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UNSTEADY MOTION OF A NON-NEWTONIAN FLUID IN THE ANNULAR SPACE BETWEEN TWO POROUS CONCENTRIC CIRCULAR CYLINDERS

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ABSTRACT

The unsteady motion of a non-Newtonian fluid, between two porous concentric circular cylinders, produced by suddenly accelerating the outer cylinder to a constant velocity, is investigated. It is assumed that the rate of injection at one pipe is equal to the rate of suction at the other. Explicit expressions for the velocity distribution have been obtained and the effect of cross-viscosity (which is effective only when there is suction and injection) on the time dependent velocity is discussed.

INTRODUCTION

1. The equations for the flow of a Reiner-Rivlin fluid[1] have yielded solutions so far mainly for steady flows or small oscillations. The object of the present note is to briefly describe a transient flow problem. We investigate the unsteady motion of a Reiner-Rivlin fluid in a porous cylindrical annulus when the outer cylinder is suddenly accelerated to a constant velocity V parallel to the common axis of the cylinders. It is assumed that the rate of suction at one cylinder is equal to the rate of injection at the other. Ghildyal[2] studied the flow of a Newtonian fluid generated by an impulsive twist given to the outer cylinder. Devi Singh[3] has generalised this problem for Newtonian fluid making the same assumptions, regarding suction and injection, as in the present paper.

EQUATIONS OF THE PROBLEM AND THEIR SOLUTION

2. The axisymmetric flow $[\partial/\partial\theta = 0]$ is governed by the following equations in cylindrical coordinates.

Equation of continuity :

$$\frac{\partial u}{\partial r} + \frac{u}{r} = 0 \cdot \qquad [2.1]$$

Equations of momentum :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\cdot\frac{\partial p}{\partial r} + v_1\left[4\frac{\partial u}{\partial r}\left(\frac{\partial^2 u}{\partial r^2} + \frac{u}{r^2}\right) - \frac{4u^2}{r^3} + 2\frac{\partial w}{\partial r}\cdot\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\left(\frac{\partial w}{\partial r}\right)^2\right], \qquad [2.2]$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} = v\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial w}{\partial r}\right) + \frac{2v_1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\cdot\frac{\partial w}{\partial r}\right), \qquad [2.3]$$

Boundary conditions :

$$t = 0, \;\; w = 0, \;\; \text{for all } r, \qquad [2.4]$$

$$\left.\begin{aligned} t > 0, \;\; w = V, \;\; r = b, \\ w = 0, \;\; r = a, \end{aligned}\right\rangle \qquad [2.5]$$

where v, v_1 are the kinematic coefficients of viscosity and cross-viscosity respectively, a, b are the radii of the inner and outer cylinders, p is the hydrostatic pressure, ρ the density, u, w are the radial and axial velocities and r is the radial distance measured from the common axis of the cylinders.

Equation [2.1] gives $u\,r$ = constant. If mV is the constant velocity at the outer cylinder due to suction or injection, we have

$$u = \frac{mVb}{r}, \qquad [2.6]$$

which is independent of time.

In terms of the non-dimensional quantities

$$\lambda = \frac{r}{b}, \;\; U = \frac{u}{V}, \;\; W = \frac{w}{V}, \;\; T = \frac{Vt}{b},$$

equations [2.1], [2.3] become

$$U = \frac{m}{\lambda}, \qquad [2.7]$$

$$\frac{\partial^2 W}{\partial \lambda^2}\left(1-\frac{2mRS}{\lambda^2}\right)+\frac{1}{\lambda}\cdot\frac{\partial W}{\partial \lambda}\left(1-mR+\frac{2mRS}{\lambda^2}\right)=R\frac{\partial W}{\partial T}, \qquad [2.8]$$

where $R=\dfrac{bV}{v}$, is the Reynolds number

and

$S=\dfrac{v_1}{b^2}$, is the cross-viscosity parameter. On making the substitutions

$$Y=R\,(\lambda^2-2mRS)^{\frac{1}{2}} \qquad [2.9]$$

and

$$W=(\lambda^2-2mRS)^{(mR)/4}\,Z, \qquad [2.10]$$

equation [2.8] becomes

$$\frac{\partial^2 Z}{\partial Y^2}+\frac{1}{Y}\cdot\frac{\partial Z}{\partial Y}-\frac{q^2\,Z}{Y^2}=\frac{1}{R}\cdot\frac{\partial Z}{\partial T}, \qquad [2.11]$$

with the boundary conditions

$$t=0,\ \ Z=0\ \text{ for all } Y, \qquad [2.12]$$

$$t>0,\ \ Z=(1-2\,mRS)^{-(q/2)}$$

when

$$Y=R\,\sqrt{(1-2\,mRS)}=Y_2, \qquad [2.13]$$

$$Z=0,\ \text{when } Y=R\,\sqrt{(\sigma^2-2\,mRS)}=Y_1, \qquad [2.14]$$

where

$$q=\frac{mR}{2}\ \text{ and }\ \sigma=\frac{a}{b}.$$

Multiplying equation [2.11] by

$$Y\,[J_{|q|}(\xi_i\,Y)\,G_{|q|}(\xi_i\,Y_2)-J_{|q|}(\xi_i\,Y_2)\,G_{|q|}(\xi_i\,Y)]$$

where $J_{|q|}$ and $G_{|q|}$ represent Bessel functions* of the first and second kind respectively and ξ_i are the roots of the equation

$$J_{|q|}(\xi_i\,Y_1)\,G_{|q|}(\xi_i\,Y_2)=J_{|q|}(\xi_i\,Y_2)\,G_{|q|}(\xi_i\,Y_1)$$

and integrating between $Y=Y_1$ and $Y=Y_2$,

we have

$$\frac{1}{R}\cdot\frac{\partial \bar{Z}}{\partial T}+\xi_i^2\,\bar{Z}+(1-2mRS)^{-(q/2)}=0, \qquad [2.15]$$

* In the notation of Sneddon, I.N. 'Fourier Transforms', (McGraw-Hill), 1951

where $\bar{Z}$ denotes the finite Hankel transform

$$\bar{Z} = \int_{Y_1}^{Y_2} ZY\,[J_{1q1}(\xi_i Y)\,G_{1q1}(\xi_i Y_2) - J_{1q1}(\xi_i Y_2)\,G_{1q1}(\xi_i Y)]\,dY. \quad [2.16]$$

On integration [2.15] gives

$$\bar{Z} = \frac{1}{\xi_i^2}(1 - 2mRS)^{-(q/2)}(e^{-R\,\xi_i^2\,T} - 1), \quad [2.17]$$

where the constant of integration has been evaluated with the help of the initial condition [2.12], so that

$$Z = 2\,(1 - 2\,mRS)^{-(q/2)} \sum_i \Bigg[(1 - e^{-R\,\xi_i^2\,T})\,J_{1q1}(\xi_i Y_1)\,J_{1q1}(\xi_i Y_2) \times \\ \times \frac{J_{1q1}(\xi_i Y)\,G_{1q1}(\xi_i Y_1) - J_{1q1}(\xi_i Y_1)\,G_{1q1}(\xi_i Y)}{J_{1q1}^2(\xi_i Y_1) - J_{1q1}^2(\xi_i Y_2)}\Bigg] \quad (2.18]$$

The velocities given by [2.10] and [2.18] are shown in Figs. I, II and IV].

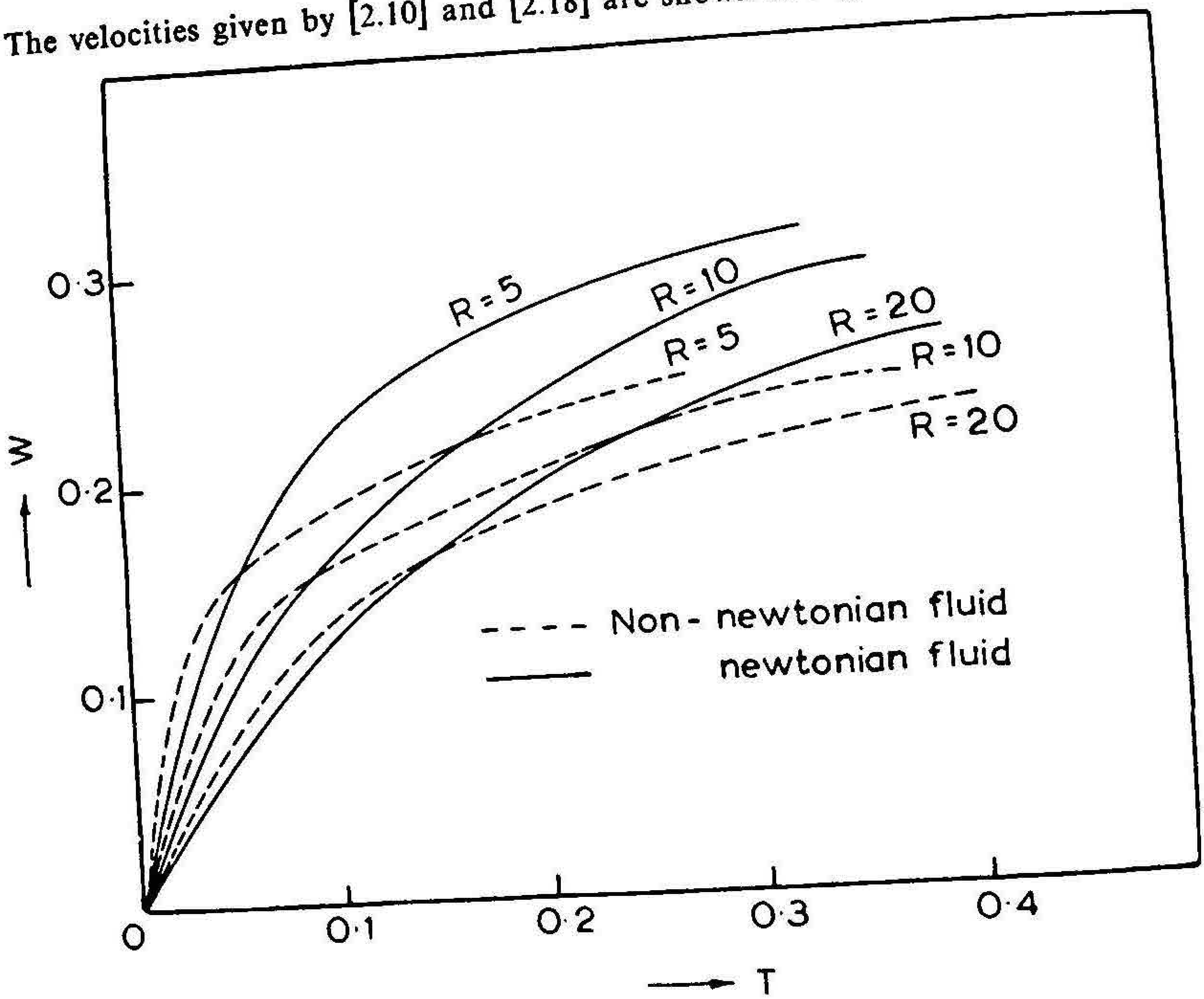

FIG. I

Suction at the Outer Pipe $mR = 5$, $S = 1/64$, $\lambda = 0{\cdot}8$

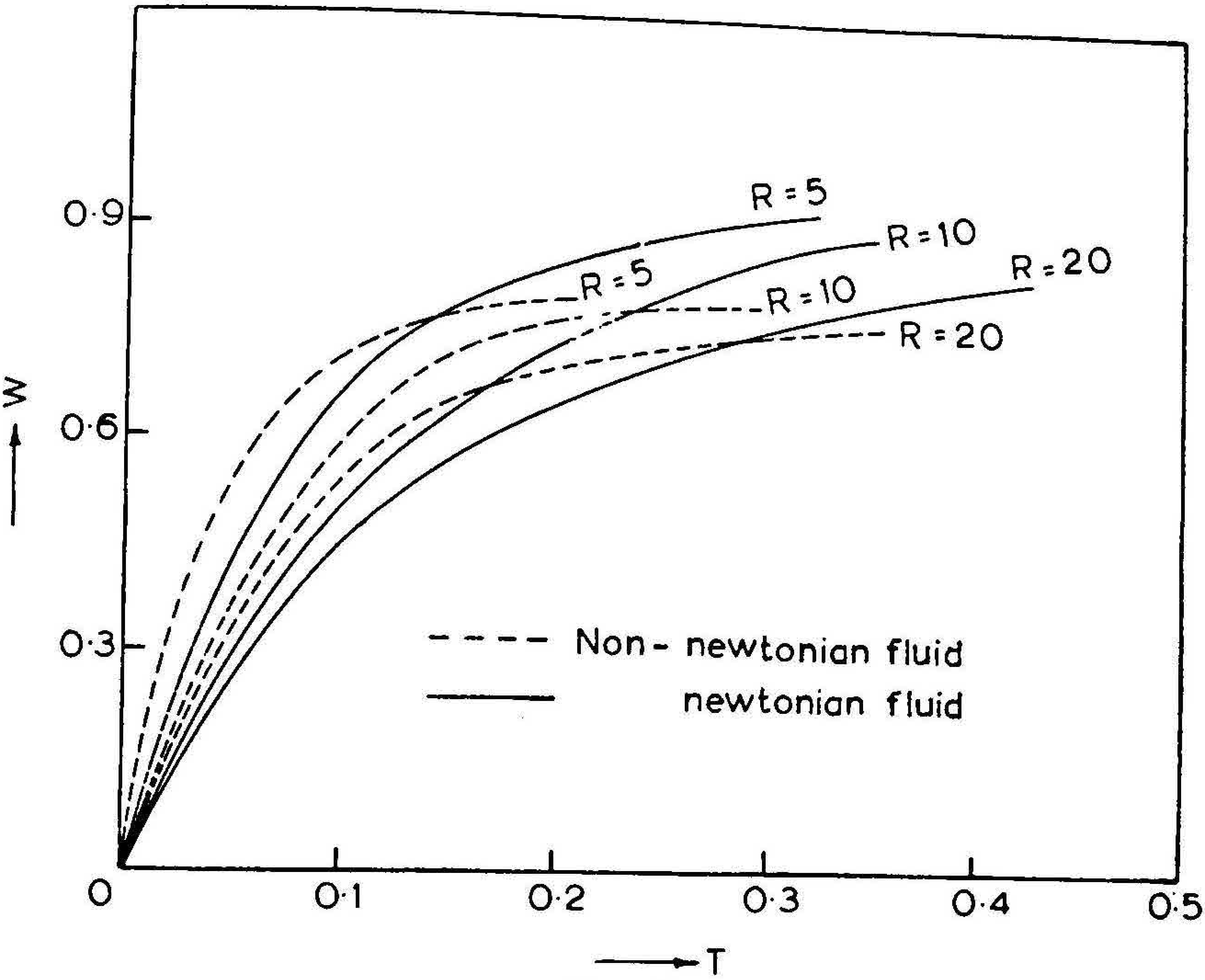

FIG. II

Injection at the Outer Pipe $mR = 5$, $S = 0{\cdot}035$, $\lambda = 0{\cdot}8$

For large values of T, the velocity approaches that of the steady state flow given by

$$W = 1 - \frac{\log \lambda}{\log \sigma} \text{ when } m = 0, \tag{2.19}$$

$$W = \frac{(\lambda^2 - 2\,mRS)^{(mR/2)} - (\sigma^2 - 2\,mRS)^{(mR/2)}}{(1 - 2\,mRS)^{(mR/2)} - (\sigma^2 - 2\,mRS)^{(mR/2)}}\,, \tag{2.20}$$

As a special case of [2·20] we have for a Newtonian fluid ($S = 0$)

$$W = \frac{\lambda^{mR} - \sigma^{mR}}{1 - \sigma^{mR}}. \tag{2.21}$$

By integrating equation [2.2] we get the pressure in the following form

$$p = \rho V^2\left[\frac{4\,m^2}{\lambda^2}\left(\frac{5}{\lambda^2} - 2\right) + \int^{\lambda}\left\{2\frac{\partial W}{\partial \lambda}\cdot\frac{\partial^2 W}{\partial \lambda^2} + \frac{1}{\lambda}\left(\frac{\partial W}{\partial \lambda}\right)^2\right\} d\lambda\right] \tag{2.22}$$

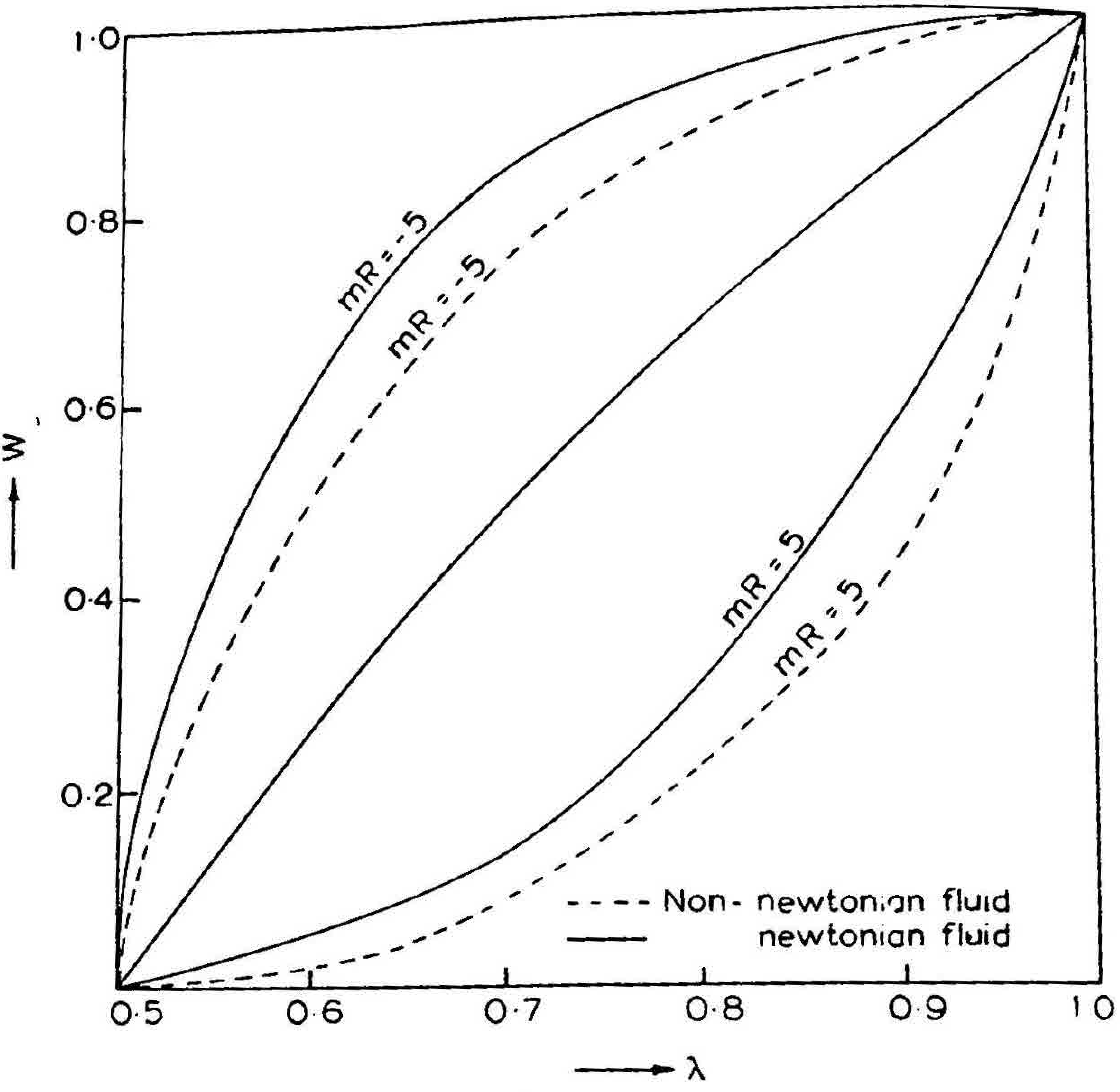

FIG. III

Terminal Velocity

DISCUSSION OF THE RESULTS

From the figures drawn for $\lambda = 0.8$, we notice the following important points in the presence of suction and injection.

(1) The initial velocity introduced by the impulsive motion of the outer cylinder is more for the non-Newtonian fluid than that of the Newtonian fluid at every point of the flow field except at the inner and outer cylinders, where they are equal.

(2) The terminal velocities ($T \to \infty$) given by [2.19] – [2.21] are attained more rapidly in the case of the non-Newtonian fluids than in the case of the Newtonian fluids.

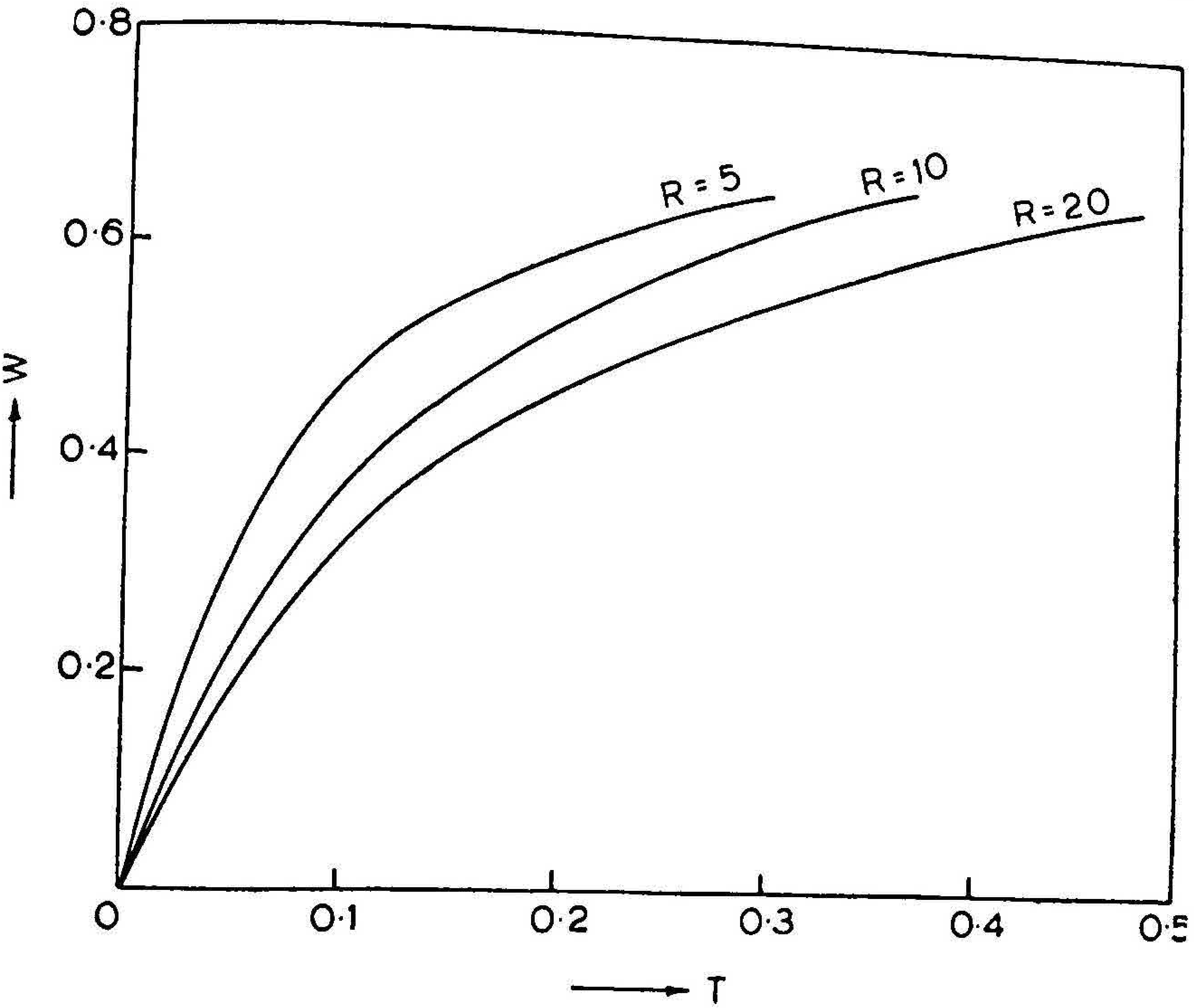

FIG. IV

No Injection or Suction with $\lambda = 0{\cdot}8$

(3) The terminal velocity for the Newtonian fluid is more than that of the non-Newtonian fluids at every point of the flow field except at the boundaries where they are equal.

Regarding the growth of the velocities of the Newtonian and non-Newtonian fluids, we may remark that in the initial stages of the motion the cross viscosity acts as additional viscosity (increases the apparent viscosity) and facilitates rapid communication of the velocity of the outer cylinders to the fluid below. With the passenge of time, the cross-viscosity retards the further growth of velocity in the case of the non-Newtonian fluid.

In the absence of suction and injection the Newtonian and the non-Newtonian fluids behave alike. From this we conclude that suction and injection play a very important role in establishing the effect of cross-viscosity in the flow. We can check this statement directly from the fundamental equations [2.1]—[2.3]. In the absenee of cross-flow ($U = 0$), the equation [2.3] determining W does not contain the cross-viscosity term and therefore under the given initial and boundary conditions the profiles for the two liquids are the same. However, from [2.2] it is clear that for a Newtonian fluid the pressure is constant while in the case of the non-Newtonian fluid the pressure is modified due to cross-viscosity. It will be of interest to note that the steady state pressure gradient flow studied by Narasimhan[4] also exhibits this particular characteristic, namely, in the absence of suction and injection the two fluids behave alike, whereas they show marked difference in the profiles in the presence of suction and injection.

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