COMPRESSIBLE BOUNDARY LAYER ON A YAWED SEMI-INFINITE FLAT PLATE WITH SUCTION AND INJECTION

By G. C. Ahuja

(Department of Applied Mathematics, Indian Institute of Science, Bangalore-12)

Received on November 21, 1963

Abstract

Boundary layer equations for compressible flow over a yawed semi-infinite flat plate in flows with pressure gradient are transformed to another plane by Howarth's transformation. Under certain assumptions mentioned in the text the spanwise flow in the transformed plane may be treated as incompressible. Pohlhausen method is used to study the correlated incompressible case. The results in respect of the location of point of separation for the chordwise flow for one case are compared with those of Stewartson's Effects of compressibility, suction and injection are studied.

1. Introduction

Incompressible boundary layer on a yawed semi-infinite flat plate with suction and injection have been investigated by Bhatnagar and Ahuja[1] and Ahuja[2] by using the 'independence principle' and a modified Pohlhausen technique. For the corresponding compressible flows, the independence principle does not hold good. In order to study the effects of compressibility, we assume that there is no flux of heat across the plate, Prandtl number of the gas is unity and the coefficient of viscosity varies directly with the absolute temperature. We transform the boundary layer equations by Howarth's transformations[3] where we assume that the spanwise flow is incompressible in the transformed plane. This assumption is based on two facts. Firstly, the angle of yaw is small and secondly, the Mach number for the chordwise flow cannot be greater than unity, since beyond that we will have to study the interaction of shock and the boundary layer, which our treatment cannot take care of. Therefore, the Mach number for the spanwise flow will be very small. Using this assumption we find that the independence principle holds in the transformed plane and the effects of compressibility for the spanwise component are taken care of through the transformations only. Crabtree[4] and Tinkler[5] have studied similar problems by using Illingworth and Stewartson's transformations[6,7] and expanding the solution in series in powers of spanwise Mach number. Here we consider four flows *viz*.,

$$U = 1 - x, \quad V = \text{const.}, \text{ (I)}; \qquad U = 1 + x, \quad V = \text{const.}, \text{ (II)}$$

$$U = \frac{1}{1+x}, \quad V = \text{const.}, \text{ (III)}; \qquad U = \frac{1}{1-x}, \quad V = \text{const.}, \text{ (IV)}$$

and compare the results of case (I) without suction with those of Stewartson's[7]. We find that the fifth degree polynomial with two boundary conditions on the plate gives good agreement even for compressible flows.

We study the effects of compressibility on all the four flows and also the effects of suction and injection through the parameter $S[\equiv(\omega_0 \Delta x)/v]$ We find that compressibility decreases the skin friction, a comparison of which is shown in Figs. 1 and 2. Suction and injecion have effects similar to the case of incompressible flows. Their effects are presented in Figs. 3, 4, 5 and 6.

EQUATIONS OF MOTION

2. The equations of motion of the steady compressible laminar boundary layer over a yawed semi-infinite plate are:

$$\rho u \frac{\partial u}{\partial x} + \rho \omega \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right), \qquad [2.1]$$

$$\rho u \frac{\partial v}{\partial x} + \rho \omega \frac{\partial v}{\partial z} = \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right), \qquad [2.2]$$

$$\frac{\partial p}{\partial z} = 0, \qquad [2.3]$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial z}(\rho \omega) = 0, \qquad [2.4]$$

and

$$\rho\left(u \frac{\partial}{\partial x} + \omega \frac{\partial}{\partial z}\right)\left(\frac{u^2 + v^2}{2} + C_p T\right) = \frac{\partial}{\partial z}\left\{\mu \frac{\partial}{\partial z}\left(\frac{u^2 + v^2}{2} + C_p T\right)\right\}, \qquad [2.5]$$

$$p = \rho R T$$

are the equations expressing conservation of energy for a Prandtl number of unity and the equation of state respectively.

We measure x, y in the chordwise and spanwise directions respectively and z is the direction perpendicular to the plate. The other symbols have their usual meanings.

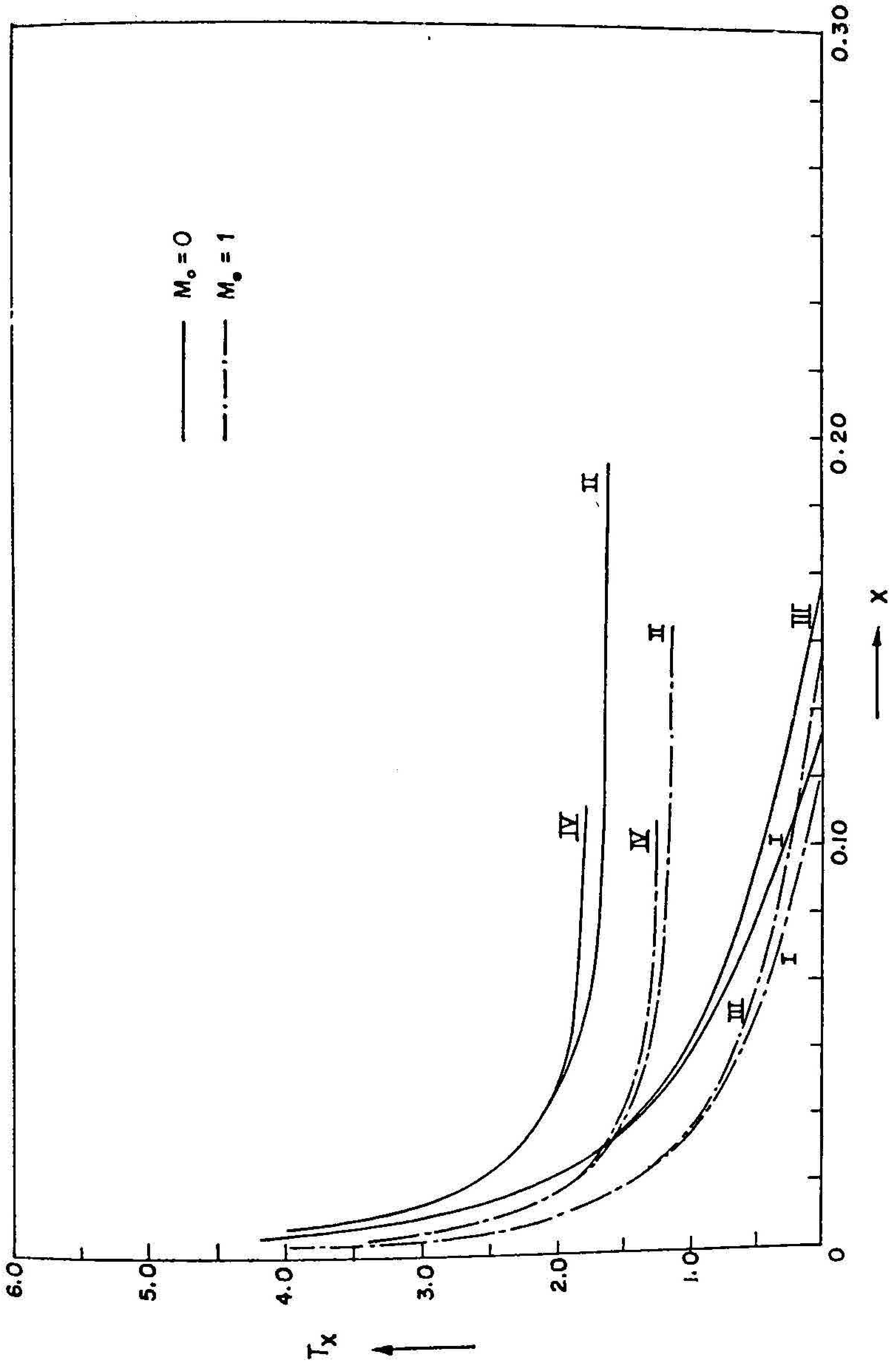

FIG. 1
Effect of Compressibility on Chordwise Skin Friction

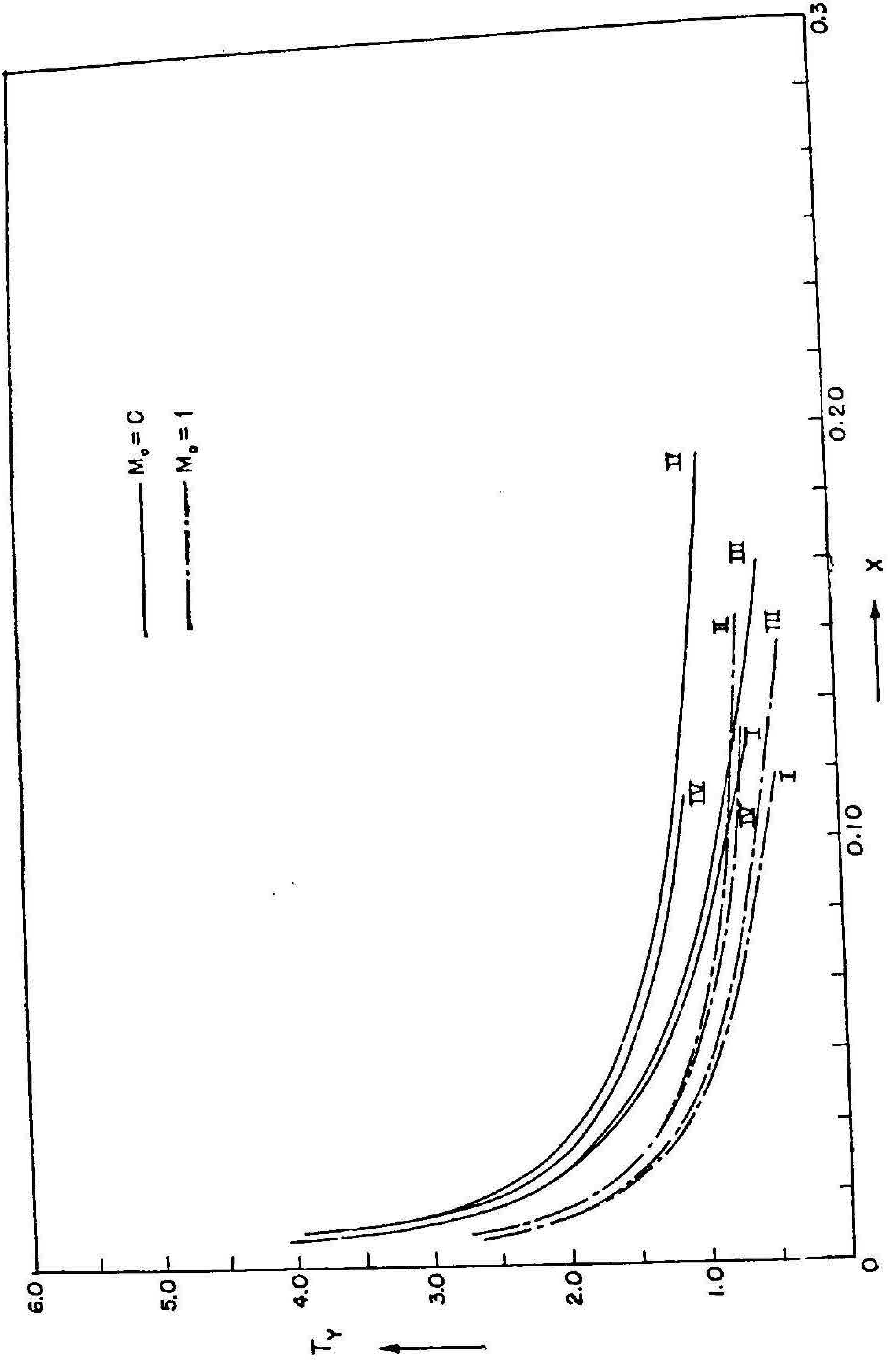

FIG. 2

Effect of Compressibility on Spanwise Skin Friction

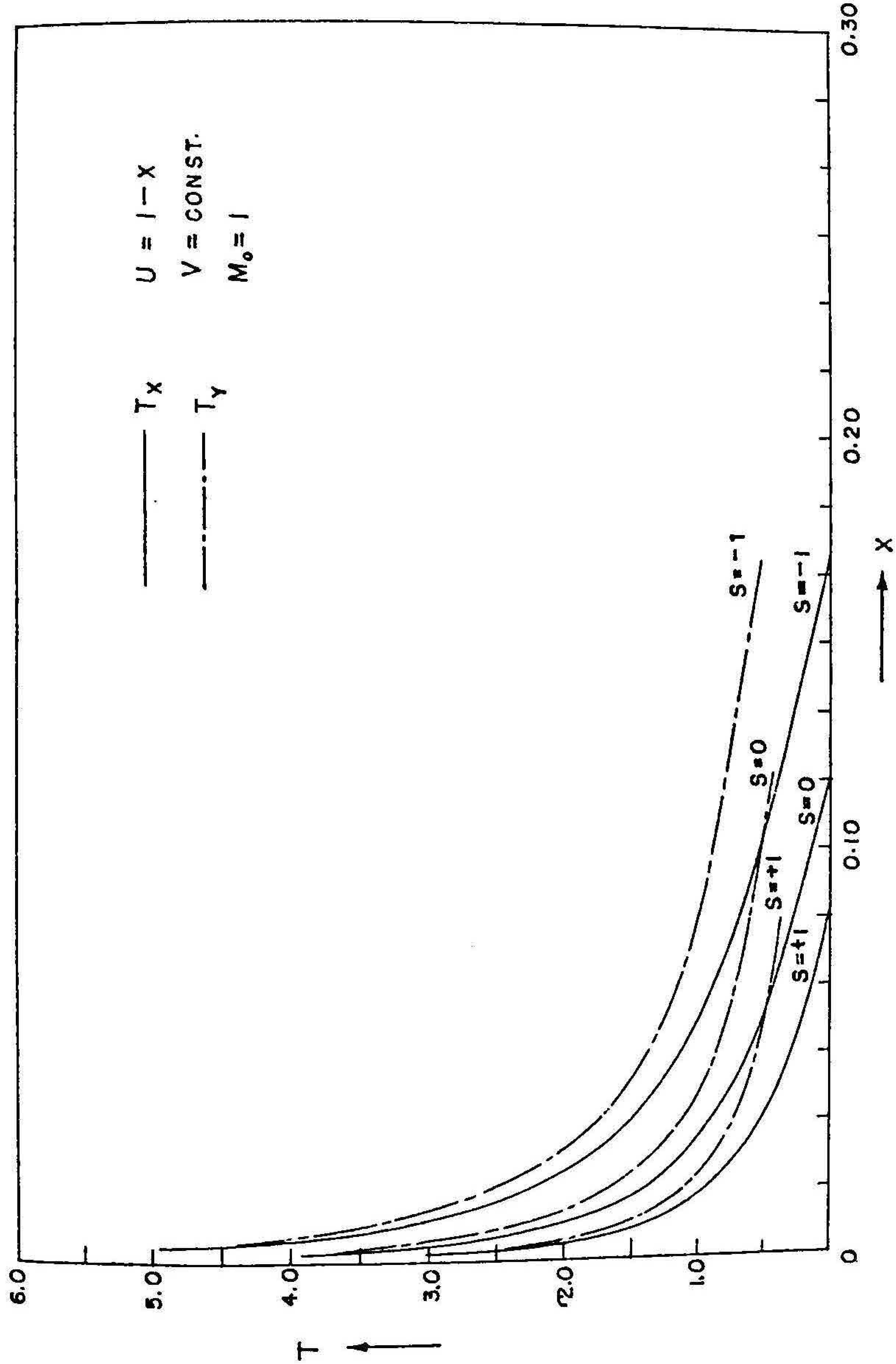

FIG. 3

Skin Friction Against Chordwise Distance

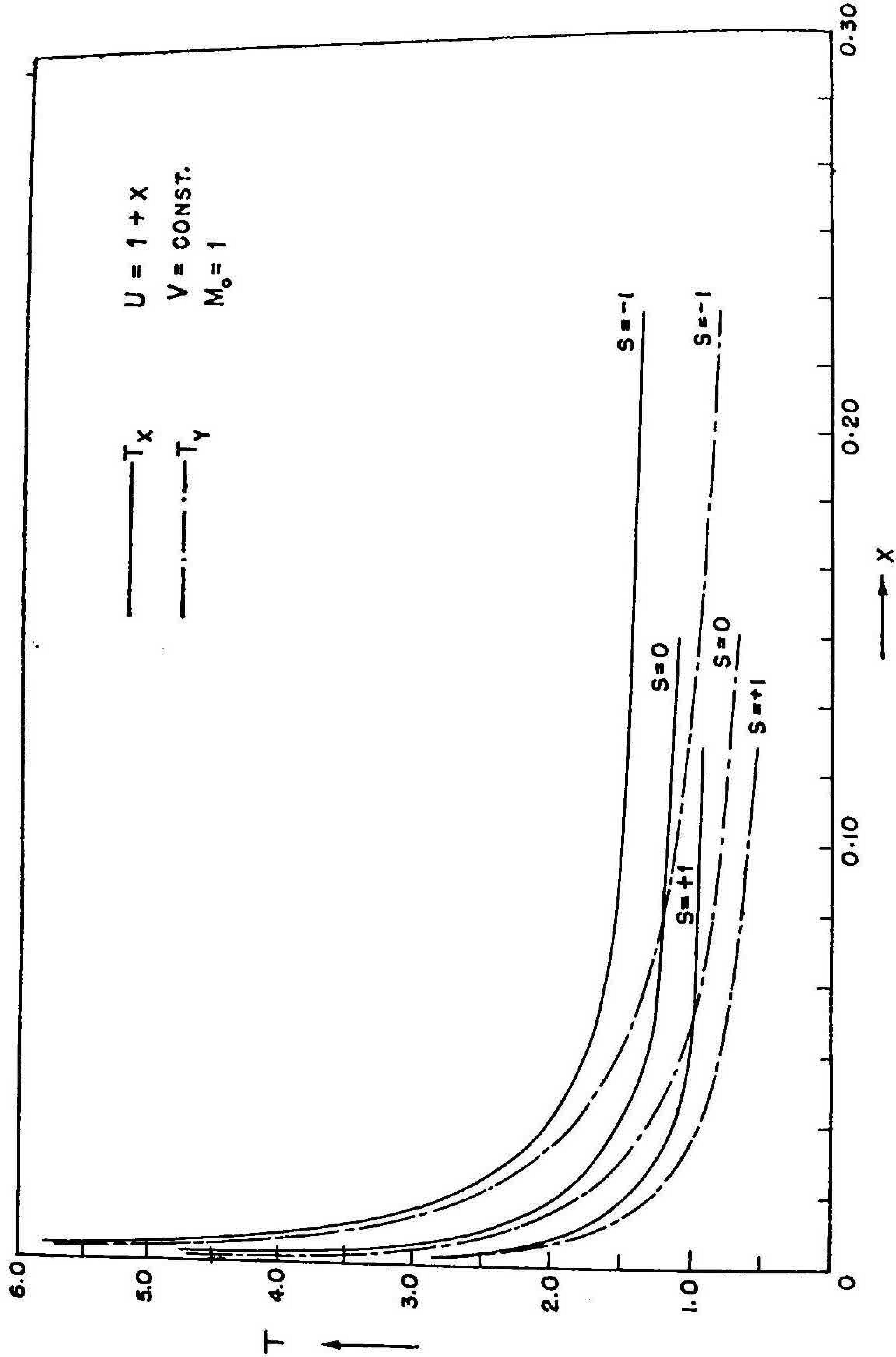

FIG. 4
Skin Friction Against Chordwise Distance

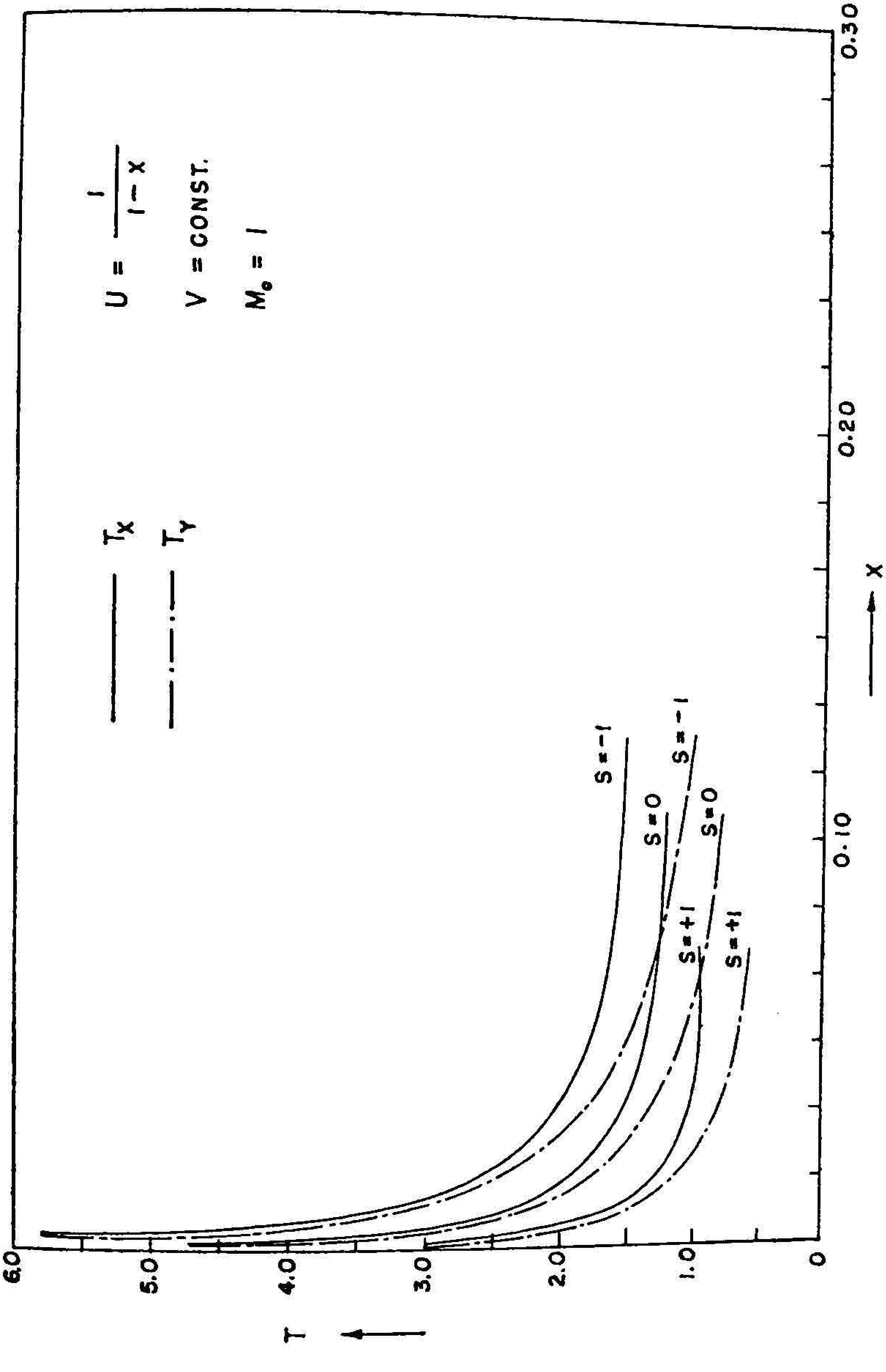

FIG. 5
Skin Friction Against Chordwise Distance

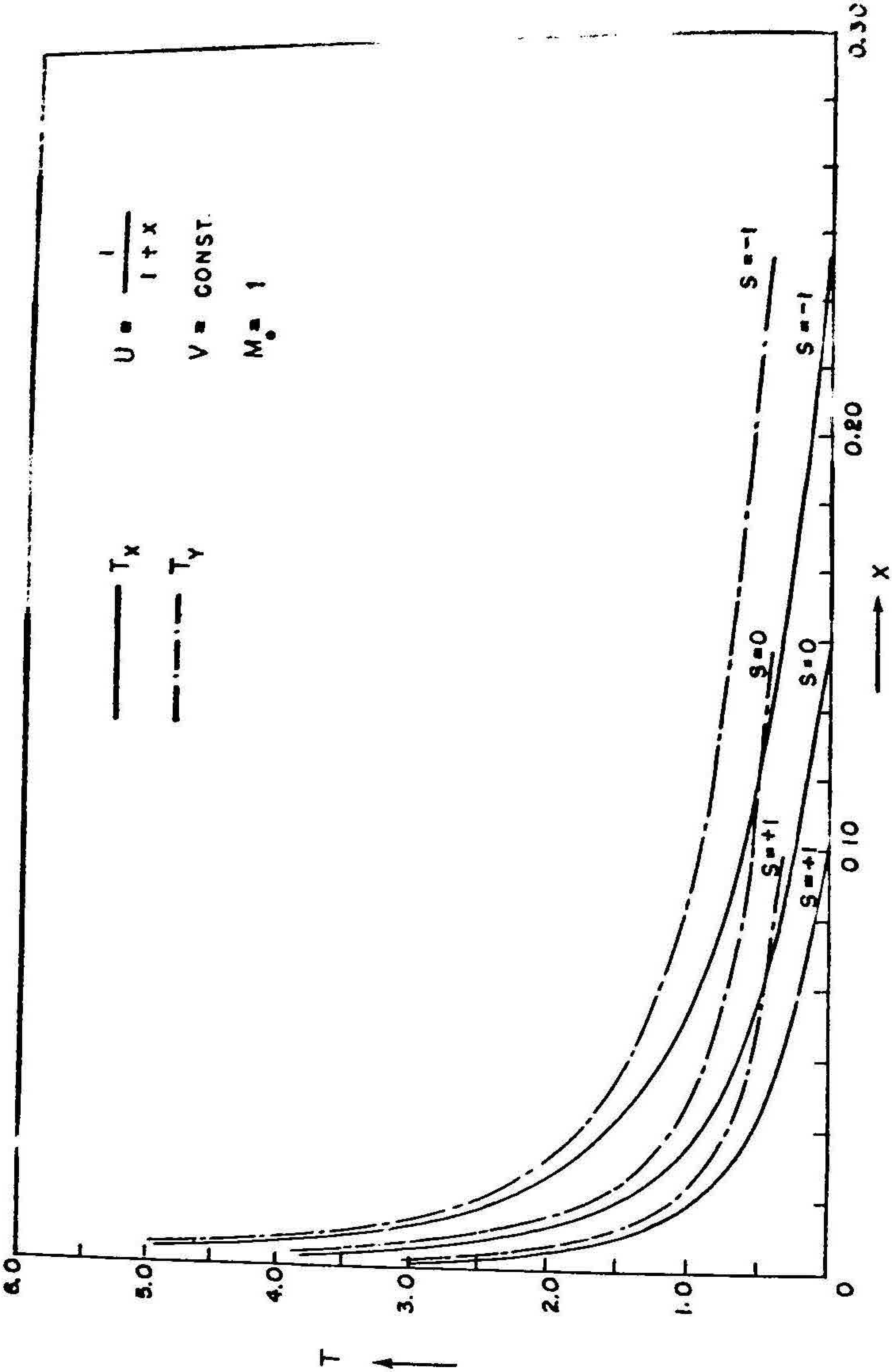

FIG. 6

Skin Friction Against Chordwise Distance

The boundary conditions are:

$$\text{At } z=0, \quad u=v=0, \quad \omega=\omega_0(x) \text{ and } \partial T/\partial z=0$$

$$\text{At } z=\infty, \quad u \to U, \quad v \to V \text{ and } T \to T_1$$

Since we are dealing with zero heat transfer between the gas and the wing surface, we use the following integral of equation [2.5].

$$C_p T + \frac{u^2+v^2}{2} = \text{constant} \qquad [2.7]$$

The chordwise velocity outside the boundary layer satisfies the following equation:

$$-\frac{1}{\rho} \cdot \frac{\partial p}{\partial x} = \frac{U dU}{dx}.$$

The velocities in the equations of motion can be replaced through the definition of a stream function;

$$\left.\begin{aligned} \rho u &= \rho_s \frac{\partial \psi}{\partial z} \\ \rho \omega &= -\rho_s \frac{\partial \psi}{\partial x} \end{aligned}\right\} \qquad [2.8]$$

so that the continuity equation is automatically satisfied. Here the suffix s denotes standard conditions of the gas at the plate.

Let us now use Howarth's transformation in changing from variables x, z to x, Z where

$$Z = \int_0^z \left(\frac{\nu_s}{\nu}\right)^{\frac{1}{2}} dz = \left(\frac{p}{p_s}\right)^{\frac{1}{2}} \int_0^z \frac{T_s}{T} dz, \qquad [2.9]$$

and introducing

$$\psi(x, z) = \left(\frac{p}{p_s}\right)^{\frac{1}{2}} X(x, Z) \qquad [2.10]$$

we get the transformed forms of equations [2.1] and [2.2] as

$$\frac{\partial^2 X}{\partial x \partial Z} \cdot \frac{\partial X}{\partial Z} - \frac{\partial^2 X}{\partial Z^2} \cdot \frac{\partial X}{\partial x} = U \frac{dU}{dx}\left[\frac{T}{T_1} - \frac{\gamma}{2a_1^2} X \frac{\partial^2 X}{\partial Z^2}\right] + \nu_s \frac{\partial^3 X}{\partial Z^3}, \qquad [2.11]$$

$$\frac{\partial X}{\partial Z}\cdot\frac{\partial v}{\partial x}-\left[\frac{\partial X}{\partial x}-\frac{\gamma}{2a_1^2}\cdot U\frac{dU}{dx}X\right]\frac{\partial v}{\partial Z}=\nu_s\frac{\partial^2 v}{\partial Z^2}, \qquad [2.12]$$

where we use the suffix '1' to denote general values in the mainstream at the edge of the boundary layer. From [2 7] we have

$$\frac{T}{T_1}=1+\frac{\gamma-1}{2a_1^2}\left[(U^2+V^2)-(u^2+v^2)\right].$$

The effects of compressibility may, therefore, be thought of as summarized in square brackets. Now we write

$$G\equiv\frac{T}{T_1}-\frac{\gamma}{2a_1^2}X\frac{\partial^2 X}{\partial Z^2}$$

$$=1+\frac{\gamma-1}{2a_1^2}\left[U^2-\left(\frac{\partial X}{\partial Z}\right)^2\right]-\frac{\gamma}{2a_1^2}X\frac{\partial^2 X}{\partial Z^2}+\frac{\gamma-1}{2}M_v^2\left\{1-\left(\frac{v}{V}\right)^2\right\}$$

where $M_v=V/a_1$, is the Mach number of the spanwise flow at the edge of the boundary layer.

The maximum value of $\omega\equiv(\gamma-1)/2\,M_v^2$ is 0.2 for air with $\gamma=1.4$ and $M_v=1$. We can take the Mach number of the chordwise flow upto unity only as our treatment cannot take into account shock wave-boundary layer interaction which may be present in the flow field for Mach numbers greater than unity. Therefore, for small angles of yaw and for the extreme value for chordwise Mach number to be unity, M_v is much smaller than unity, consequently, we neglect the term containing M_v^2. In other words, spanwise flow may be treated as incompressible. Then

$$G=1+\frac{\gamma-1}{2a_1^2}\left[U^2-\left(\frac{\partial X}{Z}\right)^2\right]-\frac{\gamma}{2a_1^2}X\frac{\partial^2 X}{\partial Z^2},$$

and the equations [2.11] and [2.12] reduce to

$$\frac{\partial^2 X}{\partial x\partial Z}\cdot\frac{\partial X}{\partial Z}-\frac{\partial^2 X}{\partial^2 Z}\cdot\frac{\partial X}{\partial x}=\left[1+\frac{\gamma-1}{2a_1^2}\left\{U^2-\left(\frac{\partial X}{Z}\right)^2\right\}-\frac{\gamma}{2a_1^2}X\frac{\partial^2 X}{\partial^2 Z}\right]U\frac{dU}{dx}+\nu_s\frac{\partial^3 X}{\partial Z^3}, \qquad [2.13]$$

and

$$\frac{\partial X}{\partial Z}\cdot\frac{\partial v}{\partial x}-\left[\frac{\partial X}{\partial x}-\frac{\gamma}{2a_1^2}U\frac{dU}{dZ}X\right]\frac{\partial v}{dZ}=\nu_s\left(\frac{\partial^2 v}{\partial Z^2}\right), \qquad [2.14]$$

where the boundary conditions are

$$\frac{\partial X}{\partial Z}=0,\quad v=0,\quad \frac{\gamma}{2a_1^2}U\frac{dU}{dx}X-\frac{\partial X}{\partial x}=\omega_0 \text{ at } Z=0,$$

and
$$\frac{\partial X}{\partial Z}\rightarrow U,\quad v\rightarrow V \quad\text{at}\quad Z\rightarrow\infty.$$

Equations [2.13] and [2.14] define an incompressible flow in the (x,Z) plane and can be solved by the modified Pohlhausen method, then we can transform the results to the physical plane by means of the inverse transformation.

The momentum integral equations in the transformed plane area :

$$\frac{d}{dx}\int_0^{\Delta x}u^2\,dZ-U\frac{d}{dx}\int_0^{\Delta x}u\,dZ$$
$$=U\frac{dU}{dx}\left[\Delta_x+\frac{M^2}{2}\left\{(\gamma-1)\int_0^{\Delta x}\left(1-\frac{u}{U}\right)dZ-\int\frac{u}{U}\left(1-\frac{u}{U}\right)dZ\right\}\right]-$$
$$-\omega_0 U-v_s\left(\frac{\partial u}{\partial Z}\right)_{Z=0}, \qquad [2.15]$$

and

$$\frac{d}{dx}\int_0^{\Delta y}u(v-V)dZ=\frac{\gamma M^2}{2U}\cdot\frac{dU}{dx}\int_0^{\Delta y}u\,(v-V)\,dZ-v_s\left(\frac{\partial v}{\partial Z}\right)_{Z=0}-\omega_0 V, \qquad [2.16]$$

where Δ_x and Δ_y are the chosdwise and spanwise boundary layer thicknesses respectively in the transformed plane and $M=\left(\frac{U}{a_1}\right)$

Method of Solution

3. Following [1] and [2], we assume

$$u/U=f(\eta_x,N,S);\quad v/V=g\,(\eta_y,K,S) \qquad [3.1]$$

where for the flows (I) and (III) we have the following velocity profiles :

$$f(\eta_x,N\ S)$$
$$=\frac{(9+S)\,N+60}{D_1}\eta_x+\frac{30\,S-18\,N}{D_1}\eta_x^2+\frac{S(\,10\,S-6\,N)}{D_1}\eta_x^3+$$
$$+\frac{(8\,N-15S-45)\,S+(18N-60)}{D_1}\eta_x^4+\frac{(24-3\,N+6S)\,S+(36-9N)}{D_1}\eta_x^5, \qquad [3.2]$$

$$g(\eta_y, KS) = \frac{60}{D_2}\eta_y + \frac{30KS}{D_2}\eta_y^2 + \frac{10K^2S^2}{D_2}\eta_y^3 - \frac{15K^2S^2+45KS+60}{D_2}\eta_y^4 + \frac{6K^2S^2+24KS+36}{D_2}\eta_y^5, \quad [3.3]$$

and for the flows (II) and (IV) we have the following sixth degree velocity profiles :

$$f(\eta_x, N, S) = \frac{(12+S)N+120}{D_3}\eta_x + \frac{30(2S-N)}{D_3}\eta_x^2 + \frac{10S(2S-N)}{D_3}\eta_x^3 + \frac{5(12N-60-36S+4SN-9S^2)}{D_3}\eta_x^4 + \frac{3(120-20N+64S-5SN+12S^2)}{D_3}\eta_x^5 - \frac{(120-18N+60S-4SN+10S^2)}{D_3}\eta_x^6 \quad [3.4]$$

$$g(\eta_y, KS) = \frac{120}{D_4}\eta_y + \frac{60KS}{D_4}\eta_y^2 + \frac{20K^2S^2}{D_4}\eta_y^3 - \frac{45K^2S^2+180KS+300}{D_4}\eta_y^4 + \frac{36K^2S^2+192KS+360}{D_4}\eta_y^5 - \frac{10K^2S^2+60KS+120}{D_4}\eta_y^6, \quad (3.5)$$

with

$$D_1 = S^2 + 9S + 36, \quad D_2 = K^2S^2 + 9KS + 36,$$

$$D_3 = S^2 + 12S + 60, \quad D_4 = K^2S^2 + 12KS + 60,$$

$$S = \frac{\omega_0}{\nu_s}\Delta_x, \quad N = \frac{U'}{\nu_s}\Delta_x^2\left[1 + \frac{\gamma-1}{2}M^2\right]$$

$$\eta_x = \frac{Z}{\Delta_x}, \quad \eta_y = \frac{Z}{\Delta_y}, \quad K = \frac{\Delta_y}{\Delta_x}.$$

It has been shown by Howarth[3] that for moderate Mach numbers, M can be replaced by $\bar{M}$, the leading edge Mach number without any loss of accuracy. Accordingly, in our subsequent working we replace M by M_0 and take $\gamma = 1.4$. As pointed out earlier we cannot take $M_0 > 1$. Therefore we restrict our analysis to $M_0 = 1$, though, for comparison we have calculated the point of separation for case (I) for the chordwise flow corresponding to $M_0 = 2, 3$.

We shall now discuss the four flows in detail.

Case I: $U = 1 - x$, $V = const.$

(*a*) *Chordwise flow*:

(i) *No suction or injection*, $M_0 = 1$

The equation [2.15] becomes

$$\frac{dN}{dx} = -\frac{U'}{U} \cdot \frac{1269\,N^3 + 44416.8N^2 - 571584\,N + 3991680}{2115\,N^2 + 4500\,N - 124000} \qquad [3.6]$$

which on integration gives

$$\begin{aligned} 5.770998 - ln(1 - x) &= 1.211550\, ln(N + 46.219215) \\ &\quad + 0.227559\, ln(N^2 - 11.217797\,N + 68.056963) \\ &\quad - 0.222399 \tan^{-1}\frac{N - 5.608899}{6.049563}. \end{aligned} \qquad [3.7]$$

Point of separation $x_s = 0.1158$.

Similarly, the point of separation, for the flow, corresponding to $M_0 = 2$ is $x_s = 0.092$ and for $M_0 = 3$, $x_s = 0.070$. From the following table it is evident that the results compare favourably with the results of Stewartson[7].

Point of separation:

	Present method	*Stewartson's results*
$M_0 = 0$	0.127	0.120
$M_0 = 1$	0.116	0.110
$M_0 = 2$	0 092	0.096
$M_0 = 3$	0.070	0.077

(ii) *Suction with* $S = -1$, $M_0 = 1$

The equation (2.15) becomes

$$\frac{dN}{dx} = -\frac{U'}{U} \cdot \frac{399N^3 + 14099.4N^2 - 109922.4N + 745113.6}{665N^2 + 2457N - 32760}, \qquad [3.8]$$

which on integration gives

$$5.623240 - ln(1-x)$$
$$= 1.233578\, ln(N + 42.794224) + 0.216544\, ln(N^2 - 7.457382N + 43.638134)$$
$$- 0.286394 \tan^{-1}\frac{N - 3.728691}{5.45980}. \qquad [3.9]$$

Point of separation $x_s = 0.170$.

(iii) *Injection with* $S = +1$, $M_0 = 1$

The equation [2.15] becomes

$$\frac{dN}{dx} = -\frac{U'}{U}\cdot\frac{7743N^3 + 275200.2N^2 - 5253534N + 40548816}{12905N^2 + 7425N - 922350}, \qquad [3.10]$$

which on integration gives

$$5.933482 - ln(1-x) = 1.194168\, ln(N + 50.894739)$$
$$+ 0.236249\, ln(N^2 - 15.352933N + 102.895360)$$
$$- 0.170068 \tan^{-1}\frac{N - 7.676466}{6.630778}. \qquad [3.11]$$

Point of separation $x_s = 0.082$

(*b*) *Spanwise flow* (i) *No suction or injection,* $M_0 = 1$

The equation [2.16] becomes

$$A\frac{\partial F}{\partial K}\cdot\frac{dK}{dN} + A\frac{\partial F}{\partial N} + BF + C = 0, \qquad [3.12]$$

where $A = -10\, UN(dN/dx)$,

$$B = -\frac{(208584N^2 - 2485920N + 19958400)}{2115N^2 + 4500N - 124000},$$

$$C = 20/K,$$

$$F = \frac{1}{498960K^5}[-166320K^6 - (8316N - 166320)K^5 + (4950N - 66000)K^4 -$$
$$- (660N - 6160)K + (243N - 2160)],$$

and where $K = 1$ when $N = 0$.

The solution of equation [3 12] is presented in the following table for selected values of N.

$-N$	0.1	0.2	0.4	0.6	0.8	1.0	2.0	3.0	4.0	5.0	6 0	6.7
K	1.0012	1.0027	1.0056	1.0083	1.0110	1.0137	1.0264	1.0390	1.0520	1.0658	1.0808	1.0920

(ii) *Suction with* $S = -1$, $M_0 = 1$

The equation [2.16] becomes

$$A\frac{\partial F}{\partial K}\cdot\frac{dK}{dN} + A\frac{\partial F}{\partial N} + BF + C = 0, \qquad [3.13]$$

where $A = -10\, N\, U\, [dN/dx]$

$$B = \frac{-(63126\, N^2 - 451332\, N + 3725568)}{665\, N^2 + 2457\, N - 32760},$$

$$C = -12\,(K^3 - 9\,K^2 + 36\,K - 60)/(K^3 - 9\,K^2 + 36\,K).$$

$$F = \frac{1}{388080\,(K^7 - 9\,K^6 + 36\,K^5)}\,[-\,194040\,K^8 - (6930\,N - 1670130)\,K^7$$
$$+ (62370\,N - 5717250)\,K^6 - (249480\,N - 4241160)\,K^5$$
$$+ (119460\,N - 1356300)\,K^4 - (4785\,N - 35145)\,K^3$$
$$+ (15525\,N - 113565)K^2 - (23820\,N - 173100)\,K + (7020\,N - 48780)]$$

and where $K = 1$ when $N = 0$.

The solution of equation [3 13] is presented in the following table for some selected values of N:

$-N$	0.1	0.2	0.4	0.6	0.8	1.0	2.0	3.0	4.0	5.0	6.0	7.0	7.5
K	1.0023	1 0044	1.0086	1.0126	1.0164	1.0202	1.0375	1.0537	1.0698	1.0867	1.1051	1.1252	1.1361

(iii) *Injection with* $S = +1$, $M_0 = 1$

The equation [2.16] becomes

$$A\frac{\partial F}{\partial K}\cdot\frac{dK}{dN} + A\frac{\partial F}{\partial N} + BF + C = 0, \qquad [3.14]$$

where $A = -10\,N\,U\,dN/dx$,

$$B = \frac{-(1353726\,N^2 - 23500620\,N + 202744080)}{12905\,N^2 + 7425\,N - 922350},$$

$$C = 12\,(K^3 + 9\,K^2 + 36\,K + 60)/(K^3 + 9\,K^2 + 36\,K),$$

$$F = \frac{1}{637560\,(K^7 + 9\,K^6 + 36\,K^5)}\,[-318780\,K^8 - (9702\,N + 2321550)\,K^7 -$$
$$-(87318\,N + 5592510)\,K^6 - (349272\,N - 8232840)\,K^5$$
$$+(265650\,N - 4136550)\,K^4 - (7095\,N - 82665)\,K^3$$
$$-(19539\,N - 228525)\,K^2 - (21396\,N - 252780)\,K$$
$$+(10476\,N - 116820)],$$

and where $K = 1$ when $N = 0$.

The solution of equatiod [3.15] is presented in the following table for some selected values of N:

−N	0.1	0.2	0.4	0.6	0.8	1.0	2.0	3.0	4.0	5.0	6.0
K	1.0010	1.0020	1.0041	1.0061	1.0081	1.0100	1.0199	1.0300	1.0406	1.0520	1.0643

Case II. $U = 1 + x$, $V = const$, $M_0 = 1$

(*a*) *Chordwise flow*:

(i) *No suction or injection*

The equation [2.15] becomes

$$\frac{dN}{dx} = -\frac{U'}{U} \cdot \frac{42\,N^3 + 2344.2\,N^2 - 48082.8\,N + 432432}{70\,N^2 + 285\,N - 9850} \qquad [3.15]$$

which on integration gives

$$6.483165 - ln\,(1 + x) = 1.209923\,ln\,(N + 73.338706)$$
$$+ 0.228372\,ln\,(N^2 - 17.524420N + 140.038974)$$
$$- 0.189331\,\tan^{-1}\frac{N - 8.762210}{7.954411}$$

Range of applicability, $x_R = 0.153$.

(ii) *Suction with* $S = -1$

The equation [2.15] becomes

$$\frac{dN}{dx} = -\frac{U'}{U} \cdot \frac{46536\,N^3 + 2574658.8\,N^2 - 36130147.2\,N + 300886185.6}{77560\,N^2 + 455742\,N - 9218160} \quad [3.16]$$

which on integration gives

$$6.359773 - ln(1+x) = 1.225092\,ln(N + 68.117468)$$
$$+ 0.220788\,ln(N^2 - 12.791294\,N + 94.919271)$$
$$- 0.250750\tan^{-1}\frac{N - 6.395647}{7.142476}.$$

Range of applicability, $x_R = 0.231$.

(iii) *Injection with* $S = +1$

The equation [2.15] becomes

$$\frac{dN}{dx} = -\frac{U'}{U} \cdot \frac{76272\,N^3 + 4350589.2\,N^2 - 121601174.4N + 1218506889.6}{127120\,N^2 + 291906\,N - 21292560} \quad [3.17]$$

which on integration gives

$$6.520083 - ln(1+x) = 1.197321\,ln(N + 79.593029)$$
$$+ 0.234673\,ln(N^2 - 22.552579N + 200.718565)$$
$$- 0.145911\tan^{-1}\frac{N - 11.276289}{8.459542}.$$

Range of applicability $x_R = 0.123$

(*b*) *Spanwise flow* :

(i) *No suction or injection* :

The equation [2.16] reduces to

$$A\frac{\partial F}{\partial K}\cdot\frac{dK}{dN} + A\frac{\partial F}{\partial N} + BF + C = 0. \quad [3.18]$$

where $A = -10\,N U \dfrac{dN}{dx}$,

$$B = \frac{-(10866N^2 - 210864\,N + 2162160)}{70\,N^2 + 285\,N - 9850},$$

$$C = 24/K,$$

$$F = \frac{1}{360360}\,[-(135\,N - 900)K^7 + (780N - 4680)K^6 - (1456N - 7280)K^5 + 3575\,NK^3 - (4290N + 42900)\,K^2]$$

and where $K = 1$ when $N = 0$.

The solution of equation [3.18] is presented in the following table for some selected values of N:

$+N$	0.1	0.2	0.4	0.6	0.8	1.0	2.0	3.0	4.0	5.0	5.5
K	0.9991	0.9984	0.9969	0.9954	0.9938	0.9922	0.9834	0.9731	0.9607	0.9454	0.9364

(ii) *Suction with* $S = -1$:

The equation [2.16] becomes

$$A\frac{\partial F}{\partial K}\cdot\frac{dK}{dN} + A\frac{\partial F}{\partial N} + BF + C = 0, \qquad [3.19]$$

where $A = -10N\,U\dfrac{dN}{dx}$,

$$B = \frac{-(11506068\,N^2 - 152996256\,N + 1504430928)}{77560\,N^2 + 455742N - 9218160}$$

$$C = -12\,(K^3 - 12\,K^2 + 60K - 120)\,/\,(K^3 - 12\,K^2 + 60\,K),$$

$$F = \frac{1}{1765764\,(K^2 - 12K + 60)}\,[-(1512N - 7560)K^9 + (21630N - 104328)K^8 - (124614N - 577206)K^7 + (323973N - 1461798)K^6 - (246480N - 1647360)K^5 - (634491N + 1647360)K^4 + (1711710N + 7207200)K^3 - (1415700N + 15444000)K^2],$$

where $K = 1$ when $N = 0$

The solution of equation [3.19] is presented in the following table for some selected values of N:

$+N$	0.1	0.2	0.4	0.6	0.8	1.0	2.0	3.0	4.0	5,0	6.0	7.0
K	0.9988	0.9978	0.9955	0.9932	0.9908	0.9883	0.9774	0.9573	0.9359	0.9096	0.8775	0.8397

(iii) *Injection with* $S = +1$

The equation [2.16] reduces to

$$A\frac{\partial F}{\partial K}\cdot\frac{dK}{dN} + A\frac{\partial F}{\partial N} + BF + C = 0, \qquad [3.20]$$

where $A = -10N\,U\dfrac{dN}{dx}$,

$$B = \frac{-(20877228N^2 - 544128192N + 6092534448)}{127120N^2 + 291906N - 21292560}$$

$$C = 12(K^3 + 12K^2 + 60K + 120)\,/\,(K^3 + 12K^2 + 60K),$$

$$F = \frac{1}{2630628(K^2 + 12K + 60)}[-(2376N - 20520)K^9 - (7470N - 75384)K^8$$

$$+ (37242N - 278154)K^7 + (0.117819N - 1192230)K^6$$

$$-(595920N - 4380480)K^5 + (541827N - 1647360)K^4 + (785070N - 7207200)K^3$$

$$-(1673100N + 15444000)K^2],$$

and where $K = 1$ when $N = 0$.

The solution of equation [3.20] is presented in the following table for some selected values of N:

$+N$	0.1	0.2	0.4	0.6	0.8	1.0	2.0	3.0	4.0	5.0	5.5
K	0.9991	0.9987	0.9976	0.9965	0.9954	0.9942	0.9881	0.9815	0.9738	0.9647	0.9594

Case III: $U = \dfrac{1}{1+x}$, $V = const$, $M_0 = 1$.

(a) *Chordwise flow*:

(i) *No suction or injection*

The equation [2.15] becomes

$$\frac{dN}{dx} = \frac{-U'}{U} \cdot \frac{423N^3 + 41416.8N^2 - 323584N + 3991680}{2115N^2 + 4500N - 124000} \qquad [3.21]$$

which on integration becomes

$$22.3300760 + ln(1+x) = 4\ 495943\ ln(N + 105.971075) + 0.252028\ ln\ (N^2 - 8.059018N + 89.048833) - 0.528910 \tan^{-1} \frac{N - 4\ 029509}{8.532988}$$

Point of separation $x_s = 0\ 146$.

(ii) *Suction with* $S = -1$:

The equation [2.15] becomes

$$\frac{dN}{dx} = -\frac{U'}{U} \cdot \frac{133\ N^3 + 12461.4\ N^2 - 44402.4\ N + 745113.6}{665\ N^2 + 2457\ N - 32760} \qquad [3\ 22]$$

which on integration gives

$$21.969780 + ln\ (1 + x) = 4.570338\ ln\ (N + 97.698835) + 0.214831\ ln\ (N^2 - 4.004098\ N + 57.343094) - 0.594751 \tan^{-1} \frac{N - 2.002049}{7.303074}$$

Point of separation $x_s = 0.243$

(iii) *Injection with* $S = +1$:

The equation [2 15] becomes

$$\frac{dN}{dx} = -\frac{U'}{U} \cdot \frac{2581\ N^3 + 270250\ 2\ N^2 - 3408434\ N + 40548816}{12905\ N^2 + 7425\ N - 922350} \qquad [3\ 23]$$

which on integration gives

$$22.778357 + ln\ (1 + x) = 4.435672\ ln\ (N + 117\ 127509) + 0\ 282164\ ln\ (N^2 - 12.419954\ N + 134.131655) - 0.473230 \tan^{-1} \frac{N - 6\ 209977}{9\ 775880}.$$

Point of separation $x_s = 0\ 098$

(*b*) *Spanwise flow*:

(i) *No suction or injection*:

The equation [2.16] becomes

$$A\frac{\partial F}{\partial K}\cdot\frac{dK}{dN} + A\frac{\partial F}{\partial N} + BF + C = 0, \qquad [3.24]$$

where $A = -10NU\dfrac{dN}{dx}$,

$$B = \frac{-(16920N^3 + 238584N - 2485920N + 19958400)}{2115\,N^2 + 4500\,N - 124000}$$

and C and F have the same values as in equation [3.12] and where $K = 1$ when $N = 0$.

The solution of equation [3.24] is presented in the following table for some selected values of N;

$-N$	0.1	0.2	0.4	0 6	0.8	1.0	2.0	3.0	4.0	5.0	6.0	6.7
K	1.0012	1.0027	1.0055	1.0082	1.0107	1.0132	1.0246	1.0353	1.0464	1.0584	1.0718	1.0823

(ii) *Suction with* $S = -1$:

The equation [2.16] becomes

$$A\frac{\partial F}{\partial K}\cdot\frac{dK}{dN} + A\frac{\partial F}{\partial N} + BF + C = 0, \qquad [3.25]$$

where $A = -10NU\dfrac{dN}{dx}$,

$$B = \frac{-(5320N^3 + 79506N^2 - 451332N + 3725568)}{665N^2 + 2457N - 32760},$$

and C and F have the same values as in equation [3.13] and where $K = 1$ when $N = 0$.

The solution of equation [3.25] is presented in the following table for some selected values of N:

$-N$	0.1	0.2	0.4	0.6	0.8	1.0	2.0	3.0	4.0	5,0	6.0	7.0	7.5
K	1.0023	1.0044	1.0084	1.0122	1.0158	1.0192	1.0341	1.0467	1.0590	1.0722	1.0875	1.1054	1.1156

(iii) *Injection with* $S = +1$:

The equation [2.16] becomes

$$A \frac{\partial F}{\partial K} \cdot \frac{dK}{dN} + A \frac{\partial F}{\partial N} + BF + C = 0\,, \qquad [3.26]$$

with $A = -10NU\dfrac{dN}{dx}$,

$$B = \frac{-(103240N^8 + 1403226N^2 - 23498620N + 202744080)}{12905N^2 + 7425N - 922350},$$

and C and F have the same values as in equation [3.14] and where $K = 1$ and $N = 0$.

The solution of equation [3.26] is presented in the following table for some selected values of N:

$-N$	0.1	0.2	0.4	0.6	0 8	1.0	2.0	3.0	4.0	5.0	6.0
K	1.0010	1.0020	1.0040	1.0060	1.0079	·1.0097	1.0189	1.0279	1.0374	1.0477	1.0591

Case IV. $U = \dfrac{1}{1-x}$, $V = \text{const.}$ $M_0 = 1$

(*a*) *Chordwise flow* :

(i) *No suction or injection* :

The equation [2.15] becomes

$$\frac{dN}{dx} = \frac{-U'}{U} \cdot \frac{14N^3 + 2154.2N^2 - 28382.8N + 432432}{70N^2 + 285N - 9850} \qquad [3.27]$$

which on integration gives

$$24.503454 + ln(1-x) = 4.469354\, ln\,(N + 167.109347)$$

$$+ 0.265323\, ln\,(N^2 - 13.237918N + 184.837047)$$

$$- 0.475055 \tan^{-1} \frac{N - 6.618959}{11\ 875455}.$$

Range of profile, $x_R = 0.106$

(ii) *Suction with* $S = -1$:

The equation [2.15] becomes

$$\frac{dN}{dx} = \frac{-U'}{U} \cdot \frac{15512N^3 + 2270830.8N^2 - 17693827\ 2N + 300886185.6}{77560N^2 + 455742N - 9218160} \quad [3.28]$$

which on integration gives

$$24.14793 + ln(1-x) = 4.521739\ ln(N + 154\ 582553)$$
$$+ 0.239130\ ln(N^2 - 8.190675N + 125.481369)$$
$$- 0\ 533049 \tan^{-1} \frac{N - 4.095337}{10.423260}.$$

Range of profile = 0 1257

(iii) *Injection with* $S = +1$:

The equation [2.15] becomes

$$\frac{dN}{dx} = -\frac{U'}{U} \cdot \frac{25424\,N^3 + 4155985.2\,N^2 - 79016054.4N + 1218506889\ 6}{127120\,N^2 + 291906\,N - 21292560} \quad [3.29]$$

which on integration gives

$$24.931661 + ln(1-x) = 4.405031\ ln(N + 182.482101)$$
$$+ 0.297484\ ln(N^2 - 19.015093N + 361.982578)$$
$$- 0.465442 \tan^{-1} \frac{N - 9.507546}{16.480933}.$$

Range of applicability $x_R = 0.073$.

(*b*) *Spanwise flow* :

The equation [2.16] becomes

$$A\frac{\partial F}{\partial K} \cdot \frac{dK}{dN} + A\frac{\partial F}{\partial N} + BF + C = 0, \quad [3.30]$$

with $A = -10\,NU\dfrac{dN}{dx}$,

$$B = \frac{-(560\,N^3 + 12766N^2 - 210864N + 2162160)}{70N^2 + 285N - 9850},$$

and C and F have the same values as in equation [3.18] and where $K = 1$ when $N = 0$.

The solution of equation [3.30] is presented in the following table for some selected values of N:

$+N$	0.1	0.2	0.4	0.6	0.8	1.0	2.0	3.0	4.0	5.0
K	0.9991	0.9984	0.9969	0.9953	0.9936	0.9919	0.9820	0.9699	0.9549	0.9367

(ii) *Suction with* $S = -1$

The equation [2.16] becomes

$$A\frac{\partial F}{\partial K}\cdot\frac{dK}{dN} + A\frac{\partial F}{\partial N} + BF + C = 0, \quad [3.31]$$

with $A = -10NU\dfrac{dN}{dx}$,

$$B = \frac{-(620480N^3 + 14544348N^2 - 152996256N + 1504430928)}{77560N^2 + 455742N - 9218160}$$

and C and F have the same values as in equation [3.19] and where $K = 1$ when $N = 0$.

The solutions of equation [3.31] is presented in the following table

$+N$	0.1	0.2	0.4	0.6	0.8	1.0	2.0	3.0	4.0	5.0
K	0.9998	0.9977	0.9955	0.9931	0.9905	0.9879	0.9724	0.9529	0.9286	0.8994

(iii) *Injection with* $S = +1$:

The equation [2.16] reduces to

$$A\frac{\partial F}{dK}\cdot\frac{dK}{dN} + A\frac{\partial F}{\partial N} + BF + C = 0, \quad [3.32]$$

with $A = -10NU\dfrac{dN}{dx}$,

$$B = \frac{-(1016960N^3 + 22823268N^2 - 544128192N + 6092534448)}{127120N^2 + 291906N - 21292560}$$

and C and F have the same values as in equation [3.20] and where $K = 1$ when $N = 0$.

The solution of equation [3.32] is presented in the following table for some selected values of N:

$+N$	0.1	0.2	0.4	0.6	0.8	1.0	2.0	3.0	4.0	5.0	6.0
K	0.9991	0 9986	0.9976	0.9964	0.9952	0.9940	0.9872	0.9792	0.9696	0.9580	0.9441

Skin Friction

4. For the flows (I) and (III), the skin-friction is given in the table below:

	$T_x \equiv \dfrac{\tau_x}{\mu_s} \sqrt{(\nu_s)}$	$T_y \equiv \dfrac{\tau_y}{\mu_s V} \sqrt{(\nu_s)}$
$S = 0$	$U \dfrac{3N+20}{12} \left(\dfrac{U'}{N}\right)^{\frac{1}{2}} \cdot \left(\dfrac{p}{p_s}\right)^{\frac{1}{2}}$	$\dfrac{5}{3K} \left(\dfrac{U'}{N}\right)^{\frac{1}{2}} \cdot \left(\dfrac{p}{p_s}\right)^{\frac{1}{2}}$
$S = -1$	$U \dfrac{2N+15}{7} \left(\dfrac{U'}{N}\right)^{\frac{1}{2}} \cdot \left(\dfrac{p}{p_s}\right)^{\frac{1}{2}}$	$\dfrac{60}{K(K^2 - 9K + 36)} \left(\dfrac{U'}{N}\right)^{\frac{1}{2}} \cdot \left(\dfrac{p}{p_s}\right)^{\frac{1}{2}}$
$S = +1$	$U \dfrac{5N+30}{23} \left(\dfrac{U'}{N}\right)^{\frac{1}{2}} \cdot \left(\dfrac{p}{p_s}\right)^{\frac{1}{2}}$	$\dfrac{60}{K(K^2 + 9K + 36)} \left(\dfrac{U'}{N}\right)^{\frac{1}{2}} \cdot \left(\dfrac{p}{p_s}\right)^{\frac{1}{2}}$

and for the flows (II) and (IV), in the table below:

	$T_x \equiv \dfrac{\tau_x}{\mu_s} \sqrt{(\nu_s)}$	$T_y \equiv \dfrac{\tau_y}{\mu_s V} \sqrt{(\nu_s)}$
$S = 0$	$U \dfrac{N+10}{5} \left(\dfrac{U'}{N}\right)^{\frac{1}{2}} \cdot \left(\dfrac{p}{p_s}\right)^{\frac{1}{2}}$	$\dfrac{2}{K} \left(\dfrac{U'}{N}\right)^{\frac{1}{2}} \cdot \left(\dfrac{p}{p_s}\right)^{\frac{1}{2}}$
$S = -1$	$U \dfrac{11N+120}{49} \left(\dfrac{U'}{N}\right)^{\frac{1}{2}} \cdot \left(\dfrac{p}{p_s}\right)^{\frac{1}{2}}$	$\dfrac{120}{K(K^2 - 12K + 60} \left(\dfrac{U'}{N}\right)^{\frac{1}{2}} \cdot \left(\dfrac{p}{p_s}\right)^{\frac{1}{2}}$
$S = +1$	$U \dfrac{13N+120}{73} \left(\dfrac{U'}{N}\right)^{\frac{1}{2}} \cdot \left(\dfrac{p}{p_s}\right)^{\frac{1}{2}}$	$\dfrac{120}{K(K^2 + 12K + 60} \left(\dfrac{U'}{N}\right)^{\frac{1}{2}} \cdot \left(\dfrac{p}{p_s}\right)^{\frac{1}{2}}$

where

$$\frac{p}{p_s} = \left[1 - \frac{\gamma - 1}{2} \cdot \frac{U^2}{a_1^2}\right]^{\gamma/\gamma - 1},$$

and the inverse transformation gives

$$\partial_x = \left(\frac{p_s}{p}\right)^{\frac{1}{2}} \Delta_x \left[1 - \frac{\gamma - 1}{2 a_s^2} U^2 \int_0^1 \frac{u^2}{U^2} d\eta_x \right]$$

$$\partial_y = K \partial_x.$$

CONCLUSIONS

(i) From Figs. (1) and (2) it is clear that the skin-friction for compressible flow is less than the corresponding value for incompressible flow. The reduced skin friction is the result of increased boundary layer thickness in compressible flow.

(ii) Figs. (3), (4), (5) and (6) show the relative values of skin-friction for the four flows corresponding to $S = 0, -1, 1$. We notice that the effects of suction and injection are similar to those for the incompressible flow.

ACKNOWLEDGMENT

The author expresses his gratitude to Prof. P. L. Bhatnagar for suggesting the investigation and for help and guidance throughout the preparation of this paper.

The author is also grateful to the Indian Institute of Science, for making a grant which enabled the numerical work to be performed on the digital computer at the Hindustan Aircraft Ltd.

REFERENCES

1. Ahuja, G. C. and Bhatnagar, P. L. .. Three dimensional boundary layer on a yawed semi-ifinite flat plate with or without suction and injection. *Proc.Ind.Acad.Sci.*, 1963, 57, 148.
2. Ahuja, G. C. .. .. .. Boundary layer on a yawed semi-infinite flat plate in converging and diverging flows with suction and injection. Communicated for publication to ZAMM.
3. Howarth, L. .. .. .. Concerning the effect of compressibility on laminar boundary layers and their separation. *Proc.Roy.Soc.,Lond.*, 1948, 194, 16
4. Grabtree, L. F. .. .. The commpressible laminar boundary layer on a yawed infinite wing. *Aero.Quart.*, 1954, 5 (85).
5. Tinkler, J. .. .. .. Three dimensional boundary layer flow. A.R.C.,F.M , 2269, 1955.
6. Illingworth, C. R. .. .. Steady flow in the laminar boundary layer of a gas. *Proc. Roy.Soc.*, 1949, 199, 533.
7. Stewartson, K. .. .. Correlated incompressible and compressible boundary layers. *Proc. Roy. Soc.*, 1949, 200, 84.