrow \infty$ as $|Y| \rightarrow \infty$. Hence, for all α and k, there are two intersections between the curves representing the right-hand side and the left-hand side of [4.10], which give two real roots such that $Y^2 > 1$.

Thus the equation [4.8] has $4n$ real roots and two purely imaginary roots. Corresponding to these $(4n + 2)$ roots, σ^2 given by [4.9] is positive. Thus the system in the presence of both magnetic field and rotation is overstable.

Case V: $\rho_0 \neq 0$, $H_0 = 0$, $\Omega = 0$.

The dispersion relation reduces to

$$4\pi G\rho\alpha\left(1 - \frac{\rho_0}{\rho}\right)\left[1 - \frac{1 - \rho_0/\rho}{\alpha(1 + \coth\alpha)}\right] = \sigma^2\left(\frac{\rho_0}{\rho} + \tanh\alpha\right). \qquad [4.11]$$

When $\rho_0/\rho < 1$, σ^2 is positive for all positive values of α. Hence the system is overstable. When $\rho_0/\rho > 1$, $\sigma^2 < 0$ and the system is unstable.

Case VI: $\rho_0 \neq 0$, $H_0 \neq 0$, $\Omega = 0$.

The dispersion relation is given by

$$4\pi G\rho\alpha\left(1 - \frac{\rho_0}{\rho}\right)\left[1 - \frac{1 - \rho_0/\rho}{\alpha(1 + \coth\alpha)}\right] + k^2 \frac{\mu H_0^2}{\rho} \tanh\alpha$$

$$= \sigma^2\left(\frac{\rho_0}{\rho} + \tanh\alpha\right). \qquad [4.12]$$

Here again, when $\rho_0/\rho < 1$, σ^2 is always positive and hence the system is overstable even in the presence of magnetic field. When $\rho_0 > \rho$, the dispersion relation [4.12] reduces to

$$\sigma^2 = \frac{1}{(\rho_0/\rho \coth \alpha + 1)}\left[\frac{k^2 \mu H_0^2}{\rho} - 4\pi G\rho\left(\frac{\rho_0}{\rho} - 1\right)\cot h\alpha\left(\alpha + \frac{\rho_0/\rho - 1}{1 + \coth \alpha}\right)\right].$$

Here the positive and negative values of σ^2 *i.e.* regions of overstability and instability are separated by its zero value and the equation

$$\frac{k^2 \mu H_0^2}{\rho} - 4\pi G(\rho_0 - \rho)\coth \alpha\left[\alpha + \frac{\rho_0/\rho - 1}{1 + \coth \alpha}\right] \qquad [4.13]$$

determines the critical value k^* of the wave number k in the direction of the external magnetic field for a given set of values of ρ, ρ_0, H_0 and l, the wave number in direction perpendicular to the magnetic field and the axis of rotation.

Case VIII : $\rho_0 \neq 0$, $H_0 = 0$, $\Omega \neq 0$.

Here the dispersion relation reduces to

$$\psi(\alpha)(X^2 - 1) = X[\tanh \alpha X + \rho_0/\rho], \qquad [4.14]$$

where $$\psi(\alpha) = \frac{\pi G\rho}{\Omega^2}\left(1 - \frac{\rho_0}{\rho}\right)\alpha\left[1 - \frac{1 - \rho_0/\rho}{\alpha(1 + \coth \alpha)}\right],$$

and $$X = \frac{\sigma}{(\sigma^2 - 4\Omega^2)^{1/2}}$$

so that $$\sigma^2 = \frac{4\Omega^2 X^2}{X^2 - 1}. \qquad (4.15)$$

Let $$f(X) = \psi(\alpha)(X^2 - 1)\cosh \alpha X$$

$$g(X) = -X[(\rho_0/\rho)\cosh \alpha X + \sinh \alpha X]$$

so that $$\underset{R_n \to \infty}{\text{Lt.}} \frac{|f(X)|}{|g(X)|} \to \infty, \text{ where } \frac{n\pi}{\alpha} < R_n < \left(n + \tfrac{1}{2}\right)\frac{\pi}{\alpha}.$$

Hence by Rouche's Theorem, the dispersion relation [4.14] has the same number of roots within the circle $|X| = R_n$ as the equation

$$\psi(\alpha)(X^2 - 1)\cosh \alpha X = 0.$$

This equation has $(2n + 2)$ roots.

We draw the graphs for the two sides of [4.14] for all positive values of X. When $\rho_0 < \rho$, $\psi(\alpha)(X^2-1)$ represents a parabola with vertex at $[0, -\psi(\alpha)]$ and latus-rectum $1/\psi(\alpha)$, and it cuts the X-axis at $(\pm 1, 0)$.

Consider

$$y = X\rho_0/\rho + X\tanh\alpha X \equiv y_1 + y_2$$

The straight line $y_1 = X\rho_0/\rho$ passes through the origin and has slope $\tan^{-1}(\rho_0/\rho)$. $y_2 = X\tanh\alpha X$ is a positive even function of X attaining zero value at the origin and tending to ∞ as $|X| \to \infty$. The straight line $y = y_1$ cuts the parabola $y = \psi(\alpha)(X^2-1)$ at two points having the abscissæ

$$X_1, X_2 = \pm\frac{\psi(\alpha)}{\psi(\alpha) - \rho_0/\rho},$$

whatever be the relative magnitude of $\psi(\alpha)$ and ρ_0/ρ one of the abscissæ of the points of intersection is always positive and greater than one. The abscissa of the other point of intersection is negative and greater than -1.

At $X = -1$

$$y_1 + y_2 = -\rho_0/\rho + \tanh\alpha.$$

If $\tanh\alpha > \rho_0/\rho \equiv \tanh\alpha^*$ *i.e.*, if $\alpha > \alpha^*$, $y_1 + y_2 > 0$.

Therefore the curve $y = y_1 + y_2$ will cut the parabola, which passes through $X = -1$, at a point whose abscissa is < -1 *i.e.* $|X| > 1$. If $\alpha < \alpha^*$, $y_1 + y_2 < 0$ and the curve $y = y_1 + y_2$ cuts the parabola at a point whose abscissa is > -1, *i.e.* $|X| < 1$. Thus from [4.15], $\sigma^2 >$ or < 0 according as $\alpha >$ or $< \alpha^*$, giving us real or pure imaginary roots for σ.

Taking $X = iY$, where Y is real, the dispersion relation reduces to

$$\psi(\alpha)(1 + Y^2) = -iY\rho_0/\rho + Y\tan\alpha Y.$$

Thus the dispersion relation cannot admit pure imaginary roots for X and hence the equation [4.14] admits $2n$ complex roots for X. Corresponding to these $2n$ complex roots, σ^2 is complex for all values of α. Thus the system is unstable for all wave-lengths.

When $\rho_0 > \rho$, $\psi(\alpha)$ is negative. One of the roots of [4.14] is now positive and less than unity. The other root is less than -1 as now $y_1 + y_2 = -\rho_0/\rho + \tanh\alpha < 0$ at $X = -1$. For the former root, $\sigma^2 < 0$ and for the latter root $\sigma^2 > 0$. Once again the other $2n$ roots are complex. Therefore the system is unstable.

5. SYMMETRICAL PERTURBATIONS

Chakraborty[3] has deduced the following dispersion relation when the displacement in the x and y directions are symmetrical about the mid-plane:

$$\left[\frac{4\pi G(\rho-\rho_0)^2}{\sigma(k^2+l^2)^{1/2}}\cdot\frac{1}{[1+\tanh(k^2+l^2)^{1/2}h]}+\frac{\rho_0(\sigma^2-4\Omega^2)^{1/2}}{(k^2+l^2)^{1/2}}+\frac{4\pi G\rho h(\rho_0-\rho)}{\sigma}\right]$$

$$\times\frac{(A^2+4\Omega^2)^{1/2}(k^2+l^2)^{1/2})}{(2\Omega l+Ak)}+\left[\frac{\rho}{k}\left\{\sigma+\frac{2\Omega i(Al-2\Omega k)}{(2\Omega l+kA)}\right\}-\frac{k\mu H_0^2}{\sigma}\right]$$

$$\times\coth\left\{\frac{A(k^2+l^2)^{1/2}h}{(A^2+4\Omega^2)^{1/2}}\right\}=0. \qquad [5.1]$$

He has discussed only the particular cases of [5.1] taking $\rho_0=0$. We shall discuss these particular cases in the presence of the surrounding non-conducting material for sake of comparison with the conclusions arrived in the last section.

Case I: $\rho_0\neq 0$, $H_0=0$, $\Omega=0$.

In this case the dispersion relation reduces to:

$$4\pi G\rho\alpha(1-\rho_0/\rho)\left[1-\frac{1-\rho_0/\rho}{\alpha(1+\tanh\alpha)}\right]=(\rho_0/\rho+\coth\alpha)\,\sigma^2. \qquad [5.2]$$

When $\rho_0<\rho$, σ^2 is positive or negative according as $\alpha>\alpha^*$ or $<\alpha^*$, where α^* is the critical wave number given by

$$\alpha^*(1+\tanh\alpha^*)=(1-\rho_0/\rho).$$

Thus the system is stable or unstable according as the wave number α is greater or less than α^*.

When $\rho_0>\rho$ the dispersion relation can be written as

$$-4\pi G\rho\alpha\left(\frac{\rho_0}{\rho}-1\right)\left[1-\frac{\rho_0/\rho-1}{\alpha(1+\tanh\alpha)}\right]=\sigma^2\left(\frac{\rho_0}{\rho}+\coth\alpha\right),$$

and σ^2 is negative for all values of α. Hence the system is unstable.

Case II: $\rho_0 \neq 0,\ H_0 \neq 0,\ \Omega = 0.$

Here the dispersion relation reduces to

$$4\pi G\rho\alpha \tanh\alpha\left(1-\frac{\rho_0}{\rho}\right)\left\{1-\frac{1-\rho_0/\rho}{\alpha(1+\tanh\alpha)}\right\}+\frac{k^2\mu H_0^2}{\rho}=\sigma^2\left(1+\frac{\rho_0}{\rho}\tanh\alpha\right). \quad [5.3]$$

When $\rho_0 < \rho$ there exists a critical wave number k^* separating the regions of stability and instability determined by the equation :

$$4\pi G(\rho-\rho_0)\,\alpha^*\tanh\alpha^*\left[1-\frac{1-\rho_0/\rho}{\alpha^*(1+\tanh\alpha^*)}\right]+\frac{k^{*2}\mu H_0^2}{\rho}=0,$$

where $$\alpha^* = h\,(k^{*2}+l^2)^{1/2},$$

for given values of the wave number l, H_0, ρ and ρ_0.

When $\rho_0 > \rho$,

$$\frac{k^2\mu H_0^2}{\rho}-4\pi G\rho\alpha\tanh\alpha\left(\frac{\rho_0}{\rho}-1\right)\left[1+\frac{\rho_0/\rho-1}{\alpha(1+\tanh\alpha)}\right]=\sigma^2\left(1+\frac{\rho_0}{\rho}\tanh\alpha\right). \quad [5.4]$$

The right-hand side of [5.4] when equated to zero will determine the critical wave number k^*, which separates the region of stability and instability.

Case III: $\rho_0 \neq 0,\ H_0 = 0,\ \Omega \neq 0.$

The dispersion relation reduces to :

$$\phi(\alpha) = \frac{X}{X^2-1}\left(\coth X - \frac{\rho_0}{\rho}\right), \quad [5.5]$$

where $$\phi(\alpha) = \pi G\rho\alpha\left(1-\frac{\rho_0}{\rho}\right)\left[1-\frac{1-\rho_0/\rho}{\alpha(1+\tanh\alpha)}\right],$$

and $$X = \frac{\sigma}{(\sigma^2-4\Omega^2)^{1/2}},$$

so that $$\sigma^2 = \frac{4\Omega^2X^2}{(X^2-1)}. \quad [5.6]$$

As in case VII of 4 equation [5.5] will have $(2n+2)$ roots within the circle $|X| = R_n$. When $\rho_0 < \rho$, $\phi(\alpha)$ is either negative or positive, but it is always negative for $\rho_0 > \rho$. For both the cases we can show as in case VII of 4, that $2n$ roots of [5.5] are complex. From [5.6] σ^2 is complex, corresponding to these roots. Hence the system is unstable for all wave lengths.

6. Conclusion

In all the special cases that we have considered we note that, if the conducting medium is surrounded by vacuum, the system is overstable for all the considered asymmetric disturbances, when rotation and magnetic field are individually or collectively present. This result may be compared with that obtained in references [1, 2] and [3], that when the perturbations are symmetrical about the mid-plane of the undisturbed conducting medium, the system is stable or unstable when the wave number is greater or less than a critical wave number.

In the presence of surrounding material with $\rho_0 < \rho$ the non-rotating system is overstable and remains in this state even in the presence of magnetic field. But, when rotation is taken into consideration the system becomes unstable for all the wave numbers. Thus we note that, in the presence of lighter surrounding non-conducting material, the presence of magnetic field does not alter the stability criterion. When $\rho_0 > \rho$, as is obvious from the physical situation, the non-rotating system in the absence of magnetic field is unstable, in the presenee of the magnetic field the regions of overstability and instability are separated by a critical wave number which depends on the strength of magnetic field; when rotation is taken into account it is unstable for all wave numbers.

It appears that under the symmetrical perturbations, the system in the perturbed state is unable to regain its original form as in the sausage type of instability. However, the possibility of occurrence of such symmetrical perturbations is rare in nature. From this point of view, the present investigation is more general and physically plausible than considered in references [1, 2] and [3]. The cause of overstability in part can be understood in view of the fact that the velocity component, v_z is even function of z, so that the upper and lower interfaces are so deformed that the crests and troughs in one correspond to the crests and troughs in the other, as if the whole layer is deformed as a rigid sheet. This sort of perturbation of the layer explains why there is no critical wave length for instability to set in as in the case of sausage-type of deformation.

We have seen in 5 that the presence of the surrounding non-conducting material is critical to some of the conclusions arrived in references [1, 2] and [3]. In particular we record that if the non-conducting material is heavier than the conducting material the system behaves alike under symmetric and asymmetric perturbations.

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